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EMBEDDINGS OF L-GROUPS

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To a real reductive group G there is attached a family of (real) groups, each of lower dimension but sharing Cartan subgroups with G (cf. [**8**]). The purpose of these groups is to provide "building blocks" (in a specific sense (cf. [**11**])) for analysis on G. Their definition is via an L-group construction; the connected component of the identity, ${}^LH^0$, in the L-group of such a group H is naturally a subgroup of ${}^LG^0$, but the requirement that H "share" Cartan subgroups with G precludes defining ${}^LH$, the full L-group of H, as a subgroup of ${}^LG$. Nevertheless, the principle of functoriality in the L-group suggests that the embeddings of ${}^LH$ in ${}^LG$ will play a role in analysis. In this paper, we study the embeddings of ${}^LH$ in ${}^LG$ in order to resolve a problem about the normalization of orbital integrals.

Our method is based on the proof of the Langlands correspondence for discrete series representations of real groups ([**7**]). Thus we attach to an embedding of ${}^LH$ in ${}^LG$ two elements in a certain vector space, and then show that these elements satisfy some congruence relations. We thereby attach to the embedding quasicharacters on various Cartan subgroups of G. The arguments for the congruences are very simply summarized in terms of the embeddings of the L-group ${}^LT$ of a Cartan subgroup T of G in ${}^LG$. Such embeddings are severely constrained; if T is common to H and G then given ${}^LT \hookrightarrow {}^LH$ and ${}^LH \hookrightarrow {}^LG$ we obtain ${}^LT \hookrightarrow {}^LG$ and so have information about ${}^LH \hookrightarrow {}^LG$.

We defer the recovery of an embedding of ${}^LH$ in ${}^LG$ from its congruences until after the normalization of orbital integrals, as the results there offer some guidance.

In order to transfer certain ("κ-" (cf. [**10**])) orbital integrals from G to stable orbital integrals on H, it is essential first to normalize the integrals on G (cf. [**10**, esp. Theorem 8.3]); thus we must specify some functions on the Cartan subgroups common to H and G. The roots in H of such a Cartan subgroup T may be identified as roots in G. We write a potential normalizing function on T as

$$\pm\, c(\gamma) \prod_{\substack{\alpha \text{ positive root,} \\ \text{not in } H, \\ \text{imaginary}}} (1 - \alpha(\gamma)^{-1}) \times \prod_{\substack{\alpha \text{ positive root,} \\ \text{not in } H, \\ \text{not imaginary}}} |\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}|, \quad \gamma \in T.$$

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The role of the term $c(\gamma)$ is to match the transformation of the function under a certain Weyl group with that of unnormalized κ-orbital integrals, and to make the various functions "compatible" as we move among the Cartan subgroups common to H and G.

In [**10**] we assumed

$$c(\gamma) = \prod_{\substack{\alpha \text{ positive root} \\ \text{not in } H, \\ \text{not imaginary}}} \alpha(\gamma)^{1/2}$$

to be well-defined (and more, to ensure compatibility) and showed that, for consistent choice of $\pm$, the resultant normalizing functions do provide a transfer of orbital integrals from G to H. While no embedding of ${}^LH$ in ${}^LG$ is present explicitly in that example, one consequence of the main result in the present paper (Theorem 8.0.1) will be that one does exist.

The assumptions above are undesirable because they fail in some simple cases, and no "natural" remedy appears available. There is also a functorial reason for not always using those $c(\gamma)$'s: given an embedding of ${}^LH$ in ${}^LG$ we may ask that the terms $c(\gamma)$ be compatible with the embedding in the sense that dual to the transfer of orbital integrals from G to H provided by the $c(\gamma)$'s we get a lifting of tempered characters from H to G, which is consistent with the map on L-packet parameters (that is, the map $\Phi(H) \to \Phi(G)$ ([**3**])) induced by ${}^LH \hookrightarrow {}^LG$ (cf. [**11**]). We do not pursue this explicitly in the present paper.

Consider an embedding of ${}^LH$ in ${}^LG$ and its attached quasicharacters. We prove two properties of the quasicharacters and then abstract these properties in the definition of a "set of correction characters". An examination of [**10**] shows that any set of correction characters can be used as $c(\gamma)$'s; that is, for consistent choice of $\pm$, the resulting set of normalizing functions provides a transfer of orbital integrals from G to H. As our terminology suggests, correction characters are the only quasicharacters which will do for the $c(\gamma)$'s.

We come then to recovering an embedding of ${}^LH$ in ${}^LG$ from its congruences. We know that these congruences must be "correction congruences"; that is, that the attached quasicharacters must form a set of correction characters. From now on, assume that G is quasi-split. We will construct an embedding whose attached quasicharacters are a given set of correction characters. We will also prove that two embeddings have the same attached quasicharacters if and only if they are Φ-equivalent in the sense that they induce the same map $\Phi(H) \to \Phi(G)$ on L-packet parameters. Our main result thus follows: a one to one correspondence between Φ-equivalence classes of embeddings of ${}^LH$ in ${}^LG$ and sets of correction characters (equivalently, correction congruences).

We have not attempted to solve correction congruences and so determine the existence of an embedding of ${}^LH$ in ${}^LG$. Recall that, according to

[**8**], there is always an embedding of ${}^LH$ in ${}^LG$ if the center of ${}^LG^0$ is connected, and a counter-example (in type $E_7 \times A_n$) if the center of ${}^LG^0$ is not connected. As an exercise, we will use one simple congruence to generate examples and counterexamples for the case that H shares a fundamental Cartan subgroup of G, and this subgroup is compact modulo the center of G.

On the other hand, a standard construction and our main result give a simple answer to the question of uniqueness for Φ-equivalence classes of embeddings of ${}^LH$ in ${}^LG$, or sets of correction characters.

The section headings indicate the organization of the paper. Notation follows [**9**] and [**10**] whenever possible.

Our arguments owe much to [**8**] and the unpublished manuscript [**7**]. It is Lemma 3.2 of [**7**] which explains why "one half the sum of the positive imaginary roots" plays a central role in the embeddings of L-groups of real groups, and which we use frequently in this paper. With the author's permission, we have included his proof of the lemma in an appendix.

1. L-groups. In this and the next section we emphasize some technical points, in order to make matters easier for the later sections. As an introduction to embeddings of L-groups, we will examine the embeddings of ${}^LT$ in ${}^LG$, for T a Cartan subgroup of G, compact modulo the center of G. The results hinge on Lemma 3.2 of [**7**].

We follow our earlier conventions for algebraic groups ([**9**]): $\underset{\sim}{G}$ will be a connected reductive linear algebraic group defined over $\mathbf{R}$ and G the group of its $\mathbf{R}$-rational points. When convenient, we identify $\underset{\sim}{G}$ with its $\mathbf{C}$-rational points. If $\underset{\sim}{T}$ is a maximal torus in $\underset{\sim}{G}$ defined over $\mathbf{R}$, we call T a Cartan subgroup, in accordance with Lie group terminology. For any torus over $\mathbf{R}$ or $\mathbf{C}$ we write $L(\ )$ for the lattice of rational characters and $L^\vee(\ )$ for the cocharacters; $\langle\ ,\ \rangle$ denotes the pairing between $L(\ )$ and $L^\vee(\ )$.

(1.1) *Notation.* For once and for all we fix data for an L-group of $\underset{\sim}{G}$:

(1.1.1) a quasi-split inner form $\underset{\sim}{G}^*$ of $\underset{\sim}{G}$ and an inner twist $\psi : \underset{\sim}{G} \to \underset{\sim}{G}^*$,

(1.1.2) a Borel subgroup $\underset{\sim}{B}^*$ of $\underset{\sim}{G}^*$ over $\mathbf{R}$ containing a maximal torus $\underset{\sim}{T}^*$ over $\mathbf{R}$,

(1.1.3) a connected reductive group ${}^LG^0$ over $\mathbf{C}$ and Borel subgroup ${}^LB^0$ containing a maximal torus ${}^LT^0$, such that $L({}^LT^0) = L^\vee(\underset{\sim}{T}^*)$ and the simple roots of ${}^LT^0$ in ${}^LB^0$ are the coroots of the simple roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{B}^*$,

(1.1.4) for each simple root $\alpha^\vee$ of ${}^LT^0$ in ${}^LB^0$, a root vector $X_{\alpha^\vee}$.

We write L for $L(\underset{\sim}{T}^*)$ and $L^\vee$ for $L^\vee(\underset{\sim}{T}^*) = L({}^LT^0)$. We denote by σ_{T^*} the Galois action on $\underset{\sim}{T}^*$ (and its usual transfer to L, $L^\vee$ and ${}^LT^0$) and by

σ_G that extension of σ_{T^*} to ${}^LG^0$ which satisfies

$$\sigma_G X_{\alpha^\vee} = X_{\sigma_G \alpha^\vee}.$$

Finally, ${}^LG = {}^LG^0 \rtimes W$, where W is the Weil group of $\mathbf{C}/\mathbf{R}$, which we realize as

$$\{z \times \tau : z \in \mathbf{C}^\times, \tau \in \langle 1, \sigma\rangle\},$$

with multiplication

$$(z_1 \times \tau_1)(z_2 \times \tau_2) = a_{\tau_1,\tau_2} z_1 \tau_1(z_2) \times \tau_1\tau_2,$$

where $a_{1,1} = a_{\sigma,1} = a_{1,\sigma} = 1$, $a_{\sigma,\sigma} = -1$; on ${}^LG^0$, $\mathbf{C}^\times \times 1$ is to act trivially and $1 \times \sigma$ by σ_G.

It is the pair $({}^LG, \psi)$ that defines an L-group for $\underset{\sim}{G}$, although we usually omit ψ in notation. When we restrict our attention to a quasi-split group (for example, the group "$\underset{\sim}{H}$" to be introduced) we may take $\underset{\sim}{G}^* = \underset{\sim}{G}$ and omit ψ altogether.

(1.2) *Standard Levi subgroups in* ${}^LG$. If $\underset{\sim}{T}$ is a maximal torus in $\underset{\sim}{G}$ defined over $\mathbf{R}$ we write $\underset{\sim}{S}_T$ (or just $\underset{\sim}{S}$) for the maximal $\mathbf{R}$-split torus in $\underset{\sim}{T}$ and $\underset{\sim}{M}_T$ (or just $\underset{\sim}{M}$) for the centralizer of $\underset{\sim}{S}_T$ in $\underset{\sim}{G}$.

We consider first a torus $\underset{\sim}{T}$ in $\underset{\sim}{G}^*$. By [**9**] there exists $g \in \mathfrak{A}(T)$ such that $\underset{\sim}{S}_{gTg^{-1}} \subseteq \underset{\sim}{S}_{T^*}$. For our purposes it will be enough to consider instead $g\underset{\sim}{T}g^{-1}$. Thus we assume also that $\underset{\sim}{S}_T \subseteq \underset{\sim}{S}_{T^*}$. Working with $(\underset{\sim}{M}, \underset{\sim}{B}^* \cap \underset{\sim}{M}, \underset{\sim}{T}^*)$ we see that ${}^LM$ is naturally a subgroup of ${}^LG$; ${}^LM^0$ is the subgroup of ${}^LG^0$ generated by ${}^LT^0$ and the coroots of the simple roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{B}^* \cap \underset{\sim}{M}$, and σ_M is the restriction of σ_G to ${}^LM^0$ (see [**7**, § 2] for more general considerations).

Passing to $\underset{\sim}{G}$, suppose now that $\underset{\sim}{T}$ is a maximal torus in $\underset{\sim}{G}$. We may fix $x \in \underset{\sim}{G}^*$ so that

$$\psi_x = \text{ad } x \circ \psi : \underset{\sim}{T} \rightarrow \underset{\sim}{G}^*$$

is defined over $\mathbf{R}$ ([**7**]). Let $\underset{\sim}{T}'$ be the image of $\underset{\sim}{T}$. We may and do require of x that $\underset{\sim}{S}_{T'} \subseteq \underset{\sim}{S}_{T^*}$. The map ψ_x is an inner twist from $\underset{\sim}{M}_T$ to $\underset{\sim}{M}_{T'}$ and $({}^L(M_{T'}), \psi_x)$ is an L-group for $\underset{\sim}{M}_T$, which we denote simply by ${}^LM_T$. We will call ${}^LM_T$ a *standard Levi subgroup* in ${}^LG$.

(1.3) *Embeddings of the L-group of a Cartan subgroup*. Suppose that $\underset{\sim}{T}$ and $\underset{\sim}{T}'$ are as in the last paragraph. Then ψ_x induces an isomorphism ${}^LT \rightarrow {}^L(T')$. By an *embedding of* ${}^LT$ *in* ${}^LM_T$ we will mean an embedding of ${}^L(T')$ in ${}^L(M_{T'})$. To study such embeddings we change notation and work under the hypothesis:

$\underset{\sim}{T}$ is a maximal torus over $\mathbf{R}$ in $\underset{\sim}{G}^*$, anisotropic modulo the center of $\underset{\sim}{G}^*$.

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First, we describe those embeddings τ of ${}^LT$ in ${}^LG$ which we will call *allowed*. There are two conditions:

$$(1.3.1) \qquad \begin{array}{ccc} {}^LT & \xrightarrow{\ \tau\ } & {}^LG \\ & {\scriptstyle \text{proj.}} \searrow \quad \swarrow {\scriptstyle \text{proj.}} & \\ & W & \end{array} \qquad \text{is commutative.}$$

For the second, we use:

Definition 1.3.2. A *pseudo-diagonalization* (p-d.) of $\underset{\sim}{T}$ is a map from $\underset{\sim}{T}$ to $\underset{\sim}{T}^*$ of the form ad $g|\underset{\sim}{T}$, $g \in \underset{\sim}{G}^*$. Our terminology comes from the fact that in examples we usually arrange that $\underset{\sim}{T}^*$ be a diagonal group. Sometimes we call the element g itself a pseudo-diagonalization.

A p-d. of $\underset{\sim}{T}$ induces isomorphisms between $L(\underset{\sim}{T})$ and L and between $L^\vee(\underset{\sim}{T})$ and $L^\vee$, and hence an isomorphism between $({}^LT)^0$, the connected component of the identity in ${}^LT$, and ${}^LT^0$, the distinguished maximal torus in ${}^LG^0$. We require, as our second condition on τ, that

(1.3.3) $\quad \tau|{}^L(T)^0$ is induced by a p-d. of $\underset{\sim}{T}$.

Suppose that we are given a p-d. g of $\underset{\sim}{T}$. We transfer the Galois action of $\underset{\sim}{T}$ to L, $L^\vee$ and ${}^LT^0$ via g (since $\underset{\sim}{T}$ is anisotropic modulo the center of $\underset{\sim}{G}^*$ the choice for g has no effect); we write the result as σ_T. To obtain an allowed embedding ${}^LT \to {}^LG$ which extends the isomorphism $({}^LT)^0 \to {}^LT^0$ induced by g, we need exactly a homomorphism $\tau_W : W \to {}^LG$ such that $\tau_W(w) = \tau_0(w) \times w$, $w \in W$, where $\tau_0(\mathbf{C}^\times \times 1) \subseteq {}^LT^0$ and $\tau_0(1 \times \sigma)$ is an element of the normalizer of ${}^LT^0$ in ${}^LG^0$, such that $\tau_0(1 \times \sigma) \times (1 \times \sigma) = \tau_W(1 \times \sigma)$ acts on ${}^LT^0$ as σ_T. Note that any element n of the normalizer of ${}^LT^0$ in ${}^LG^0$ which maps the positive roots of ${}^LT^0$ in ${}^LG^0$, that is, the roots of ${}^LT^0$ in ${}^LB^0$, to negative ones, has the property that $n \times (1 \times \sigma)$ acts on ${}^LT^0$ as σ_T.

It is an easy consequence of [**8**, Lemma 4] that such a homomorphism τ_W exists; note that for this existence there is no need to assume that $\underset{\sim}{T}$ is anisotropic modulo the center of $\underset{\sim}{G}^*$. Alternatively, we may construct τ_W quite explicitly, via Lemma 3.2 of [**7**], the lemma critical to the proof of the Langlands correspondence for discrete series representations.

Thus suppose that we have an element n of ${}^LG^0$ normalizing ${}^LT^0$ and such that $n \times (1 \times \sigma)$ acts on ${}^LT^0$ as σ_T, together with a homomorphism $\eta : \mathbf{C}^\times \to {}^LT^0$ such that

$$\eta(\bar{z}) = \sigma_T(\eta(z)), \quad z \in \mathbf{C}^\times.$$

Then $n\sigma_G(n) \in {}^LT^0$ and we may write

$$\lambda^\vee(\eta(z)) = z^{\langle \mu, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_T \mu, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee,$$

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for some (unique) $\mu \in L \otimes \mathbf{C}$ with $\mu - \sigma_T\mu \in L$. Thus

$$\tau_W(z \times 1) = \eta(z) \times z, \quad z \in \mathbf{C}^\times,$$
$$\tau_W(1 \times \sigma) = n \times (1 \times \sigma)$$

defines a homomorphism $\tau_W : W \to {}^LG$ if and only if

$$(1.3.4) \quad \lambda^\vee(n\sigma_G(n)) = (-1)^{\langle \mu - \sigma_T\mu, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee.$$

In place of n we could have chosen tn, $t \in {}^LT^0$. To pick the correct n for (1.3.4) we need just the following information: let $\lambda \in L \otimes \mathbf{C}$ be such that

$$\lambda^\vee(n) = e^{2\pi i \langle \lambda, \lambda^\vee \rangle}$$

for any $\lambda^\vee \in L^\vee$ which extends to a rational character on ${}^LG^0$. Although λ is not uniquely determined, an argument as in Lemma 3.3.2 to follow, shows that λ may be replaced only by elements of

$$\lambda + L + \sum_{\substack{\alpha \text{ root} \\ \text{of } \underset{\sim}{T}^* \text{ in } \underset{\sim}{G}^*}} \mathbf{C}\alpha.$$

As a consequence of Lemma 3.2 of [**7**] (cf. § 10) we have:

PROPOSITION 1.3.5. (1.3.4) *is satisfied if and only if*

$$(1.3.6) \quad \tfrac{1}{2}(\mu - \sigma_T\mu) + \iota \equiv (\lambda + \sigma_T\lambda) \bmod L,$$

where ι is one half the sum of the roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{B}^*$.

Proof. The lemma cited computes $\lambda^\vee(n\sigma_G(n))$ as

$$(-1)^{\langle 2\iota, \lambda^\vee \rangle}(\lambda^\vee + \sigma_T\lambda^\vee)(n) = (-1)^{2\langle \iota + \lambda + \sigma_T\lambda, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee.$$

The rest is immediate.

Note that (1.3.6) is easily solved. For example, we obtain an embedding if $\mu = \iota$ and $\lambda = 0$, that is, if we "twist" $\mathbf{C}^\times \times 1$ by ι and choose for n any element of the normalizer of ${}^LT^0$ in the derived group of ${}^LG^0$, which maps positive roots to negative ones. It is a simple exercise to describe the remaining allowed embeddings of ${}^LT$ in ${}^LG$; we omit the details.

If $\tau : {}^LT \to {}^LG$ is an allowed embedding and (μ, λ) are parameters attached to τ_W as above then we write $\tau = \tau(\mu, \lambda)$, some underlying p-d. being understood.

2. (T, κ)-groups.

The groups $\underset{\sim}{H}$ of the introduction were called (T, κ)-groups in [**10**]. We review their definition ([**8**]) in (2.1). First ${}^LH$ is defined. There are some choices but, in any case, ${}^LH^0$ is a subgroup of ${}^LG^0$ and a simple argument shows that the action of $1 \times (1 \times \sigma) \in {}^LH$ on ${}^LH^0$ can be achieved by conjugation with respect to a suitable element of ${}^LG$.

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In (2.2) we introduce standard position for ${}^LH$ in order to realize the action of $1 \times (1 \times \sigma) \in {}^LH$ on ${}^LH^0$ by (conjugation with respect to) an element of a standard Levi subgroup in ${}^LG$ and, further, to make this subgroup as small as possible. Then, after refining the procedure of § 6 in [**10**] for selecting embeddings of Cartan subgroups of H in G ((2.3)), we will be able to carry out many arguments "inside" standard Levi subgroups.

In (2.4) we review the definitions formalizing the notion that various objects attached to $\underset{\sim}{G}$ "come from $\underset{\sim}{H}$."

(2.1) *Definitions.* ([**8**], cf. [**10**], [**11**]). If $\underset{\sim}{T}$ is a maximal torus in $\underset{\sim}{G}$, defined over $\mathbf{R}$, we denote by $\Xi_T \subset L(\underset{\sim}{T})$ the set of roots of $\underset{\sim}{T}$ in $\underset{\sim}{G}$, and by $\Xi_T{}^\vee \subseteq L^\vee(\underset{\sim}{T})$ the set of coroots. By definition, κ is a quasicharacter on $\langle \Xi_T{}^\vee \rangle$, the span of $\Xi_T{}^\vee$ in $L^\vee(\underset{\sim}{T})$, and is trivial on

$$\mathscr{L}(T) = \{\mu^\vee \in \langle \Xi_T{}^\vee \rangle : \mu^\vee = \lambda^\vee - \sigma_T\lambda^\vee,\ \lambda^\vee \in L^\vee(\underset{\sim}{T})\},$$

σ_T denoting the Galois action of $\underset{\sim}{T}$.

Recall the twist $\psi : \underset{\sim}{G} \to \underset{\sim}{G}^*$. We fix some map

$$\psi_x = \operatorname{ad} x \circ \psi : \underset{\sim}{T} \to \underset{\sim}{T}^*,$$

with $x \in \underset{\sim}{G}^*$, and use it to transfer κ and σ_T to $L^\vee$, without change in notation. Thus σ_T is now an automorphism of $L^\vee$ and κ a quasicharacter on $\langle \Xi^\vee \rangle$, the span of the roots $\Xi^\vee$ of ${}^LT^0$ in ${}^LG^0$, trivial on

$$\mathscr{L} = \{\mu^\vee \in \langle \Xi^\vee \rangle : \mu^\vee = \lambda^\vee - \sigma_T\lambda^\vee,\ \lambda^\vee \in L^\vee\}.$$

Such a κ extends to a σ_T-invariant quasicharacter on $L^\vee$. In fact, denoting by Z the center of ${}^LG^0$ and by $(-)^{\sigma_T}$ the σ_T-invariant elements of $-$, we have a commutative diagram (2.1.1):

$$\begin{array}{ccc}
(\operatorname{Hom}(L^\vee, \mathbf{C}^\times))^{\sigma_T} & \xrightarrow[\sim]{\text{canonical}} & ({}^LT^\circ)^{\sigma_T} \\
\Big\downarrow {\scriptstyle \text{restriction, followed by projection}} & & \Big\downarrow {\scriptstyle \text{projection from } {}^LG^\circ \text{ to adjoint group}} \\
\operatorname{Hom}(\langle \Xi^\vee \rangle/\mathscr{L}, \mathbf{C}^\times) & \xrightarrow{\sim} & Z({}^LT^\circ)^{\sigma_T}/Z = ({}^LT^\circ)^{\sigma_T}/Z \cap ({}^LT^\circ)^{\sigma_T}
\end{array}$$

Thus we may regard κ as an element of ${}^LT^0$, some choice being required. The centralizer of κ in ${}^LG^0$ is independent of that choice and ${}^LH^0$ is the connected component of the identity in this centralizer.

Fix a Borel subgroup ${}^LB_H{}^0$ of ${}^LH^0$ containing ${}^LT^0$, and for each simple root $\alpha^\vee$ of ${}^LT^0$ in ${}^LB_H{}^0$ a root vector $Y_{\alpha^\vee}$. We define σ_H, and so complete the

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definition of ${}^LH$, as follows. On ${}^LT^0$, σ_H induces that automorphism of the simple roots of ${}^LT^0$ in ${}^LB_H{}^0$ which differs from σ_T by an element of the Weyl group of ${}^LT^0$ in ${}^LH^0$, and on root vectors we have

$$\sigma_H Y_{\alpha^\vee} = Y_{\sigma_H \alpha^\vee}.$$

The group $\underset{\sim}{H} = \underset{\sim}{H}(T, \kappa)$ is any quasi-split group with L-group ${}^LH$.

Given $\underset{\sim}{H} = \underset{\sim}{H}(T, \kappa)$ we choose a Borel subgroup $\underset{\sim}{B}_H$ over $\mathbf{R}$ and a maximal torus $\underset{\sim}{T}_H$ in $\underset{\sim}{B}_H$, also over $\mathbf{R}$, in the usual way.

We denote by $\underset{\sim}{T}_N'$ (for later purposes) the torus with same underlying complex torus as $\underset{\sim}{T}^*$, but with Galois action σ_H. To conserve notation we will always assume that $\underset{\sim}{H}$, $\underset{\sim}{B}_H$, $\underset{\sim}{T}_H$ are chosen so that

(2.1.2) $\quad \underset{\sim}{T}_H = \underset{\sim}{T}_N'.$

Remark. As an immediate consequence of (2.1.1) we have:

PROPOSITION 2.1.3. *If $\underset{\sim}{G}$ is a simply-connected, semisimple group and $\underset{\sim}{T}$ is anisotropic over* $\mathbf{R}$ *then for each map $\psi_x : \underset{\sim}{T} \to \underset{\sim}{T}^*$ we have a one to one correspondence between the non-trivial (quasi-) characters κ attached to $\underset{\sim}{T}$ and the elements of order two in ${}^LT^0$.*

These correspondences allow us to generate examples for ${}^LH$ without describing κ, ψ_x and all the attendant notation (see (3.2)).

(2.2) *Standard position.* By changing the choice of x in the map ψ_x, we may change ${}^LH$ within its isomorphism class and, because of (2.1.2), our choice of $\underset{\sim}{H}$. Suppose that we follow ψ_x by ω, an element of $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$. On $L^\vee$, ω acts as an element of $\Omega({}^LG^0, {}^LT^0)$ and is thus realized by an element w of ${}^LG^0$. A possible replacement for $({}^LH^0, {}^LB_H{}^0, {}^LT^0, \{Y_{\alpha^\vee}\}, \sigma_H)$ is

$$(({}^LH^0)^w, ({}^LB_H{}^0)^w, {}^LT^0, \{wY_{\alpha^\vee}\}, \omega\sigma_H\omega^{-1});$$

in particular, we may replace σ_H on ${}^LT^0$ by a conjugate under $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$.

According to [**8**] (cf. [**10**, § 6]) we can find $g \in \underset{\sim}{G}^*$ such that

$$\underset{\sim}{T}_N' \xrightarrow{\text{id}} \underset{\sim}{T}^* \xrightarrow{\text{ad } g} \underset{\sim}{G}^*$$

is defined over $\mathbf{R}$. Let $\underset{\sim}{T}_N$ be the image of $\underset{\sim}{T}_N'$. We may and do assume that $\underset{\sim}{S}_{T_N} \subset \underset{\sim}{S}_{T^*}$. We fix a p-d. m_N of $\underset{\sim}{T}_N$ in $\underset{\sim}{M}_N = \underset{\sim}{M}_{T_N}$ and use m_N to transfer the Galois action of $\underset{\sim}{T}_N$ to L and $L^\vee$; we denote the result, which is independent of the choice for m_N, by σ_N. On L or $L^\vee$ we have that

$$\sigma_H = \omega\sigma_N\omega^{-1},$$

where ω, as element of $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$, is realized by $(m_N g)^{-1}$.

Note that if $\underset{\sim}{T}$ is any torus in $\underset{\sim}{G}^*$ with $\underset{\sim}{S}_T \subset \underset{\sim}{S}_{T^*}$ and we define σ_T as we did $\sigma_N = \sigma_{T_N}$, then σ_H is conjugate to σ_T under $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$ if and only if $\underset{\sim}{T}$ is stably conjugate to $\underset{\sim}{T}_N$ (cf. [**8**]), that is, if and only if $\underset{\sim}{T}$ could have been chosen in place of $\underset{\sim}{T}_N$. Also, if $\underset{\sim}{S}_{T_N} = \underset{\sim}{S}_{T^*}$ then $\sigma_N = \sigma_{T^*}$ and we may as well take $\underset{\sim}{T}_N = \underset{\sim}{T}^*$ in that case. In general, however, the choice of $\underset{\sim}{T}_N$ affects σ_N.

We can change ψ_x so that:

(2.2.1) σ_H coincides with σ_N on L and $L^\vee$.

We then say that ${}^LH$ is in *standard position with respect to* $\underset{\sim}{T}_N$. The (chosen) group $\underset{\sim}{T}_N$ plays a major role in later sections.

PROPOSITION 2.2.2. *Suppose that ${}^LH$ is in standard position with respect to $\underset{\sim}{T}_N$. Then*

(i) $\underset{\sim}{T}_H = \underset{\sim}{T}_N' \xrightarrow{\text{id}} \underset{\sim}{T}^* \xrightarrow{m_N} \underset{\sim}{T}_N$ *is defined over* $\mathbf{R}$, *for any* p-d. m_N *of* $\underset{\sim}{T}_N$ *in* $\underset{\sim}{M}_N$, *and*

(ii) *there exists $m \in {}^LM_N{}^0$ such that $m \times (1 \times \sigma)$ normalizes ${}^LH^0$ and acts on ${}^LH^0$ as σ_H.*

Proof. (i) is immediate. For (ii), consider first σ_N acting on L. There is $\omega \in \Omega(\underset{\sim}{M}_N, \underset{\sim}{T}^*)$ such that $\sigma_N = \omega\sigma_G$. This equation remains true on $L^\vee$ if we replace ω by its contragredient, that is, if we regard ω as an element of $\Omega({}^LM_N{}^0, {}^LT^0)$. Hence we may choose $m \in {}^LM_N{}^0$ such that $m \times (1 \times \sigma)$ acts on ${}^LT^0$ as $\sigma_N = \sigma_H$. Then $m \times (1 \times \sigma)$ normalizes ${}^LH^0$ and clearly $\operatorname{ad}(m \times (1 \times \sigma))$ acts on ${}^LH^0$ as $\operatorname{ad} t \circ \sigma_H$, for some $t \in {}^LT^0$. The proposition thus follows.

(2.3) *Framework of Cartan subgroups.* We assume that ${}^LH$ is in standard position with respect to $\underset{\sim}{T}_N$ and that $\underset{\sim}{H}$ satisfies (2.1.2).

In H, choose a complete set of representatives $T_0', \dots, T_N'$ (T_N' as in (2.1.2)) for the conjugacy classes of Cartan subgroups of H, such that

(2.3.1a) $\underset{\sim}{S}_{T_n'} \subset S_{T_N'}, \quad n = 0, \dots, N-1,$

and for each T_n' a p-d. m_n' in $\underset{\sim}{M}_n' = \underset{\sim}{M}_{T_n'}$ (with respect to $\underset{\sim}{T}_H = \underset{\sim}{T}_N'$) such that

(2.3.1b) m_N' is the identity map.

Note that the indices $0, \dots, N-1$ bear no relation to the ordering on the conjugacy classes of Cartan subgroups, and that N plays a different role in [**10**].

In G^*, choose Cartan subgroups $T_0, \dots, T_N$ ($\underset{\sim}{T}_N$ as above) and for each n a p-d. m_n of $\underset{\sim}{T}_n$ in $\underset{\sim}{M}_n = \underset{\sim}{M}_{T_n}$, such that

(2.3.2a) $S_{T_n} \subseteq S_{T_N} \subseteq S_{T^*}$ for $n = 0, \dots, N-1$,

(2.3.2b) $\underset{\sim}{T}_n' \xrightarrow{m_n'} \underset{\sim}{T}^* \xrightarrow{m_n^{-1}} \underset{\sim}{T}_n$ is defined over $\mathbf{R}$, and

(2.3.2c) $T_n = T_p$ if T_n is conjugate to T_p,
$T_N = T^*$ if T_N is conjugate to T^* (cf. (2.2)).

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For each T_n' originating in G in the sense that some $\psi^{-1} \circ \operatorname{ad} g$, $g \in \underset{\sim}{G}^*$, maps $\underset{\sim}{T}_n$ into $\underset{\sim}{G}$ over $\mathbf{R}$, choose a Cartan subgroup $T_n{}^G$ of G and element g_n of $\underset{\sim}{G}^*$ such that

(2.3.3a) $\psi_n = \operatorname{ad} g_n \circ \psi : \underset{\sim}{T}_n{}^G \to \underset{\sim}{T}_n$ is defined over $\mathbf{R}$,

(2.3.3b) if $T_n{}^G$ is conjugate to $T_p{}^G$ then $T_n{}^G = T_p{}^G$ and $\psi_n = \psi_p$ (cf. (2.3.2c)),

(2.3.3c) our fixed Cartan subgroup T of G defining ${}^LH$ and $\underset{\sim}{H}$ is included among the $T_n{}^G$ and

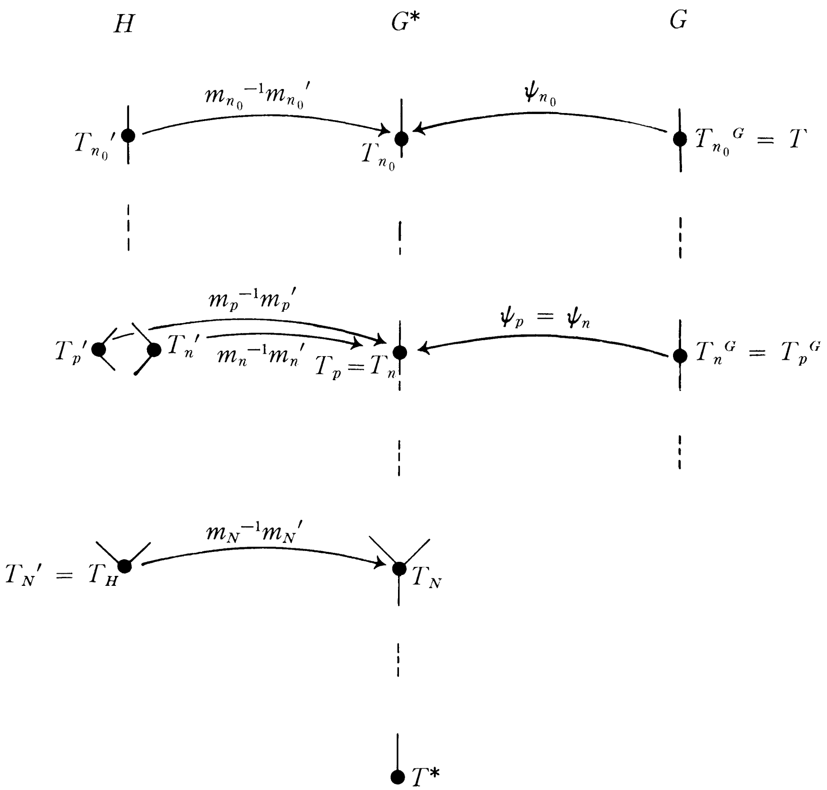
(2.3.3d) for some n such that $T_n{}^G = T$, ψ_n is such that the element

$$\omega : \underset{\sim}{T}^* \xrightarrow{m_n^{-1}} \underset{\sim}{T}_n \xrightarrow{\psi_n^{-1}} \underset{\sim}{T} \xrightarrow{\psi_x} \underset{\sim}{T}^*$$

of $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$ acts on $L^\vee$ as an element of $\Omega({}^LH^0, {}^LT^0)$. Recall that ψ_x is the map from $\underset{\sim}{T}$ to $\underset{\sim}{T}^*$ fixed in the definition of ${}^LH$ and $\underset{\sim}{H}$.

We write $\underset{\sim}{M}_n{}^G$ for $\underset{\sim}{M}_{T_n^G}$ and use ψ_n to define the L-group for $\underset{\sim}{M}_n{}^G$.

We can summarize our framework of Cartan subgroups enclosing T_N in a diagram (cf. [**10**]):



We have to check that (2.3.1a)–(2.3.3d) are possible. First we pick $T_0', \ldots, T_N', m_0', \ldots, m_N'$ satisfying (2.3.1a) and (2.3.1b). For $T_0, \ldots, T_N, m_0, \ldots, m_N$, we know that there is $y_n \in \underset{\sim}{G}^*$ such that

$$\underset{\sim}{T}_n' \xrightarrow{m_n'} \underset{\sim}{T}^* \xrightarrow{\text{ad } y_n} \underset{\sim}{G}^*$$

is defined over $\mathbf{R}$. We can adjust the image, $\underset{\sim}{T}_n$, so that (2.3.2a) and (2.3.2c) are satisfied. Then the quasi-split group $\underset{\sim}{M}_n = \underset{\sim}{M}_{T_n}$ contains $\underset{\sim}{T}^*$ and, because ${}^LH$ is in standard position with respect to T_N, we can argue in $\underset{\sim}{M}_n$ to find $\bar{m}_n$ in ad $\underset{\sim}{M}_n$ so that

$$\underset{\sim}{T}_n' \xrightarrow{m_n'} T^* \xrightarrow{\bar{m}_n^{-1}} M_n$$

is defined over $\mathbf{R}$. The image under this map is, like $\underset{\sim}{T}_n$, fundamental in $\underset{\sim}{M}_n$. Thus we can follow $\bar{m}_n$ by an element of ad M_n to obtain

$$\underset{\sim}{T}_n' \xrightarrow{m_n'} \underset{\sim}{T}^* \xrightarrow{m_n^{-1}} \underset{\sim}{T}_n$$

defined over $\mathbf{R}$. Next, (2.3.3a) (2.3.3b) and (2.3.3c) present no difficulties; we choose the T_n^G and ψ_n as desired. However to satisfy (2.3.3d) as well, we may have to modify some ψ_n. Suppose that $\{T_n', m_n', T_n, M_n, T_n^G, \psi_n; n = 0, \ldots, N\}$ satisfies all but (2.3.3d). By (2.3.3c), our fixed Cartan subgroup T is $T_{n_0}^G$ for some n_0. Then $\psi_{n_0} : \underset{\sim}{T} \rightarrow \underset{\sim}{T}_{n_0}$ is defined over $\mathbf{R}$. We may write ψ_x, the given p-d. of $\underset{\sim}{T}$, as $g \circ \psi_{n_0}$, where g is a p-d. of T_{n_0} Note that the transfer of σ_T to ${}^LT^0$ via ψ_x coincides with the transfer of $\sigma_{T_{n_0}}$ to ${}^LT^0$ via g. Thus [**8**] implies that there exists $h \in$ ad $\underset{\sim}{H}$ such that

$$\underset{\sim}{T}_{n_0} \xrightarrow{g} \underset{\sim}{T}^* \xrightarrow{h^{-1}} \underset{\sim}{H}$$

is defined over $\mathbf{R}$. We may assume that the image is some $\underset{\sim}{T}_n'$. By (2.3.2c) we then have that $\underset{\sim}{T}_n = \underset{\sim}{T} = \underset{\sim}{T}_{n_0}$ and thus $\psi_n = \psi_{n_0}$ ((2.3.3b)), for both

$$\underset{\sim}{T}_n' \xrightarrow{m_n'} \underset{\sim}{T}^* \xrightarrow{m_n^{-1}} \underset{\sim}{T}_n \text{ and}$$

$$\underset{\sim}{T}_n' \xrightarrow{h} \underset{\sim}{T}^* \xrightarrow{g^{-1}} \underset{\sim}{T}$$

are defined over $\mathbf{R}$, causing T_n and T to be conjugate in G (cf. [**10**]). We may write $m_n^{-1}m_n'$ as $wg^{-1}h$, where $w \in$ ad $\underset{\sim}{G}^*$ and $w : \underset{\sim}{T}_n \rightarrow \underset{\sim}{T}_n$ is defined over $\mathbf{R}$. Thus

$$\psi_x = (h(m_n')^{-1})m_n(w^{-1}\psi_n)$$

and so if we replace ψ_n by $w\psi_n$ then all conditions (2.3.1a)–(2.3.3d) are satisfied. In (2.4) we explain why (2.3.3d) is demanded.

(2.4) *Data "from H"*. Our starting point is the fact that the roots of ${}^LT^0$ in ${}^LH^0$ form a subsystem of the roots of ${}^LT^0$ in ${}^LG^0$. We make the

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natural identification of $(L^\vee)^\vee = L^\vee({}^LT^0)$ with $L = L(\underset{\sim}{T}^*)$, and thus of the coroots for the roots of ${}^LT^0$ in ${}^LG^0$ with the roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{G}^*$, writing $(\alpha^\vee)^\vee = \alpha$. At the same time, we identify the coroots for the roots of ${}^LT^0$ in ${}^LH^0$ with the roots of $\underset{\sim}{T}_H$ (or $\underset{\sim}{T}^*$, since at this point we are working over $\mathbf{C}$) in $\underset{\sim}{H}$. A root of $\underset{\sim}{T}_H$ in $\underset{\sim}{H}$ is therefore identified, as element of L, with a root of $\underset{\sim}{T}^*$ in $\underset{\sim}{G}^*$; the roots of $\underset{\sim}{T}_H$ in $\underset{\sim}{H}$ do not, in general, form a subsystem of the roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{G}^*$. Nevertheless, $\Omega(\underset{\sim}{H}, \underset{\sim}{T}_H)$ is naturally embedded in $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$.

Analogous results hold if we replace $(\underset{\sim}{G}^*, {}^LG^0, \underset{\sim}{H}, {}^LH^0)$ by $(\underset{\sim}{M}_n, {}^LM_n{}^0, \underset{\sim}{M}_n', {}^L(M_n')^0)$, as provided by our framework of Cartan subgroups.

We write L_n for $L(\underset{\sim}{T}_n)$, $L_n^\vee$ for $L^\vee(T_n)$, σ_n for the Galois action of $\underset{\sim}{T}_n$ and its transfer to L and $L^\vee$ by m_n; L_n', $(L_n')^\vee$ and σ_n' are similarly defined for $\underset{\sim}{T}_n'$. On L and $L^\vee$ we have $\sigma_n' = \sigma_n$; a root $\alpha^\vee$ of ${}^LT^0$ in ${}^LG^0$ belongs to ${}^LM_n{}^0$ if and only if $\sigma_n\alpha^\vee = -\alpha^\vee$.

Using m_n, m_n' and the identifications of the first paragraph we embed the roots of $\underset{\sim}{T}_n'$ in $\underset{\sim}{H}$ in the roots of $\underset{\sim}{T}_n$ in $\underset{\sim}{G}^*$; a root of $\underset{\sim}{T}_n$ "comes from $\underset{\sim}{H}$" if it lies in the image of this map. Similarly we map $\Omega(\underset{\sim}{H}, \underset{\sim}{T}_n')$ into $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ and an element of $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ may "come from $\underset{\sim}{H}$" (see [**10**, § 6] for further details).

Recall that $\underset{\sim}{H} = \underset{\sim}{H}(T, \kappa)$. If $T = T_n{}^G$ then we transfer κ to κ_n for $\underset{\sim}{T}_n$ via ψ_n. By (2.3.3b), κ_n is well-defined. If n is as in (2.3.3d) then κ_n coincides with the transfer of κ to $\underset{\sim}{T}^*$ via ψ_x as in the definition of ${}^LH$, and thence to $\underset{\sim}{T}_n$ via m_n. We may therefore regard H as defined by T_n, κ_n and m_n, instead of by T, κ and ψ_x. Next, we transfer κ to κ_p for T_p, $p = 0, \ldots, N$, via $m_pm_n^{-1}$; T_p, κ_p and m_p again define the same $\underset{\sim}{H}$; a root α of $\underset{\sim}{T}_p$ comes from $\underset{\sim}{H}$ if and only if $\kappa_p(\alpha^\vee) = 1$ (cf. [**10**, § 7]). Note that if κ_p^- is the restriction of κ_p to the span of the coroots for $\underset{\sim}{M}_p$ then $\underset{\sim}{M}_p'$ is a (T_p, κ_p^-)-group for $\underset{\sim}{M}_p$.

3. Admissible embeddings of ${}^LH$ in ${}^LG$.

(3.1) *Introduction*. Given (T, κ), consider first any ${}^LH$ attached as in (2.1). We wish to extend the inclusion of ${}^LH^0$ in ${}^LG^0$ to an admissible (cf. [**3**]) embedding of ${}^LH$ in ${}^LG$; that is, we seek a homomorphism $\xi : {}^LH \to {}^LG$ such that $\xi(h \times w) = h\xi(w)$, $h \in {}^LH^0$, $w \in W$, and $\xi(1 \times w) \in {}^LG^0 \times w$, $w \in W$. Equivalently we seek a homomorphism $\xi^W : W \to {}^LG$ such that

(3.1.1) $\xi^W(w) = \xi_0(w) \times w$, some $\xi_0(w) \in {}^LG^0$, and $\xi^W(w)$ stabilizes ${}^LH^0$, acting on ${}^LH^0$ as $1 \times w \in {}^LH$, $w \in W$.

Thus $\xi^W(\mathbf{C}^\times \times 1)$ is to act trivially on ${}^LH^0$, and $\xi^W(1 \times \sigma)$ as σ_H.

We omit the superscript W from ξ^W and use ξ in both contexts.

PROPOSITION 3.1.2. *Suppose that* $\xi : W \to {}^LG$ *is a homomorphism satis-*

fying (3.1.1). *Then* $\xi_0(\bar{z} \times 1) = \sigma_H(\xi_0(z \times 1))$, $z \in \mathbf{C}^\times$, *and* $\xi_0(\mathbf{C}^\times \times 1)$ *is contained in* $Z({}^LH^0)$, *the center of* ${}^LH^0$.

Proof. This is immediate.

Conversely, suppose that $\xi_0 : \mathbf{C}^\times \rightarrow Z({}^LH^0)$ is some homomorphism such that $\xi_0(\bar{z}) = \sigma_H(\xi_0(z))$, $z \in \mathbf{C}^\times$. Pick $n \in {}^LG^0$ such that $n \times (1 \times \sigma)$ acts as σ_H on ${}^LH^0$. An argument as in the proof of Proposition 2.2.2 shows that this is possible. A chosen element n may be replaced only by zn, $z \in Z({}^LH^0)$. Set

$$\xi(z \times 1) = \xi_0(z) \times z, \quad z \in \mathbf{C}^\times, \text{ and}$$
$$\xi(1 \times \sigma) = n \times (1 \times \sigma).$$

Then $\xi : W \rightarrow {}^LG$ is a homomorphism (satisfying (3.1.1)) if and only if

(3.1.3) $\quad n\sigma_G(n) = \xi_0(-1)$.

Replacing n by zn, $z \in Z({}^LH^0)$; multiplies $n\sigma_G(n)$ on the left by $z\sigma_H(z)$.

(3.2) *Examples.* The following simple examples are of particular interest in later sections.

(3.2.1) Let ${}^LG^0 = PGL_3(\mathbf{C})$. We write A_* for the image of $A \in GL_3(\mathbf{C})$ in $PGL_3(\mathbf{C})$. Let ${}^LB^0$ be the image of the upper triangular matrices and

$${}^LT^0 = \{\mathrm{diag}(x_1, x_2, x_3)_*\}.$$

Take as attached root vectors,

$$X_{x_1-x_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$X_{x_2-x_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let σ_G act by

$$\mathrm{diag}(x_1, x_2, x_3)_* \rightarrow \mathrm{diag}(x_3^{-1}, x_2^{-1}, x_1^{-1})_* \text{ on } {}^LT^0,$$

and by

$$\sigma_G X_{x_1-x_2} = X_{x_2-x_3}.$$

Set

$${}^LH^0 = \mathrm{Cent}^0\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_*\right),$$

the connected component of the identity in the centralizer of

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_*$$

in ${}^LG^0$, ${}^LB_H{}^0 = {}^LB^0 \cap {}^LH^0$ and $Y_{x_2-x_3} = X_{x_2-x_3}$; let σ_H act on ${}^LT^0$ by

$$\mathrm{diag}(x_1, x_2, x_3)_* \to \mathrm{diag}(x_1^{-1}, x_3^{-1}, x_2^{-1})_*$$

(following the remark in (2.1), we take σ_H on ${}^LT^0$ to be that automorphism which induces an automorphism of the Dynkin diagram of $({}^LH^0, {}^LT^0)$ and differs from $t \to t^{-1}$ by an element of $\Omega({}^LH^0, {}^LT^0)$), and set

$$\sigma_H Y_{x_2-x_3} = Y_{x_2-x_3}.$$

Then we embed ${}^LH$ in ${}^LG$ by ξ_λ ($\lambda \in \mathbf{Z}$):

$$\xi_\lambda(h \times (1 \times 1)) = h \times (1 \times 1), \quad h \in {}^LH^0,$$

$$\xi_\lambda(1 \times (z \times 1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (z/\bar{z})^{(2\lambda+1)/2} & 0 \\ 0 & 0 & (z/\bar{z})^{(2\lambda+1)/2} \end{bmatrix}_* \times (z \times 1),$$

$$z \in \mathbf{C}^\times,$$

$$\xi_\lambda(1 \times (1 \times \sigma)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_* \times (1 \times \sigma).$$

It would have been easier to consider the following isomorphic ${}^LH$ (cf. (2.2)). Let

$${}^LH^0 = \mathrm{Cent}^0\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_*\right),$$

$${}^LB_H{}^0 = {}^LB^0 \cap {}^LH^0 \quad \text{and} \quad Y_{x_1-x_3} = [X_{x_1-x_2}, X_{x_2-x_3}].$$

Set $\sigma_H = \sigma_G$ on ${}^LT^0$ and $\sigma_H Y_{x_1-x_3} = Y_{x_1-x_3}$. Note that

$$\sigma_G Y_{x_1-x_3} = -Y_{x_1-x_3}.$$

We embed ${}^LH$ in ${}^LG$ by ξ_λ ($\lambda \in \mathbf{Z}$):

$$\xi_\lambda(h \times (1 \times 1)) = h \times (1 \times 1), \quad h \in {}^LH^0,$$

$$\xi_\lambda(1 \times (z \times 1)) = \begin{bmatrix} (z/\bar{z})^{(2\lambda+1)/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (z/\bar{z})^{(2\lambda+1)/2} \end{bmatrix}_*, \quad z \in \mathbf{C}^\times,$$

$$\xi_\lambda(1 \times (1 \times \sigma)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_* \times (1 \times \sigma).$$

(3.2.2) Let ${}^LG^0 = PSp_4(\mathbf{C})$; this time A_* denotes the image of $A \in Sp_4(\mathbf{C})$ in ${}^LG^0$. In place of ${}^LB^0$, we specify

$${}^LT^0 = \{\mathrm{diag}(x_1, x_2, x_1^{-1}, x_2^{-1})_*\}$$

and the positive system $2x_1$, $2x_2$, $x_1 \pm x_2$ for the roots of ${}^LT^0$. Fix root

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vectors for ${}^LG^0$ and require that σ_G act trivially. Set

$$ {}^LH^0 = \mathrm{Cent}(\mathrm{diag}(i, -i, -i, i)_*)^0 $$

and choose ${}^LB_H{}^0 \supset {}^LT^0$ by requiring that $x_1 + x_2$ be a root in ${}^LB_H{}^0$. Set

$$ Y_{x_1+x_2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

On ${}^LT^0$, σ_H is the automorphism

$$ \mathrm{diag}((x_1, x_2, x_1^{-1}, x_2^{-1}))_* \to \mathrm{diag}(x_2, x_1, x_2^{-1}, x_1^{-1})_*; $$

set $\sigma_H Y_{x_1+x_2} = Y_{x_1+x_2}$. Then we embed ${}^LH$ in ${}^LG$ by:

$$ \xi_\lambda(h \times (1 \times 1)) = h \times (1 \times 1), \quad h \in {}^LH^0, $$

$$ \xi_\lambda(1 \times (z \times 1)) = \begin{bmatrix} \left(\frac{z}{\bar z}\right)^\lambda & 0 & 0 & 0 \\ 0 & \left(\frac{\bar z}{z}\right)^\lambda & 0 & 0 \\ 0 & 0 & \left(\frac{\bar z}{z}\right)^\lambda & 0 \\ 0 & 0 & 0 & \left(\frac{z}{\bar z}\right)^\lambda \end{bmatrix}_* \times (z \times 1), \quad z \in \mathbf{C}^\times, $$

$$ \xi_\lambda(1 \times \sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_* \times (1 \times \sigma). $$

(3.3) *Data attached to an embedding*. In the term *admissible embedding* $\xi : {}^LH \hookrightarrow {}^LG$ we will understand that $\xi|{}^LH^0$ is the inclusion mapping. Until (9.2), we will assume that ${}^LH$ is in standard position with respect to a Cartan subgroup T_N, as described in (2.2). We will attach to ξ a pair (μ^*, λ^*) of elements in the vector space $L \otimes \mathbf{C}$, and write $\xi = \xi(\mu^*, \lambda^*)$.

As before, ξ also denotes the restriction of ξ to W. Set $\xi(w) = \xi_0(w) \times w$, $w \in W$. First consider $\xi_0|\mathbf{C}^\times \times 1$. There exist $\mu^*, \nu^* \in L \otimes \mathbf{C}$ with $\mu^* - \nu^* \in L$ and such that

$$ \lambda^\vee(\xi_0(z \times 1)) = z^{\langle \mu^*, \lambda^\vee \rangle} \bar z^{\langle \nu^*, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee, z \in \mathbf{C}^\times. $$

PROPOSITION 3.3.1. (i) $\nu^* = \sigma_H \mu^*$.
(ii) $\langle \mu^*, \alpha^\vee \rangle = 0$ *for each root* $\alpha^\vee$ *of* ${}^LH^0$.

Proof. This is immediate.

Next, we exploit the fact that ${}^LH$ is in standard position with respect to

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T_N. By Proposition 2.2.2 we have that $\xi_0(1 \times \sigma)$ lies in ${}^LM_N{}^0$. We pick $\lambda^* \in L \otimes \mathbf{C}$ such that

$$\lambda^\vee(\xi_0(1 \times \sigma)) = e^{2\pi i\langle\lambda^*,\lambda^\vee\rangle}$$

for each $\lambda^\vee \in L^\vee$ which extends to a rational character on ${}^LM_N{}^0$.

PROPOSITION 3.3.2. *For given* ξ,
(i) μ^* *is uniquely determined,*
(ii) λ^* *may be replaced only by elements of*

$$\lambda^* + L + \sum_{\substack{\alpha \text{ root of} \\ \underset{\sim}{T}^* \text{ in } \underset{\sim}{M}_N}} \mathbf{C}\alpha.$$

Proof. (i) is immediate. For (ii), suppose that

$$e^{2\pi i\langle\lambda_1^*,\lambda^\vee\rangle} = e^{2\pi i\langle\lambda^*,\lambda^\vee\rangle}$$

for all $\lambda^\vee$ extending to ${}^LM_N{}^0$. Set $\lambda_2^* = \lambda^* - \lambda_1^*$ and pick $t \in {}^LT^0$ such that

$$\lambda^\vee(t) = e^{2\pi i\langle\lambda_2^*,\lambda^\vee\rangle}, \quad \lambda^\vee \in L^\vee.$$

Then $t \in {}^LT^0 \cap ({}^LM_N{}^0)_{\text{der}}$ since every rational character on ${}^LM_N{}^0$ annihilates t. Thus we can pick

$$\lambda_3^* \in \sum_{\substack{\alpha \text{ root of} \\ \underset{\sim}{T}^* \text{ in } \underset{\sim}{M}_N}} \mathbf{C}\alpha$$

such that

$$\lambda^\vee(t) = e^{2\pi i\langle\lambda_3^*,\lambda^\vee\rangle}, \quad \lambda^\vee \in L^\vee.$$

Clearly $\lambda_2^* - \lambda_3^* = \lambda^* - (\lambda_1^* + \lambda_0^*) \in L$, and the proposition is proved.

(3.4) *Congruences.* From now on, we enclose T_N in a framework of Cartan subgroups. Consider the attached standard Levi subgroups ${}^LM_n{}'$, $n = 0, \ldots, N$, in ${}^LH$, and ${}^LM_n$ in ${}^LG$. An admissible embedding $\xi : {}^LH \hookrightarrow {}^LG$ (as always, in standard position) induces, by restriction, an admissible embedding

$$\xi^{(n)} : {}^LM_n{}' \hookrightarrow {}^LM_n, \quad n = 0, \ldots, N.$$

Recall that $\underset{\sim}{M}_n{}'$ is a $(T_n, \kappa_n{}^-)$-group for $\underset{\sim}{M}_n$. Clearly ${}^LM_n{}'$ is in standard position with respect to T_N and $\xi^{(n)} = \xi^{(n)}(\mu^*, \lambda^*)$. Let ι_n denote one-half the sum of the roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{M}_n \cap \underset{\sim}{B}^*$ and $\iota_n{}'$ one-half the sum of the roots of $\underset{\sim}{T}_H$ in $\underset{\sim}{M}_n{}' \cap \underset{\sim}{B}_H$. Then:

THEOREM 3.4.1.

$$\tfrac{1}{2}(\mu^* - \sigma_n\mu^*) + \iota_n - \iota_n{}' \equiv (\lambda^* + \sigma_n\lambda^*) \bmod L, \quad n = 0, \ldots, N.$$

Proof. Fix an allowed embedding $\tau : {}^LT_n{}' \hookrightarrow {}^LM_n{}'$ with underlying p-d. m_n'; suppose that $\tau = \tau(\mu, \lambda)$ (cf. (1.3)). Then we have:

$$\begin{array}{ccc} {}^L T_n' & \xrightarrow{\tau(\mu,\lambda)} & {}^L M_n' \\ \wr \downarrow & & \downarrow \xi^{(n)}(\mu^*, \lambda^*) \\ {}^L T_n & \xrightarrow{\tau(\mu+\mu^*, \lambda+\lambda^*)} & {}^L M_n \end{array}$$

where the left vertical arrow is induced by the **R**-isomorphism $m_n^{-1}m_n'$, and the p-d. m_n underlies the bottom horizontal arrow which is defined by commutativity of the diagram. From Proposition 1.3.5 we know that

$$\tfrac{1}{2}(\mu - \sigma_n\mu) + \iota_n' \equiv (\lambda + \sigma_n\lambda) \bmod L \quad \text{and}$$
$$\tfrac{1}{2}((\mu + \mu^*) - \sigma_n(\mu + \mu^*)) + \iota_n \equiv ((\lambda + \lambda^*) + \sigma_n(\lambda + \lambda^*)) \bmod L.$$

Subtracting, we obtain the theorem.

Note that Proposition 3.3.2 shows that the congruences do not depend on the choice for λ^*.

4. Quasicharacters attached to an admissible embedding.

(4.1) *Congruences and quasicharacters*. Obtaining a quasicharacter on T_n from a congruence as in Theorem 3.4.1 is a step in the Langlands correspondence for real tori (cf. [**7**, § 2]). We recall some of the details. Let $\underset{\sim}{T}$ be a torus over **R**, with Galois action σ; in the usual manner, we identify the Lie algebra $\underset{\sim}{t}$ of $\underset{\sim}{T}(= \underset{\sim}{T}(\mathbf{C}))$ with $L^\vee(\underset{\sim}{T}) \otimes \mathbf{C}$ and write an element of T as $\exp X$, where $\lambda(\exp X) = e^{\langle\lambda, X\rangle}$, $\lambda \in L(T)$; $\exp X_1 = \exp X_2$ if and only if $X_1 - X_2 \in 2\pi i L^\vee(\underset{\sim}{T})$. An element $\exp X$ of $\underset{\sim}{T}$ belongs to T $(= \underset{\sim}{T}(\mathbf{R}))$ if and only if $\sigma\bar{X} - X \in 2\pi i L^\vee(\underset{\sim}{T})$, where $\bar{X}$ denotes that element of $L^\vee(T) \otimes \mathbf{C}$ satisfying $\langle\lambda, \bar{X}\rangle = \overline{\langle\lambda, X\rangle}$, $\lambda \in L(\underset{\sim}{T})$ (recall that for $\underset{\sim}{t} \to L^\vee(\underset{\sim}{T}) \otimes \mathbf{C}$ to respect Galois action, σ must act on both $L^\vee(\underset{\sim}{T})$ and **C**). Suppose that $\exp X \in T$. We write $X = X_{\mathbf{R}} + X_{\mathbf{I}}$, where

$$X_{\mathbf{R}} = \tfrac{1}{2}(X + \sigma\bar{X}) \quad \text{and} \quad X_{\mathbf{I}} = \tfrac{1}{2}(X - \sigma\bar{X}).$$

Then $X_{\mathbf{R}} \in$ t, the Lie algebra of T, and $X_{\mathbf{I}}$ is a σ-invariant element of $i\pi L^\vee(T)$. We thus decompose $\exp X$ as h_1h_2, where $h_1 \in T^0$, the (euclidean) connected component of the identity in T, and

$$h_2 \in F = \{\exp i\pi\lambda^\vee : \lambda^\vee \in L^\vee(T), \sigma\lambda^\vee = \lambda^\vee\}.$$

We then obtain $T = T^0F$, with

$$T^0 \cap F = \{\exp i\pi\lambda^\vee = \exp i\pi(\mu^\vee - \sigma\mu^\vee); \lambda^\vee = \mu^\vee + \sigma\mu^\vee, \mu^\vee \in L^\vee(\underset{\sim}{T})\}.$$

Given a pair (μ, λ) of elements in $L(\underset{\sim}{T}) \otimes \mathbf{C}$ we set

$$\chi(\mu, \lambda)(\exp X) = e^{\langle\mu, X_{\mathbf{R}}\rangle + 2\langle\lambda, X_{\mathbf{I}}\rangle}, \quad \exp X \in T.$$

Then $\chi(\mu, \lambda)$ is a well-defined quasicharacter on T if and only if

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$\frac{1}{2}(\mu - \sigma\mu) + \lambda + \sigma\lambda \in L(\underset{\sim}{T})$ or both $\mu - \sigma\mu \in L(\underset{\sim}{T})$ and

$$\tfrac{1}{2}(\mu - \sigma\mu) \equiv (\lambda + \sigma\lambda) \bmod L(\underset{\sim}{T});$$

$\chi(\mu', \lambda') = \chi(\mu, \lambda)$ if and only if $\mu' = \mu$ and

$$\lambda' \equiv \lambda \bmod (L(\underset{\sim}{T}) + \{\nu - \sigma\nu : \nu \in L(\underset{\sim}{T}) \otimes \mathbf{C}\})$$

and, moreover, every quasicharacter on T is of this form.

(4.2) *Quasicharacters* $\chi^{(n)}_{(\xi,\ )}$. We return to our groups $\underset{\sim}{G}$, $\underset{\sim}{G}^*$ and $\underset{\sim}{H}$. First we transfer the congruences of Theorem 3.4.1 from $L \otimes \mathbf{C}$ to $L_n \otimes \mathbf{C}$.

A p-d. $m_n : \underset{\sim}{T}_n \to \underset{\sim}{T}$ has been fixed; using this we transfer μ^* and λ^* to elements of $L_n \otimes \mathbf{C}$ and σ_n back to the Galois action of $\underset{\sim}{T}_n$, without change in notation. Note that μ^* depends on the choice for m_n; λ^* may depend on that choice but $\frac{1}{2}(\lambda^* + \sigma_n\lambda^*)$, which is all that matters for the congruences, does not.

If we transfer ι_n and ι_n' to $L_n \otimes \mathbf{C}$ via m_n then we obtain, respectively, one-half the sum of the positive roots of $\underset{\sim}{T}_n$ in $\underset{\sim}{M}_n$, one-half the sum of the positive roots of $\underset{\sim}{T}_n$ in $\underset{\sim}{M}_n$ coming from $\underset{\sim}{M}_n'$, under certain fixed orderings. It is convenient to change notation here. Thus we now use ι_n to denote one-half the sum of the roots in any prescribed positive system I_n^+ for the roots of $\underset{\sim}{T}_n$ in $\underset{\sim}{M}_n$, and ι_n' to denote one-half the sum of those roots in I_n^+ which come from $\underset{\sim}{M}_n'$. With these conventions, we have easily that

$$\tfrac{1}{2}(\mu^* - \sigma_n\mu^*) + \iota_n - \iota_n' \equiv (\lambda^* + \sigma_n\lambda^*) \bmod L_n.$$

Definition 4.2.1. If $\xi = \xi(\mu^*, \lambda^*)$ is an admissible embedding of ${}^LH$ in ${}^LG$ then $\chi^{(n)}_{(\xi, I_n^+)}$ is the quasicharacter $\chi(\mu^* + \iota_n - \iota_n', \lambda^*)$.

Clearly, $\chi^{(n)}_{(\xi, I_n^+)}$ does not depend on which choice we make for λ^*. We transfer $\chi^{(n)}_{(\xi, I_n^+)}$ to T_n^G in G, whenever T_n^G exists, via ψ_n and without change in notation. For the present, however, we work on G^*, and ignore G.

Example 4.3.1. Let

$$\underset{\sim}{G} = \underset{\sim}{G}^* = SU\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right),$$

a real form of $SL_3(\mathbf{C})$; let $\underset{\sim}{B}^*$ be the group of upper triangular matrices in $\underset{\sim}{G}^*$, and $\underset{\sim}{T}^*$ be the diagonal subgroup; we write $\mathrm{diag}(t_1, t_2, t_3)$ for the generic element of $\underset{\sim}{T}^*$. We take ${}^LG$ as in (3.2.1), using the identification of $L^\vee(\underset{\sim}{T}^*)$ with $L({}^LT^0)$ induced by the pairing $\langle t_i, x_j\rangle = \delta_{ij}$, $i, j = 1, 2, 3$, between the vector spaces $\mathbf{C}t_1 + \mathbf{C}t_2 + \mathbf{C}t_3$ and $\mathbf{C}x_1 + \mathbf{C}x_2 + \mathbf{C}x_3$. Let κ be that character attached to $\underset{\sim}{T}^*$ which satisfies $\kappa(x_1 - x_3) = 1$, $\kappa(x_2 - x_3) = -1$. Then on L, $\sigma_H = \sigma_{T^*}$, and the second group ${}^LH$ of

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(3.2.1) is attached to (T^*, κ) and sits in standard position with respect to T^*. We take as $\underset{\sim}{H}$ the subgroup of $\underset{\sim}{G}^*$ consisting of matrices of the form $\begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix}$; thus $\underset{\sim}{H} = U(1, 1)$. We use the inclusion of H in G to define a framework of Cartan subgroups. Thus we pick

$$T_0' = T_0 = \left\{ r(\theta, \varphi) = \begin{bmatrix} \frac{1}{2}(e^{i\theta} + e^{i\varphi}) & 0 & \frac{1}{2}(e^{i\varphi} - e^{i\theta}) \\ 0 & e^{-i(\theta+\varphi)} & 0 \\ \frac{1}{2}(e^{i\varphi} - e^{i\theta}) & 0 & \frac{1}{2}(e^{i\theta} + e^{i\varphi}) \end{bmatrix} \right\}$$

and

$$T_1' = T_1 = T^* = \{a(\theta, t) = \operatorname{diag}(e^{i\theta+t}, e^{-2i\theta}, e^{i\theta-t}); t \in \mathbf{R}\}$$

and

$$m_0' = m_0 = \operatorname{ad}\left(\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \right), \quad m_1' = m_1 = 1.$$

Let I_0^+ be the system of positive roots for $\underset{\sim}{T}_0$ induced by $(m_0, \underset{\sim}{B}^*)$. Then, if $\xi = \xi_\lambda : {}^LH \hookrightarrow {}^LG$ as in (3.2.1) we have that $\mu^* = \frac{1}{2}(2\lambda + 1)(t_1 + t_3)$. Thus:

$$\chi^{(0)}_{(\xi_\lambda, I_0^+)}(r(\theta, \varphi)) = e^{i((\lambda+1)\theta+\lambda\varphi)}$$
$$\chi^{(1)}_{(\xi_\lambda, -)}(a(\theta, t)) = e^{i(2\lambda+1)\theta}.$$

Example 4.3.2. Let $\underset{\sim}{G} = \underset{\sim}{G}^* = Sp_4$; for $\underset{\sim}{T}^*$ we take the diagonal subgroup

$$\{\operatorname{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})\},$$

and for $\underset{\sim}{B}^*$ the Borel subgroup generated by $\underset{\sim}{T}^*$ and the 1-parameter subgroups for $2t_1, 2t_2, t_1 \pm t_2$. We may take ${}^LG$ as in (3.2.2), where $L^\vee(\underset{\sim}{T}^*)$ is identified with $L({}^LT^0)$ via the pairing

$$\langle t_1, x_1 \rangle = \langle t_1, x_2 \rangle = \langle t_2, x_1 \rangle = \tfrac{1}{2}, \quad \langle t_2, x_2 \rangle = -\tfrac{1}{2}$$

(so that $x_1 + x_2 = (2t_1)^\vee$, $x_1 - x_2 = (2t_2)^\vee$, $2x_1 = (t_1 + t_2)^\vee$, $2x_2 = (t_1 - t_2)^\vee$). We will choose a κ not attached to $\underset{\sim}{T}^*$. Let T_0 be the Cartan subgroup

$$\left\{ r(\theta, \varphi) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\varphi & 0 & \sin\varphi \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \right\}$$

and T_1 the Cartan subgroup

$$\left\{ a(\alpha, \varphi) = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \cos\varphi & 0 & \sin\varphi \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & -\sin\varphi & 0 & \cos\varphi \end{bmatrix}, \alpha \in \mathbf{R}^\times \right\};$$

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we diagonalize $\underset{\sim}{T}_0$ by

$$g_0 = \text{ad}\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{bmatrix},$$

and $\underset{\sim}{T}_1$ by

$$g_1 = \text{ad}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & -i/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Let κ be that character attached to $\underset{\sim}{T}_0$, for which the transfer by g_0 to the coroots of $\underset{\sim}{T}^*$ and thence to the roots of ${}^LT^0$, satisfies $\kappa(x_1 + x_2) = 1$, $\kappa(x_1 - x_2) = -1$. Then as ${}^LH$ we take the group of (3.2.2); this group is in standard position with respect to $\underset{\sim}{T}_1$. We realize $\underset{\sim}{H}$ not as a subgroup of $\underset{\sim}{G}^*$, but as a group satisfying (2.1.2). Thus $\underset{\sim}{H}$ will be the subgroup

$$\left\{\begin{bmatrix} a & 0 & b & 0 \\ 0 & t & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix} : ad - bc = 1\right\}$$

of $GL_4(\mathbf{C})$, with σ_H acting by $a \to \bar{a}$, $b \to \bar{b}$, $c \to \bar{c}$, $d \to \bar{d}$, $t \to \bar{t}^{-1}$. For our framework enclosing T_1 we take

$$T_0' = \left\{\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & e^{-i\varphi} \end{bmatrix}\right\}, \ T_0 \text{ as above}$$

$$m_0' = \text{ad}\left(\begin{bmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right), m_0 = g_0,$$

$$T_1' = \left\{\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & e^{-i\varphi} \end{bmatrix} \ \alpha \in \mathbf{R}^\times\right\}, \ T_1 \text{ as above},$$

$m_1' = 1$, $m_1 = g_1$. Let I_0^+ be the system of positive roots for $\underset{\sim}{T}_0$ induced by $(g_0, \underset{\sim}{B}^*)$, and I_1^+ the system of imaginary roots of $\underset{\sim}{T}_1$ induced by $(g_1, \underset{\sim}{B}^*)$. Then if $\xi = \xi_\lambda : {}^LH \hookrightarrow {}^LG$ as in (3.2.2) we obtain $\mu^* = 2\lambda t_2$ and

$\frac{1}{2}t_1$ as a choice for λ^*. Hence:

$$\chi^{(0)}_{(\xi, I_0{}^+)}(r(\theta, \varphi)) = e^{i(\theta+(2\lambda+1)\varphi)}$$
$$\chi^{(1)}_{(\xi, I_1{}^+)}(a(\alpha, \varphi)) = \operatorname{sgn}\alpha\, e^{i(2\lambda+1)\varphi}.$$

(4.4) *Assumptions in* [**10**]. In [**10**] we have used $\iota_n - \iota_n'$ for the normalization of κ_n-orbital integrals in the sense of the introduction, assuming that

(4.4.1) $\quad \iota_n - \iota_n' \in I_n$

and, on transferring to L,

(4.4.2) $\quad (\iota_n - \iota_n') - (\iota_p - \iota_p')$ is an integral combination of roots.

In general, (4.4.1) may fail, as in the groups of (4.3.1), or if (4.4.1) is true then (4.4.2) may fail, as in (4.3.2). These examples show more, namely that $\iota_n - \iota_n'$ need not define, by restriction, a character on T_n, or if it does then that character need not have the desired properties for orbital integrals (cf. [**10**], Proposition 9.4, or direct calculation). However it is easily seen that in each example we can use $\chi^{(n)}_{(\xi, I_n{}^+)}$ in place of $\iota_n - \iota_n'$. We proceed now to prove this in general.

(4.5) $(\iota_n - \iota_n')$-*type*. By definition (cf. [**10**]), $\Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ is the subgroup of $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ consisting of those elements which commute with σ_n, that is, which are realized in $\mathfrak{A}(T_n)$. If $\omega \in \Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ and ω comes from $\underset{\sim}{H}$ (cf. (2.4)) then ω is the image of an element of $\Omega_0(\underset{\sim}{H}, \underset{\sim}{T}_n')$. Thus $\omega(\iota_n - \iota_n') - (\iota_n - \iota_n')$ is an integral combination of roots of $\underset{\sim}{T}_n$ and hence an element of L_n.

Definition 4.5.1. A quasicharacter χ on T_n is of $(\iota_n - \iota_n')$-*type* if

$$\chi(\gamma^{\omega^{-1}}) = (\omega(\iota_n - \iota_n') - (\iota_n - \iota_n'))(\gamma)\chi(\gamma), \quad \gamma \in T_n,$$

for each $\omega \in \Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ coming from $\underset{\sim}{H}$.

Section 5 will be devoted to the proof of:

THEOREM 4.5.2. *$\chi^{(n)}_{(\xi, I_n{}^+)}$ is of $(\iota_n - \iota_n')$-type.*

5. Proof of theorem 4.5.2. In this section we abbreviate $\chi^{(n)}_{(\xi, I_n{}^+)}$, writing just $\chi^{(n)}$, $n = 0, \ldots, N$. By definition, $\chi^{(n)}$ is of $(\iota_n - \iota_n')$-type if and only if

$$\chi(\omega(\mu^* + \iota_n - \iota_n'), \omega\lambda^*) = \chi((\omega(\iota_n - \iota_n') - (\iota_n - \iota_n')) + \mu^* + \iota_n - \iota_n', \lambda^*)$$

for each $\omega \in \Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ coming from $\underset{\sim}{H}$. Consider first the restriction of

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$\chi^{(n)}$ to the connected component of the identity in T_n. If $\gamma = \exp X$, $X \in \mathfrak{t}_n$, then

$$\chi^{(n)}(\gamma) = e^{\langle \mu^* + \iota_n - \iota_n', X\rangle}.$$

Since $\langle \mu^*, \alpha^\vee \rangle = 0$ for each $\alpha^\vee$ from $\underset{\sim}{H}$ we have immediately:

PROPOSITION 5.0.1. *If T_n is connected then $\chi^{(n)}$ is of $(\iota_n - \iota_n')$-type.*

Thus we have:

Example 5.0.2 (cf. (4.3.1)). If $G = SU(p, q)$ then each $\chi^{(n)}$ is of $(\iota_n - \iota_n')$-type.

In general, it remains to show that

(5.0.3) $\quad \omega\lambda^* \equiv \lambda^* \bmod (L_n + \{\nu - \sigma_n\nu : \nu \in L_n \otimes \mathbf{C}\}).$

(5.1) *Some reductions.*

REDUCTION 5.1.1. *It is sufficient to prove* (5.0.3) *for the case $n = N$.*

Proof: From [**10**, § 7] we recall that there is a diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Omega(\mathbf{M}_n, \mathbf{T}_n) & \longrightarrow & \Omega_0(\mathbf{G}^*, \mathbf{T}_n) & \longrightarrow & \mathfrak{W}_n & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Omega(\mathbf{M}_n', T_n') & \longrightarrow & \Omega_0(\mathbf{H}, \mathbf{T}_n') & \longrightarrow & \mathfrak{W}_n' & \longrightarrow & 1
\end{array}$$

where $\mathfrak{W}_n^{(\prime)}$ denotes the restricted Weyl group relative to $\underset{\sim}{S}_n^{(\prime)}$ (the maximal R-split torus in $\underset{\sim}{T}_n^{(\prime)}$). If ω comes from $\Omega(\underset{\sim}{M}_n', \underset{\sim}{T}_n')$ then clearly (5.0.3) is satisfied. It follows then that, in general, the coset of $\omega\lambda^* - \lambda^*$ in

$$L_n \otimes \mathbf{C}/L_n + \{\nu - \sigma_n\nu : \nu \in L_n \otimes \mathbf{C}\}$$

depends only on the image $\tilde{\omega}$ of ω in $\mathfrak{W}_n$.

Suppose that ω comes from $\omega' \in \Omega_0(\underset{\sim}{H}, \underset{\sim}{T}_n')$ whose image in $\mathfrak{W}_n'$ is $\tilde{\omega}'$. There exists $\tilde{\omega}_N' \in \mathfrak{W}_N'$ whose restriction to $\underset{\sim}{S}_n'$ is $\tilde{\omega}'$, by the definition of $\mathfrak{W}_n'$. Set $\tilde{\omega}_N$ equal to the image of $\tilde{\omega}_N'$ in $\mathfrak{W}_N$ and choose ω_N in $\Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_N)$ coming from $\underset{\sim}{H}$ and with image $\tilde{\omega}_N$ in $\mathfrak{W}_N$. Let $\lambda_n^* = \frac{1}{2}(\lambda^* + \sigma_n\lambda^*)$. We transfer everything to $L \otimes \mathbf{C}$ (via m_n, m_n', m_N, m_N') without change in notation. From definitions, it follows that

$$\omega\lambda^* - \lambda^* \equiv \tilde{\omega}\lambda_n^* - \lambda_n^* \bmod \{\nu - \sigma_n\nu : \nu \in L \otimes \mathbf{C}\}$$

and

$$\tilde{\omega}\lambda_n^* - \lambda_n^* \equiv \omega_N\lambda_N^* - \lambda_N^* \bmod \{\nu - \sigma_n\nu : \nu \in L \otimes \mathbf{C}\}.$$

Since

$$\tilde{\omega}_N\lambda_N^* - \lambda_N^* \equiv \omega_N\lambda^* - \lambda^* \bmod \{\nu - \sigma_N\nu : \nu \in L \otimes \mathbf{C}\}$$

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and $\{\nu - \sigma_N \nu\} \subseteq \{\nu - \sigma_n \nu\}$, we have

$$\omega\lambda^* - \lambda^* \equiv \omega_N \lambda^* - \lambda^* \bmod \{\nu - \sigma_n \nu : \nu \in L \otimes \mathbf{C}\}$$

and the reduction follows.

LEMMA 5.1.2. (5.0.3) *holds provided that*

(5.1.3) $\langle \lambda^*, \alpha^\vee \rangle \alpha \in L + \{\nu - \sigma_H \nu : \ \nu \in L \otimes \mathbf{C}\}$

for all simple roots $\alpha^\vee$ *of* ${}^L H^0$ *fixed by* σ_H.

COROLLARY 5.1.4. *If*

(5.1.5) $\langle \lambda^*, \alpha^\vee \rangle \in \mathbf{Z}$ *for all simple roots* $\alpha^\vee$ *of* ${}^L H^0$ *fixed by* σ_H

then (5.0.3) *holds.*

We will prove Theorem 4.5.2 by verifying (5.1.5) where possible, and by going directly to (5.1.3) for the few exceptions.

Proof of Lemma 5.1.2. Dual to the simple system of roots for $({}^L H^0, {}^L T^0)$ as prescribed by ${}^L B_H{}^0$, we have a simple system for $(\underset{\sim}{H}, \underset{\sim}{T}_H = \underset{\sim}{T}_N{}')$, prescribing $\underset{\sim}{B}_H$; we use this latter system to define a simple system for the restricted roots of $\underset{\sim}{T}_N{}'$. The group $\mathfrak{W}_N{}'$ is generated by simple reflections, which we classify as being of type A, B, or C as in [**10**, § 7]. Suppose that $\omega \in \Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_N)$ comes from $\omega' \in \Omega_0(\underset{\sim}{H}, \underset{\sim}{T}_N{}')$ which has image ω' in $\mathfrak{W}_N{}'$.

If $\tilde{\omega}'$ is of type A then there is a real root α of $\underset{\sim}{T}_N (\sigma_N \alpha = \alpha)$ coming from a simple root of $\underset{\sim}{H}$, such that ω_α has the same image in $\mathfrak{W}_N$ as ω. Thus (5.0.3) is satisfied by ω if (5.1.3) is true.

If $\tilde{\omega}'$ is of type B then there is a root α of $\underset{\sim}{T}_N$ coming from a simple root of $\underset{\sim}{H}$, satisfying $\langle \alpha, \sigma_N \alpha^\vee \rangle = 0$, and such that $\omega_\alpha \omega_{\sigma_N \alpha}$ has the same image in $\mathfrak{W}_N$ as ω. Clearly,

$$\omega_\alpha \omega_{\sigma_N \alpha} \lambda^* - \lambda^* \equiv -\langle \lambda^* + \sigma_N \lambda^*, \alpha^\vee \rangle \alpha \bmod (L_N + \{\nu - \sigma_N \nu\}).$$

Also, by (3.3.1) and (3.4.1), we have

$$\langle \lambda^* + \sigma_N \lambda^*, \alpha^\vee \rangle \equiv \langle \iota_N, \alpha^\vee \rangle \bmod \mathbf{Z}.$$

Thus (5.0.3) for ω follows from:

PROPOSITION 5.1.6. *If* α *is a root of* $\underset{\sim}{T}_N$ *such that* $\langle \alpha, \sigma_N \alpha^\vee \rangle = 0$ *then* $\langle \iota_N, \alpha^\vee \rangle \in \mathbf{Z}$.

Proof. We may assume $\underset{\sim}{G}^*$ absolutely simple (cf. proof of (5.1.7)). Further, we may exclude type G_2 since direct computation shows that, in that case, $\iota_N = 0$; that is, $T_N = T^*$, for all $\underset{\sim}{H}$.

We assume that $\langle \iota_N, \alpha^\vee \rangle \equiv \frac{1}{2} \bmod \mathbf{Z}$ and obtain a contradiction. First note that $\underset{\sim}{M}_N$ is of type $A_1 \times \ldots \times A_1$. Indeed, ${}^L(M_N{}')^0 = {}^L T^0$; that is, there are no roots of ${}^L M_N{}^0$ annihilated by κ. Hence if $\alpha^\vee$, $\beta^\vee$ are roots of ${}^L M_N{}^0$ then $\alpha^\vee \pm \beta^\vee$ are not roots, for $\kappa(\alpha^\vee) = \kappa(\beta^\vee) = -1$ and

$\kappa(\alpha^\vee \pm \beta^\vee) = 1$. Therefore ${}^L M_N{}^0$ is of type $A_1 \times A_1 \times \ldots \times A_1$; σ_G must act trivially and $\underset{\sim}{M}_N$ be of type $A_1 \times \ldots \times A_1$ over $\mathbf{R}$.

Let $\omega = \omega_\alpha \omega_{\sigma_N \alpha}$, an element of $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}_N)$. Clearly ω commutes with σ_N, and so permutes the roots of $\underset{\sim}{M}_N$. Thus

$$\iota_N - \omega \iota_N = \langle \iota_N, \alpha^\vee \rangle (\alpha - \sigma_N \alpha)$$

is an integral linear combination of roots of $\underset{\sim}{M}_N$. We claim that $1/2(\alpha - \sigma_N \alpha)$ is also an integral linear combination of such roots. To verify this it is enough to show that $\alpha - \sigma_N \alpha$ itself is such a combination. But since $\langle \iota_N, \alpha^\vee \rangle \equiv 1/2 \bmod \mathbf{Z}$ we have $\langle \beta, \alpha^\vee \rangle = 1$ for some root β of $\underset{\sim}{M}_N$. Then, with ω as above, we obtain

$$\alpha - \sigma_N \alpha = \beta - \omega \beta;$$

$\omega\beta$ is also a root of $\underset{\sim}{M}_N$, and so the claim is proved.

Let $\beta_1 = 1/2(\alpha - \sigma_N \alpha)$ and $\beta_2 = 1/2(\alpha + \sigma_N \alpha)$. Then $\beta_2 \neq 0$ and the length of α is greater than that of β_1. On the other hand, $\underset{\sim}{M}_N$ is of type $A_1 \times \ldots \times A_1$. Hence β_1 must be a root of $\underset{\sim}{M}_N$. Then β_2 is a root of $\underset{\sim}{G}^*$ and β_1, β_2 generate a root system of type C_2. Thus $\beta_1{}^\vee = \alpha^\vee - \sigma_N \alpha^\vee$. This implies that $\kappa_N(\beta_1{}^\vee) = 1$, a contradiction since β_1 is a root of $\underset{\sim}{M}_N$. Hence Proposition 5.1.6 is proved.

We return to the proof of Lemma 5.1.2. If $\tilde{\omega}'$ is of type C then there is a root α of $\underset{\sim}{T}_N$, coming from a simple root of $\underset{\sim}{T}_N{}'$, such that $\alpha + \sigma_N \alpha$ is a root and ω has the same image in $\mathfrak{W}_N$ as $\omega_{\alpha + \sigma_N \alpha}$. Either $\langle \alpha, \sigma_N \alpha^\vee \rangle = 0$ or $\langle \alpha, \sigma_N \alpha^\vee \rangle < 0$, since α is simple. If $\langle \alpha, \sigma_N \alpha^\vee \rangle = 0$ then $\omega_{\alpha + \sigma_N \alpha}$ has the same image in $\mathfrak{W}_N$ as $\omega_\alpha \omega_{\sigma_N \alpha}$. Thus

$$\begin{aligned}
\omega\lambda^* - \lambda^* &\equiv (\omega_\alpha \omega_{\sigma_N \alpha} \lambda^* - \lambda^*) \bmod (L_N + \{\nu - \sigma_N \nu\}) \\
&\equiv \langle \lambda^* + \sigma_N \lambda^*, \alpha^\vee \rangle \alpha \bmod (L_N + \{\nu - \sigma_N \nu\}) \\
&\equiv \langle \iota_N, \alpha^\vee \rangle \alpha \bmod (L_N + \{\nu - \sigma_N \nu\}) \\
&\equiv 0 \bmod (L_N + \{\nu - \sigma_N \nu\})
\end{aligned}$$

by (3.3.1), (3.4.1) and (5.1.6); (5.0.3) now follows. On the other hand, if $\langle \alpha, \sigma_N \alpha^\vee \rangle < 0$ then $(\alpha + \sigma_N \alpha)^\vee = \alpha^\vee + \sigma_N \alpha^\vee$ since α and $\sigma_N \alpha$ have the same length. Then

$$\begin{aligned}
\omega\lambda^* - \lambda^* &\equiv (\omega_{\alpha + \sigma_N \alpha} \lambda^* - \lambda^*) \bmod (L_N + \{\nu - \sigma_N \nu\}) \\
&\equiv \langle \lambda^* + \sigma_N \lambda^*, \alpha^\vee \rangle (\alpha + \sigma_N \alpha) \bmod (L_N + \{\nu - \sigma_N \nu\}) \\
&\equiv 2 \langle \iota_N, \alpha^\vee \rangle \alpha \bmod (L_N + \{\nu - \sigma_N \nu\}) \\
&\equiv 0 \bmod (L_N + \{\nu - \sigma_N \nu\}),
\end{aligned}$$

by (3.3.1) and (3.4.1). Again (5.0.3) follows. This completes the proof of Lemma 5.1.2.

By a *simple factor of* $\underset{\sim}{G}^*$ we will mean an $\mathbf{R}$-simple factor of the simply-connected covering group of the derived group of $\underset{\sim}{G}^*$.

REDUCTION 5.1.7. (i) *To prove* (5.1.5) *for* $\underset{\sim}{G}^*$ *it is sufficient to prove it for each simple factor of* $\underset{\sim}{G}^*$.

(ii) *To prove* (5.1.3) *for* $\underset{\sim}{G}^*$ *it is sufficient to prove it for each simple factor of* $\underset{\sim}{G}^*$, *but with* L *replaced by the span of the roots in that factor.*

We denote this stronger version of (5.1.3) by (5.1.8).

Proof. We may regard $\underset{\sim}{H}$ as $\underset{\sim}{H}(T, \kappa)$ for any (T, κ) among $\{(T_n, \kappa_n) : n = 0, \ldots, N\}$. Let $\underset{\sim}{G}^\dagger$ be a simple factor of $\underset{\sim}{G}^*$ (in the sense above), $\underset{\sim}{T}^\dagger$ be the preimage of $\underset{\sim}{T}^*$ in $\underset{\sim}{G}^\dagger$, and $\underset{\sim}{T}_n{}^\dagger$ the preimage of $\underset{\sim}{T}_n$. Then $L^\vee(T^\dagger)$ is naturally identified as a submodule of $L^\vee$ and $L^\vee(\underset{\sim}{T}_n{}^\dagger)$ as a submodule of $L_n{}^\vee$. We may thus identify κ_n as a quasicharacter attached to $\underset{\sim}{T}_n{}^\dagger$. If $\underset{\sim}{H}^\dagger = \underset{\sim}{H}^\dagger(T_n{}^\dagger, \kappa_n)$ then the Lie algebra of ${}^L(H^\dagger)^0$ is a summand of the Lie algebra of ${}^LH^0$, assuming all choices are in correct position.

We extend the natural map $L \to L(\underset{\sim}{T}^\dagger)$ to a $\mathbf{C}$-linear map $L \otimes \mathbf{C} \to L(\underset{\sim}{T}^\dagger) \otimes \mathbf{C}$. Recall that $\xi : {}^LH \hookrightarrow {}^LG$ is $\xi(\mu^*, \lambda^*)$. Let $(\mu^*)^\dagger$ be the image of μ^* in $L(\underset{\sim}{T}^\dagger) \otimes \mathbf{C}$ and $(\lambda^*)^\dagger$ be the image of λ^*. Then $((\mu^*)^\dagger, (\lambda^*)^\dagger)$ are parameters for the embedding $\xi^\dagger$ of ${}^L(H^\dagger)$ in ${}^L(G^\dagger)$ obtained by mapping ${}^L(H^\dagger)^0$ to itself by the identity and $1 \times w$ to the image of $\xi(1 \times w)$ under the natural map ${}^LG \to {}^L(G^\dagger)$ (cf. [**3**, § 2.5]), $w \in W$. If $\alpha^\vee$ is a root of both ${}^LH^0$ and ${}^L(G^\dagger)^0$ then $\langle(\lambda^*)^\dagger, \alpha^\vee\rangle = \langle\lambda^*, \alpha^\vee\rangle$. Thus (i) follows; (ii) also follows easily.

Note. For a simple factor of $\underset{\sim}{G}$ which is not absolutely simple, (5.1.5) and (5.1.8) are vacuously true. Thus to prove (5.0.3), and hence Theorem 4.5.2, we need consider only absolutely simple factors. In (5.2) by "simple group" we will mean an absolutely simple group.

(5.2) *Computations in* ${}^LG$. We start with the case that the "most split" Cartan subgroup T^* of G^* is also a Cartan subgroup of H; that is, the case that $T_N = T^*$.

LEMMA 5.2.1. *If* G^* *is split modulo its center and* H *contains* T^* *then* $\langle\lambda^*, \alpha^\vee\rangle \in \mathbf{Z}$ *for all roots* $\alpha^\vee$ *of* ${}^LH^0$.

Proof. By assumption, $\sigma_H = \sigma_G$ on ${}^LT^0$; also σ_G acts trivially on the root vectors for ${}^LG^0$, and σ_H trivially on the root vectors for ${}^LH^0$. Recall that $\xi(1 \times \sigma) = t \times (1 \times \sigma)$, where, since ${}^LM_N{}^0 = {}^LT^0$, we have

$$t \in {}^LT^0 \quad \text{and} \quad \lambda^\vee(t) = e^{2\pi i\langle\lambda^*, \lambda^\vee\rangle}, \quad \lambda^\vee \in L^\vee.$$

Thus if $Y_{\alpha^\vee}$ is a simple root vector for ${}^LH^0$ then the fact that $\xi(1 \times \sigma)$ acts on ${}^LH^0$ as σ_H implies that

$$t \times (1 \times \sigma) Y_{\alpha^\vee} = e^{2\pi i\langle\lambda^*, \alpha^\vee\rangle} Y_{\alpha^\vee} = Y_{\alpha^\vee},$$

and the lemma follows.

Example 5.2.2. Theorem 4.5.2 is now proved for $\underset{\sim}{G}^*$ of type G_2, as $T_N = T^*$, for all $\underset{\sim}{H}$, in that case.

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LEMMA 5.2.3. *If G^* is a simple nonsplit group of type other than A_{2n} and H contains T^* then $\langle \lambda^*, \alpha^\vee \rangle \in \mathbf{Z}$ for all simple roots $\alpha^\vee$ of ${}^LH^0$ fixed by σ_H.*

Proof. In order to imitate the proof of Lemma 5.2.1 we show that $\sigma_G Y_{\alpha^\vee} = Y_{\alpha^\vee}$ for each simple root $\alpha^\vee$ of ${}^LH^0$ satisfying $\sigma_H \alpha^\vee = \alpha^\vee (= \sigma_G \alpha^\vee)$.

From Lemma 3 of [**8**] we obtain that

$$\sigma_G Y_{\alpha^\vee} = (-1)^l Y_{\alpha^\vee},$$

where l is the number of ${}^LG^0$-simple roots $\beta^\vee$ for which $\beta^\vee \neq \sigma_G \beta^\vee$ and $\langle \beta, \sigma_G \beta^\vee \rangle \neq 0$, counted according to multiplicity in the ${}^LG^0$-simple expansion of $\alpha^\vee$. We claim that because we have excluded type A_{2n} we have $l = 0$. This is checked by inspection of the possibilities (cf. [**5**]).

The lemma thus follows.

Example 5.2.4. If G^* is a simple nonsplit group of type A_{2n} and H contains T^* then (5.1.5) may fail (cf. Example 3.2.1); however (5.1.8) is true. This is a simple computation: if $\alpha^\vee$ is any root of ${}^LG^0$ for which $\sigma_G \alpha^\vee = \alpha^\vee$ and $\sigma_G Y_{\alpha^\vee} = -Y_{\alpha^\vee}$ then α is of the form $\beta + \sigma_G \beta = \beta + \sigma_H \beta$, β a root, so that

$$\tfrac{1}{2}\alpha = \beta \bmod (\{\nu - \sigma_H \nu : \nu \in L \otimes \mathbf{C}\}),$$

and (5.1.8) holds.

LEMMA 5.2.5. *Suppose that G^* is simple, not of type B_n, C_n, F_4 or A_{2n}-nonsplit, and that H does not contain T^*. Then $\langle \lambda^*, \alpha^\vee \rangle \in \mathbf{Z}$ for all simple roots $\alpha^\vee$ of ${}^LH^0$ fixed by σ_H.*

In case of type F_4, we can show that, in fact, $T_N = T^*$ for all H, by observing how T_N is obtained from T^* (cf. (5.1.6) and (6.1.3)) and examining the possibilities.

Proof. We write $\xi(1 \times \sigma)$ as $m \times (1 \times \sigma)$ with $m \in {}^LM_N{}^0$, and m as $t_1 m_1$, with t_1 in the connected center of ${}^LM_N{}^0$ and m_1 in ${}^L\mathscr{M}^0 = ({}^LM_N{}^0)_{\text{der}}$. Let $\alpha^\vee$ be a simple root of ${}^LH^0$ fixed by σ_H. Then $\sigma_G \alpha^\vee = \alpha^\vee$ also. In the proof of Lemma 5.2.3 we showed that $\sigma_G Y_{\alpha^\vee} = Y_{\alpha^\vee}$. Hence

$$m \times (1 \times \sigma) Y_{\alpha^\vee} = t_1 m_1 Y_{\alpha^\vee}.$$

We have only to show that $m_1 Y_{\alpha^\vee} = Y_{\alpha^\vee}$ for then

$$t Y_{\alpha^\vee} = e^{2\pi i \langle \lambda^*, \alpha^\vee \rangle} Y_{\alpha^\vee} = Y_{\alpha^\vee},$$

since $\alpha^\vee$ extends to a rational character on ${}^LM_N{}^0$, and $\langle \lambda^*, \alpha^\vee \rangle \in \mathbf{Z}$.

First, because $\alpha^\vee$ extends to ${}^LM_N{}^0$, we have that $m_1 Y_{\alpha^\vee} = m_2 Y_{\alpha^\vee}$ for any $m_2 \in {}^L\mathscr{M}^0$ such that $m_2 \times (1 \times \sigma)$ normalizes ${}^LT^0$ and maps each root of ${}^L\mathscr{M}^0$ to its negative. We have seen that ${}^L\mathscr{M}^0$ is of type $A_1 \times \ldots \times A_1$ and that σ_G acts trivially on ${}^L\mathscr{M}^0$ (cf. (5.1.6)). Thus if $\alpha_1^\vee, \ldots, \alpha_d^\vee$ are the positive roots of ${}^L\mathscr{M}^0$ we may replace m_1 by any element of ${}^L\mathscr{M}^0$

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realizing $\omega_{\alpha_1^\vee}\omega_{\alpha_2^\vee} \ldots \omega_{\alpha_d^\vee}$. By excluding types B_n, C_n, F_4 we have ensured that $\alpha^\vee \pm \alpha_i^\vee$ are not roots, $i = 1, \ldots, d$. Therefore, by using the element

$$\exp\left(\sum_{i=1}^{d} \tfrac{1}{2} i\pi (X_{\alpha_i^\vee} + X_{-\alpha_i^\vee})\right),$$

we see that $m_1 Y_{\alpha^\vee} = Y_{\alpha^\vee}$, and the lemma is proved.

LEMMA 5.2.6. *If G^* is simple, of type B_n, C_n, F_4 or A_{2n}-nonsplit and H does not contain T^* then* (5.1.8) *is satisfied.*

Proof. Consider type A_{2n}-nonsplit first. Suppose that $\sigma_H \alpha^\vee = \alpha^\vee$. Then $\sigma_G \alpha^\vee = \alpha^\vee$. Suppose that $\sigma_G Y_{\alpha^\vee} = -Y_{\alpha^\vee}$. Then we have

$$\tfrac{1}{2}\alpha \equiv \beta \bmod (\{\nu - \sigma_G \nu : \nu \in L \otimes \mathbf{C}\}),$$

for some root β (cf. (5.2.4)). Since $\{\nu - \sigma_G \nu\} \subseteq \{\nu - \sigma_H \nu\}$ we obtain (5.1.8). If $\sigma_G Y_{\alpha^\vee} = Y_{\alpha^\vee}$ then we can argue as in the proof of Lemma 5.2.5 to obtain $\langle \lambda^*, \alpha^\vee \rangle \in \mathbf{Z}$.

If $\underset{\sim}{G}^*$ is of type B_n, C_n or F_4 then $\langle \lambda^*, \alpha^\vee \rangle \in \mathbf{Z}$ unless we have the following: there is a root $a_i^\vee$ such that $\sigma_N \alpha_i^\vee = \sigma_H \alpha_i^\vee = -\alpha_i^\vee$, $\langle \alpha_i, \alpha^\vee \rangle = 0$ and $\alpha_i^\vee + \alpha^\vee$ is a root. Then $\frac{1}{2}(\alpha_i + \alpha)$ is a root of $\underset{\sim}{T}_N$, with coroot $\alpha_i^\vee + \alpha^\vee$, and $\frac{1}{2}\alpha = \frac{1}{2}(\alpha + \alpha_i) - \frac{1}{2}\alpha_i$ so that (5.1.8) is true.

This completes the proof of Theorem 4.5.2.

6. Quasicharacters continued, correction characters.

(6.1) *Compatibility.* We come to formulating and proving the compatibility of the quasicharacters $\chi^{(n)} = \chi^{(n)}_{(\xi, I_n{}^+)}$. This is the key to our main result, Theorem 8.0.1. First we recall some definitions and simple results about Cayley transforms (cf. [**9**], [**10**]).

Suppose that T is a Cartan subgroup of G^* and α an imaginary root of $\underset{\sim}{T}$. Then by a *Cayley transform* with respect to α we mean a map $s : \underset{\sim}{T} \to \underset{\sim}{G}^*$, obtained by restriction to $\underset{\sim}{T}$ of an inner automorphism of $\underset{\sim}{G}^*$ and with the property that $\sigma(s^{-1})s = \omega_\alpha$, the Weyl reflection with respect to α. Because $\underset{\sim}{G}^*$ is quasi-split there exists a Cayley transform with respect to each imaginary root α of $\underset{\sim}{T}$ (cf. [**10**]). The image $\underset{\sim}{T}_s$ of $\underset{\sim}{T}$ under s is defined over $\mathbf{R}$ and $s\alpha$ is a real root of T_s; if s is another Cayley transform with respect to α then s is of the form ad $w \circ s$, $w \in \mathfrak{A}(T_s)$ ([**9**]). Note also that if $\gamma \in T$, $\alpha(\gamma) = 1$ then $s(\gamma) = \gamma^s$ belongs to T_s.

Suppose that α' is an imaginary root of one of our fixed Cartan subgroups T_n' of H. If s' is a Cayley transform with respect to α' and with image $\underset{\sim}{T}_p'$ then

$$s : \underset{\sim}{T}_n \to \underset{\sim}{T}_n' \xrightarrow{s'} \underset{\sim}{T}_p' \to \underset{\sim}{T}_p$$

is easily shown to be a Cayley transform with respect to the image α of α'

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in the roots of $\underset{\sim}{G}^*$ (cf. (2.4)). We call s a *Cayley transform from* $\underset{\sim}{H}$; with respect to any imaginary root from $\underset{\sim}{H}$ there is a Cayley transform from $\underset{\sim}{H}$.

Continuing with the same α', s', α, s, if I_n^+ is a positive system for the imaginary roots of $\underset{\sim}{T}_n$ then $(I_n^+)_s = \{\beta : s^{-1}\beta \in I_n^+\}$ is a positive system for the imaginary roots of $\underset{\sim}{T}_p$. We say that I_n^+ is *adapted to* α if $\langle \alpha, \beta^\vee \rangle > 0$ implies that $\beta \in I_n^+$; $\langle \iota_n - \iota_n', \alpha^\vee \rangle$ (cf. (4.2)) is independent of the choice of I_n^+ adapted to α.

The quasicharacters $\chi^{(n)}_{(\xi, I_n^+)}$ are "compatible" in the following sense:

THEOREM 6.1.1. *Suppose that* $s : \underset{\sim}{T}_n \rightarrow \underset{\sim}{T}_p$ *is a Cayley transform from* $\underset{\sim}{H}$, *with respect to the root* α *from* $\underset{\sim}{H}$. *Then if* I_n^+ *is adapted to* α *and* $\gamma \in T$ *satisfies* $\alpha(\gamma) = 1$ *we have*

$$\chi^{(n)}_{(\xi, I_n^+)}(\gamma) = \chi^{(p)}_{(\xi, (I_n^+)_s)}(\gamma^s).$$

Proof. Write β for the real root $s\alpha$ of $\underset{\sim}{T}_p$. To compute $\chi^{(p)}(\gamma^s)$ we decompose γ^s as in (4.1). Let

$$\gamma^s = \exp X = \exp X_{\mathbf{R}} \exp X_{\mathbf{I}},$$

where $\sigma_p X_{\mathbf{R}} = \bar{X}_{\mathbf{R}}$ and $X_{\mathbf{I}} = i\pi\lambda^\vee$, $\lambda^\vee \in L_p^\vee$ and $\sigma_p \lambda^\vee = \lambda^\vee$. Then $\beta(\gamma^s) = 1$ implies that $\langle \beta, X_{\mathbf{R}} \rangle = 0$ and $\langle \beta, X_{\mathbf{I}} \rangle \in 2\pi i\, \mathbf{Z}$. It is therefore enough to consider those γ for which:

(i) $\gamma^s \in T_p^0$; that is, $\gamma^s = \exp X$, $\sigma_p X = \bar{X}$,

(ii) $\gamma^s = \exp i\pi\lambda^\vee$, $\lambda^\vee \in L_p^\vee$, $\sigma_p\lambda^\vee = \lambda^\vee$, $\langle \beta, \lambda^\vee \rangle = 0$,

or

(iii) $\gamma^s = \exp i\pi\beta^\vee$.

Suppose (i). Then $\sigma_n(s^{-1}X) = \overline{s^{-1}X}$ (recall that $\langle \beta, X \rangle = 0$) so that $s^{-1}X = (s^{-1}X)_{\mathbf{R}}$). Then

$$\chi^{(n)}(\gamma) = e^{\langle \mu^* + \iota_n - \iota_n', s^{-1}X \rangle} \quad \text{and } \chi^{(p)}(\gamma) = e^{\langle \mu^* + \iota_p - \iota_p', X \rangle}.$$

We claim that $s^{-1}\mu^* = \mu^*$. Recall that we use μ^* to denote the transfer of $\mu^* \in L \otimes \mathbf{C}$ to $L_n \otimes \mathbf{C}$ by m_n, as well as its transfer to $L_p \otimes \mathbf{C}$ by m_p. Thus our claim follows from the fact that s "comes from H". Also, $(\iota_n - \iota_n') - s^{-1}(\iota_p - \iota_p')$ is a half-integer multiple of α and so we obtain the assertion of the theorem.

Suppose (ii). Then

$$\gamma = \exp(i\pi s^{-1}\lambda^\vee) \quad \text{and} \quad \sigma_n(s^{-1}\lambda^\vee) = s^{-1}\lambda^\vee.$$

Thus

$$\chi^{(n)}(\gamma) = e^{2\pi i \langle \lambda^*, s^{-1}\lambda^\vee \rangle} \quad \text{and} \quad \chi^{(p)}(\gamma^s) = e^{2\pi i \langle \lambda^*, s^{-1}\lambda^\vee \rangle}.$$

Once again, λ^* in the first equation denotes the transfer of $\lambda^* \in L \otimes \mathbf{C}$ to $L_n \otimes \mathbf{C}$ by m_n and λ^* in the second equation denotes the transfer to

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$L_p \otimes \mathbf{C}$ by m_p. Let $\tilde{s} = m_p \circ s \circ m_n^{-1}$. Then to prove the theorem for case (ii) it will be enough to show that

$$\langle \tilde{s}\lambda^*, \mu^\vee \rangle \equiv \langle \lambda^*, \mu^\vee \rangle \bmod \mathbf{Z}$$

for all $\mu^\vee \in L^\vee$ satisfying $\sigma_n \mu^\vee = \mu^\vee$.

PROPOSITION 6.1.2. *There exists h in the normalizer of $\underset{\sim}{T}_N'$ in H such that the action of* ad h *on $\underset{\sim}{S}_n'$ (the maximal* $\mathbf{R}$*-split torus in $\underset{\sim}{T}_N'$), when transferred to S_n, coincides with $\tilde{s}$; that is, $\tilde{s}$ acts on $\underset{\sim}{S}_n$ as an element in the image of $\mathfrak{W}_n'$ in $\mathfrak{W}_n$.*

Proof. We can choose a Cayley transform s_0' in $\underset{\sim}{M}_n'$ with respect to the root α' from which α originates, such that $s' = \text{ad } h' \circ s_0'$ for some $h' \in H$, where s' is the Cayley transform in $\underset{\sim}{H}$ from which s originates. Then, on $\underset{\sim}{S}_n'$, s' acts as ad h'. If $\underset{\sim}{T}'$ is the image of $\underset{\sim}{T}_n'$ under s_0' then ad h' maps $\underset{\sim}{M}_{T'}$ to $\underset{\sim}{M}_p'$. We can modify h' by an element of M_p' to obtain h as desired.

Suppose now (iii). Since $\sigma_n(i\pi\alpha^\vee) = i\pi\alpha^\vee$ we have to prove:

LEMMA 6.1.3. *If α comes from H, $\sigma_n\alpha = -\alpha$, s is a Cayley transform from $\underset{\sim}{H}$ and with respect to α, and I_n^+ is adapted to α, then*

(6.1.4) $\quad \frac{1}{2}\langle \iota_n - \iota_n', \alpha^\vee \rangle \equiv \langle \lambda^*, s\alpha^\vee \rangle \bmod \mathbf{Z}.$

Proof. First we remark that we may assume that $\underset{\sim}{S}_p' \supset \underset{\sim}{S}_n'$. Then $\underset{\sim}{M}_n \supset \underset{\sim}{M}_p$. Since $\underset{\sim}{M}_n'$ is a (T_n, κ_n^-)-group for $\underset{\sim}{M}_n$ we may now replace $\underset{\sim}{G}^*$ by $\underset{\sim}{M}_n$. Thus it is enough to work under the hypothesis that T_n is compact modulo the center of G^*. We may also assume $\underset{\sim}{G}^*$ absolutely simple and simply-connected (cf. (5.1.7)).

Unless ${}^LG^0$ is of type $B_l (l \geqq 1)$ and $\alpha^\vee$ a short root, there is s root $\alpha_0^\vee$ of ${}^LG^0$ such that $\langle \alpha, \alpha_0^\vee \rangle = 1$ (cf. [**4**]). Thus, except in that case, on $L_n^\vee$ we have

$$\alpha^\vee = \alpha_0^\vee - \omega_{\alpha^\vee}(\alpha_0^\vee)$$

and on $L_p^\vee$,

$$\beta^\vee = s\alpha^\vee = (s\alpha_0^\vee) + \sigma_p(s\alpha_0^\vee).$$

Thus exp $i\pi\beta^\vee$ lies in T_p^0 and (6.1.4) follows from what we have already proved.

Suppose now that ${}^LG^0$ is of type $B_l (l \geqq 1)$; (6.1.4) certainly holds for SL_2, so that we may assume that $l \geqq 2$. We list the roots of $(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$ as $\{\pm e_i \pm e_j, \pm 2e_i; 1 \leqq i, j \leqq l\}$ and the roots of $({}^LG^0, {}^LT^0)$ as $\{\pm e_i \pm e_j, \pm e_i; 1 \leqq i, j \leqq l\}$, with $\langle\ ,\ \rangle$ given by $\langle e_i, e_j \rangle = \delta_{ij}$, $1 \leqq i, j \leqq l$ (cf. [**4**]). We transfer roots of $(\underset{\sim}{G}^*, \underset{\sim}{T}^*)$ to roots of $(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ and roots of $({}^LG^0, {}^LT^0)$ to coroots of $(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ via m_n, without change in notation.

To verify (6.1.4) we essentially describe the possibilities for ${}^LH^0$. Suppose first that $T_N = T^*$. Then Lemma 5.2.1 implies that $\langle \lambda^*, s\alpha^\vee \rangle$ is

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an integer. Thus we have to show that $\langle \iota_n - \iota_n', \alpha^\vee \rangle$ is an even integer. Suppose that $\alpha^\vee$ is long, say $\alpha^\vee = e_i - e_j$. Then $\kappa_n(e_i - e_j) = 1$, so that $\kappa_n(e_i) = \kappa_n(e_j)$. Consider the set R_α of roots of $(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ not from H, not perpendicular to α, and positive with respect to a system adapted to α. The short roots in R_α are of the form $e_i \pm e_k$, $-e_j \pm e_k$, where $k \neq i, j$; note that for given k either all roots $e_i \pm e_k$, $-e_j \pm e_k$ belong to R_α or none does. The subset of long roots in R_α is either $\{2e_i, -2e_j\}$ or the empty set. Clearly then

$$\langle \iota_n - \iota_{n'}{}'\alpha^\vee \rangle = \tfrac{1}{2} \sum_{\beta \in R_\alpha} \langle \beta, \alpha^\vee \rangle$$

is even. Suppose now that $\alpha^\vee$ is short, say $\alpha^\vee = e_i$. Then R_α, as defined above, contains only short roots; these roots are of the form $e_i \pm e_k$, $k \neq i$, and $e_i + e_k \in R_\alpha$ if and only if $e_i - e_k \in R_\alpha$. Also $e_i \pm e_k \in R_\alpha$ if and only if $\kappa_n(e_k) = -1$, since $\kappa_n(e_i) = 1$. Thus we have to show that, under our assumption that $T_N = T^*$, there are an even number of roots not from H among $2e_1, \ldots, 2e_l$. Relabel these roots so that $2e_1, \ldots, 2e_r$ are not from H and $2e_{r+1}, \ldots, 2e_l$ are from H; that is, $\kappa_n(e_1) = \ldots = \kappa_n(e_r) = -1$ and $\kappa_n(e_{r+1}) = \ldots = \kappa_n(e_l) = 1$. Suppose that r is odd. Clearly $e_1 \pm e_2, e_3 \pm e_4, e_{r-2} \pm e_{r-1}$ are from H. Since T_n is compact and $T_N = T^*$ is split we have that the automorphism $e_j \rightarrow -e_j$, $1 \leqq j \leqq l$, belongs to $\Omega(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ and is from H. On the other hand,

$$\omega = \omega_{e_1 - e_2}\omega_{e_1 + e_2} \ldots \omega_{e_{r-2} - e_{r-1}}\omega_{e_{r-2} + e_{r-1}}\omega_{2e_{r+1}} \ldots \omega_{2e_n}$$

maps e_j to $-e_j$ for $j \neq r$ and fixes e_r. We conclude that ω_{2e_r} is from H. Then

$$\kappa_n(\omega_{e_{2r}}) = \kappa_n(e_{2r}) = 1$$

since e_{2r}, being long, is noncompact (cf. [**10**, Propositions 2.1, 7.4]). This is a contradiction. Hence r is even and (6.1.4) is proved in the case that $T_N = T^*$.

Suppose that $T_N \neq T^*$. We claim that T_N has exactly one positive imaginary root and this root is long; that is, that $({}^L M_N{}^0, {}^L T^0)$ has exactly one positive root and that this root is short. Indeed, the roots of $({}^L M_N{}^0, {}^L T^0)$ form a subsystem of the roots of $({}^L G^0, {}^L T^0)$, of type $A_1 \times A_1 \times \ldots \times A_1$ (cf. proof of Proposition 5.1.6). Clearly such a subsystem contains at most one of the roots $e_1, \ldots, e_l$. We have then only to show that no long root is a root of $({}^L M_N{}^0, {}^L T^0)$. Suppose that $e_i + e_j$ is a root of $({}^L M_N{}^0, {}^L T^0)$. Then, after transfer to coroot of $(\underset{\sim}{G}^*, \underset{\sim}{T}_N)$, we have

$$\sigma_N(e_i + e_j) = -(e_i + e_j) \quad \text{and} \quad \kappa_N(e_i + e_j) = -1.$$

On the other hand, it is easily seen that $\sigma_N(e_i - e_j) = e_i - e_j$. Hence $\sigma_N e_i = -e_j$, so that

$$e_i + e_j = e_i - \sigma_N e_i.$$

This implies that $\kappa_N(e_i + e_j) = 1$, a contradiction. Similarly we obtain a contradiction if we assume that $e_i - e_j$ is a root of $({}^LM_N{}^0, {}^LT^0)$. Hence $({}^LM_N{}^0, {}^LT^0)$ has exactly one positive root and this root is short, as claimed.

To verify (6.1.4), we proceed as for the case $T_N = T^*$. If $\alpha^\vee$ is long then the proof of Lemma 5.2.6 shows that $\langle \lambda^*, s\alpha^\vee \rangle$ is an integer. That $\langle \iota_n - \iota_n', \alpha^\vee \rangle$ is an even integer follows from arguments already given. If $\alpha^\vee$ is short then on transfer to ${}^LT^0$ we have that $\sigma_H(s\alpha^\vee) = s\alpha^\vee$ and that $(s\alpha^\vee) \pm \beta^\vee$ are roots, where $\beta^\vee$ is the positive root of $({}^LM_N{}^0, {}^LT^0)$. The proof of Lemma 5.2.6 then shows that $\langle \lambda^*, s\alpha \rangle \equiv 1/2 \bmod \mathbf{Z}$. Thus we have to show that $\langle \iota_n - \iota_n', \alpha^\vee \rangle$ is an odd integer. Arguing as for the case $T_N = T^*$, we find it sufficient to show that there are an odd number of the roots $2e_1, \ldots, 2e_l$ of $(\underset{\sim}{G}^*, \underset{\sim}{T}_n)$ not from $\underset{\sim}{H}$. Suppose that there are an even number not from $\underset{\sim}{H}$. Then our earlier argument shows that the automorphism $e_i \to -e_i$, $1 \leqq i \leqq l$, belongs to $\Omega({}^LH^0, {}^LT^0)$, and hence that $T_N = T^*$. This completes the proof of Lemma 6.1.3.

(6.2) *Correction Characters.*

Definition 6.2.1. A set of correction characters for H is the set of quasi-characters $\chi(\mu^* + \iota_n - \iota_n', \lambda^*)$ attached, in the manner of (4.2), to a pair $(\mu^*, \underline{\underline{\lambda}}^*)$, $\mu^* \in L \otimes \mathbf{C}$, $\underline{\underline{\lambda}}^* \in L \otimes \mathbf{C}/L + \{\nu - \sigma_H\nu : \nu \in L \otimes \mathbf{C}\}$ satisfying

(i) $\mu^* - \sigma_H\mu^* \in L$, $\langle \mu^*, \alpha^\vee \rangle = 0$ if $\alpha^\vee$ is a root of ${}^LH^0$,

(ii) $\frac{1}{2}(\mu^* - \sigma_n\mu^*) + \iota_n - \iota_n' \equiv (\lambda^* + \sigma_n\lambda^*) \bmod L$,
$n = 0, \ldots, N$, $\lambda^* \in \underline{\underline{\lambda}}^*$,

(iii) $\omega\lambda^* \equiv \lambda^* \bmod (L + \{\nu - \sigma_H\nu : \nu \in L \otimes \mathbf{C}\})$,
$\lambda^* \in \underline{\underline{\lambda}}^*$, $\omega \in \Omega_0(\underset{\sim}{G}^*, \underset{\sim}{T}_N)$ and from H,

(iv) $\langle \lambda^*, s\alpha^\vee \rangle \equiv \frac{1}{2}\langle \iota_n - \iota_n', \alpha^\vee \rangle \bmod \mathbf{Z}$

for each imaginary root α of T_n from H, $I_n{}^+$ adapted to α, and $\lambda^* \in \underline{\underline{\lambda}}^*$.

Here ι_n, ι_n' are as in (4.2); ι_n, ι_n', λ^* move between $L \otimes \mathbf{C}$ and $L_n \otimes \mathbf{C}$ (via m_n) without change in notation. Clearly correction characters are of $(\iota_n - \iota_n')$-type and compatible in the sense of Theorem 6.1.1.

A set of correction characters allows us to transfer orbital integrals from G^* to H, in the sense of the introduction to this paper; that is, if in §§8–10 of [**10**] we replace G by G^*, "Schwartz function" by a slightly more general notion, and $\iota_n - \iota_n'$ by $\chi(\mu^* + \iota_n - \iota_n', \lambda^*)$, omitting the assumptions on $\iota_n - \iota_n'$, then Theorem 10.2 of [**10**] remains true. Note that "$(\iota_n - \iota_n')$-type" is used in Theorem 8.3 and "compatibility" in Proposition 9.4 of that paper. This will be discussed further elsewhere.

We indicate briefly why correction characters are the only replacements for $\iota_n - \iota_n'$. Thus, suppose that $\{\chi(\mu_n + \iota_n - \iota_n', \lambda_n)\}$ may replace

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$\{\iota_n - \iota_n'\}$ in Theorem 10.2 of [**10**]. Then it is easily checked that these quasicharacters must be of $(\iota_n - \iota_n')$-type and compatible. Transfer μ_n, λ_n to L via m_n, $n = 0, \ldots, N$. We claim that for some n, $\langle \mu_n, \alpha^\vee \rangle = 0$ for all roots $\alpha^\vee$ of ${}^LH^0$. Indeed, by "$(\iota_n - \iota_n')$-type", we know that for each n, $\omega\mu_n = \mu_n$ for all $\omega \in \Omega(\underset{\sim}{H}, \underset{\sim}{T}^*)$ commuting with σ_n. To prove the claim we may assume $\underset{\sim}{H}$ simple. Then unless $\underset{\sim}{H}$ is of type D_{2l} or is obtained by restriction of scalars from $\mathbf{C}$, there is some n such that every element of $\Omega(\underset{\sim}{H}, \underset{\sim}{T}^*)$ commutes with σ_n (by inspection), and the claim is proved. For the remaining groups a simple computation on the fundamental Cartan subgroup gives the result; we omit the details. We argue now that

$$\mu_p = \mu_n \quad \text{for all } p = 0, 1, \ldots, N \quad \text{and}$$
$$\lambda_p \equiv \lambda_N \bmod (L + \{\nu - \sigma_p\nu : \nu \in L \otimes \mathbf{C}\}),$$

by compatibility (cf. (6.1)). Thus writing $\mu^\dagger$ for μ_N, $\lambda^\dagger$ for λ_N we have that $\{\chi(\mu_n + \iota_n - \iota_n', \lambda_n)\}$ is just the set of correction characters $\{\chi(\mu^\dagger + \iota_n - \iota_n', \lambda^\dagger)\}$.

We have considered correction characters for G^*, rather than for G, the group with which we started and whose orbital integrals we wish to transfer to H. To move to G we use the embeddings $\psi_n : T_n{}^G \to T_n$, prescribed for those $T_n{}^G$ originating in H, to define the relevant notions ("element of $\Omega_0(\underset{\sim}{G}, \underset{\sim}{T}_n{}^G)$ from H", "Cayley transform from $\underset{\sim}{H}$", etc. (cf. [**10**])). We then conclude that Theorem 10.2 of [**10**] remains true for G when we omit the assumptions on $\iota_n - \iota_n'$ and replace $\iota_n - \iota_n'$ by a correction character $\chi(\mu^* + \iota_n - \iota_n', \lambda^*)$, especially $\chi^{(n)}_{(\xi, I_n^+)}$ transferred to $T_n{}^G$.

7. Φ-equivalence. We recall the set $\Phi(G)$ of [**7**]. A homomorphism $\varphi : W \to {}^LG$ is *admissible* if $\varphi(w)$ is of the form $\varphi_0(w) \times w$, $w \in W$, where $\varphi_0(w)$ is a semisimple element of ${}^LG^0$, and the image of φ is contained only in parabolic subgroups of ${}^LG$ which are relevant to G ([**3**]). We will consider G^* in place of G; all parabolic subgroups of ${}^LG$ are relevant to G^*. Two homomorphisms φ, φ' are *equivalent* if there is $g \in {}^LG^0$ such that $\varphi' = \operatorname{ad} g \circ \varphi$. The set $\Phi(G)$ consists of the equivalence classes of admissible homomorphisms of $\varphi : W \to {}^LG$.

Clearly an admissible embedding $\xi : {}^LH \hookrightarrow {}^LG$ induces a mapping $\xi^\Phi : \Phi(H) \to \Phi(G^*)$.

Definition 7.0.1. Two admissible embeddings $\xi, \xi' : {}^LH \to {}^LG$ are Φ-*equivalent* if and only if $\xi^\Phi = (\xi')^\Phi$.

We denote this equivalence by $\overset{\Phi}{\sim}$.

THEOREM 7.0.2. $\xi \overset{\Phi}{\sim} \xi'$ *if and only if*

$$\chi^{(n)}_{(\xi, I_n^+)} = \chi^{(n)}_{(\xi', I_n^+)} \textit{ for all } n,\ I_n^+.$$

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Proof. Let $\xi = \xi(\mu^*, \lambda^*)$, $\xi' = \xi'((\mu^*)', (\lambda^*)')$. Then

$$\chi^{(n)}_{(\xi, I_n{}^+)} = \chi^{(n)}_{(\xi', I_n{}^+)} \text{ for all } n,\ I_n^+$$

if and only if

(7.0.3) $(\mu^*)' = \mu^*$ and
$(\lambda^*)' \equiv \lambda^* \bmod (L + \{\nu - \sigma_H \nu : \nu \in L \otimes \mathbf{C}\})$.

As in the proof of Theorem 3.4.1, we will find the L-groups of Cartan subgroups useful, although we could easily argue with congruences alone.

For each $n = 0, \ldots, N$ fix an allowed embedding

$$\tau_n = \tau_n(\mu_n, \lambda_n) : {}^L(T_n') \hookrightarrow {}^L(M_n')$$

as in (1.3). Recall that we have identified ${}^L(M_n')$ as a subgroup of ${}^LH$. Thus τ_n induces a map

$$\Phi(T_n') \xrightarrow{\tau_n^\Phi} \Phi(H).$$

A class in the image has a representative $\varphi : W \to {}^LH$ satisfying

(7.0.4) $\varphi_0(\mathbf{C}^\times) \subset {}^LT^0$

and

(7.0.5) $\varphi(1 \times \sigma)$ normalizes ${}^LT^0$ and acts on ${}^LT^0$ as σ_n.

Conversely, any class with such a representative factors through τ_n^Φ. This is easily seen as follows. Given such a φ, write φ as $\varphi(M_n', \zeta, \eta)$ where

(7.0.6) $\lambda^\vee(\varphi_0(z)) = z^{\langle \zeta, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_n \zeta, \lambda^\vee \rangle} \quad z \in \mathbf{C}^\times, \lambda^\vee \in L^\vee$,

(7.0.7) $\lambda^\vee(\varphi_0(1 \times \sigma)) = e^{2\pi i \langle \eta, \lambda^\vee \rangle}$ for all rational characters $\lambda^\vee$ on ${}^LT^0$ which extend to ${}^L(M_n')^0$.

Then, because φ defines an allowed embedding of ${}^L(T_n')$ in ${}^L(M_n')$ (see (1.3)) we have $\zeta - \sigma_n \zeta \in L$ and

(7.0.8) $\frac{1}{2}(\zeta - \sigma_n \zeta) + \iota_n' \equiv (\eta + \sigma_n \eta) \bmod L$,

where ι_n' denotes half the sum of the roots of $\underset{\sim}{T}_H$ in $\underset{\sim}{M}_n' \cap \underset{\sim}{B}_H$. Thus the class of φ in $\Phi(H)$ is the image of the class of $\varphi(T_n', \zeta - \mu_n, \eta - \lambda_n)$ in $\Phi(T_n')$. We remark that $\Phi(T_n')$ consists exactly of the classes of $\varphi(T_n', \zeta, \eta)$ (defined by (7.0.6) and (7.0.7) with T_n' replacing M_n'), where $\zeta - \sigma_n \zeta \in L$ and

$$\tfrac{1}{2}(\zeta - \sigma_n \zeta) \equiv (\eta + \sigma_n \eta) \bmod L.$$

PROPOSITION 7.0.9.

$$\Phi(H) = \bigcup_{n=0}^{N} \tau_n(\Phi(T_n')).$$

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Proof. In [**7**] it is shown that every class in $\Phi(H)$ has a representative φ such that $\varphi_0(\mathbf{C}^\times) \subseteq {}^LT^0$ and $\varphi(1 \times \sigma)$ normalizes ${}^LT^0$. Because H is quasi-split, a little further argument using [**13**, Theorem 1.7], shows that $\varphi(1 \times \sigma)$ acts as σ_T, where $\underset{\sim}{T}$ is some maximal torus over $\mathbf{R}$ in $\underset{\sim}{H}$ and σ_T denotes the Galois action of $\underset{\sim}{T}$ transferred to L via some p-d. Replacing φ by an equivalent homomorphism if necessary, we can assume that $\varphi(1 \times \sigma)$ acts as σ_n, as well as that $\varphi_0(\mathbf{C}^\times)$ is contained in ${}^LT^0$. This proves the proposition.

We move now to $\Phi(G^*)$. The image of $\Phi(H)$ under ξ^Φ, or $(\xi')^\Phi$, consists of all those classes in $\Phi(G^*)$ with a representative φ satisfying (7.0.4) and (7.0.5), for some n. To check this, we write such a homomorphism φ as $\varphi(M_n, \zeta, \eta)$ where ζ, η are defined as in (7.0.6) and (7.0.7) (with ${}^L(M_n')^0$ replaced by ${}^LM_n{}^0$). In place of (7.0.8) we now have

(7.0.10) $\frac{1}{2}(\zeta - \sigma_n\zeta) + \iota_n \equiv (\eta + \sigma_n\eta) \bmod L,$

where ι_n is half the sum of the roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{M}_n \cap \underset{\sim}{B}^*$, and the class of $\varphi(M_n, \zeta, \eta)$ is the image under ξ^Φ of the class of $\varphi(M_n', \zeta - \mu^*, \eta - \lambda^*)$.

It follows that if (7.0.3) holds then $\xi \overset{\Phi}{\sim} \xi'$ for, clearly, $\varphi(M_n, \zeta, \eta)$ is equivalent to $\varphi(M_n, \zeta', \eta')$ if

$$\zeta = \zeta' \quad \text{and} \quad \eta \equiv \eta' \bmod(L + \{\nu - \sigma_n\nu : \nu \in L \otimes \mathbf{C}\}).$$

Conversely, suppose that $\xi(\mu^*, \lambda^*) \overset{\Phi}{\sim} \xi'((\mu^*)', (\lambda^*)')$. Then from

$$\Phi(T_n') \xrightarrow{\tau_n^\Phi} \Phi(H) \xrightarrow{\xi^\Phi,\ (\xi')^\Phi} \Phi(G^*)$$

we obtain that

$$\varphi = \varphi(M_n, \underline{\underline{\zeta}} + \mu_n + \mu^*, \underline{\underline{\eta}} + \lambda_n + \lambda^*)$$

is equivalent to

$$\varphi' = \varphi(M_n, \underline{\underline{\zeta}} + \mu_n + (\mu^*)', \underline{\underline{\eta}} + \lambda_n + (\lambda^*)')$$

for all $\underline{\underline{\zeta}}, \underline{\underline{\eta}} \in L \otimes \mathbf{C}$ satisfying

$$\underline{\underline{\zeta}} - \sigma_n\underline{\underline{\zeta}} \in L, \quad \tfrac{1}{2}(\underline{\underline{\zeta}} - \sigma_n\underline{\underline{\zeta}}) \equiv (\underline{\underline{\eta}} + \sigma_n\underline{\underline{\eta}}) \bmod L.$$

For convenience, we may take $\mu_n = \iota_n', \lambda_n = 0$. Because $\underline{\underline{\zeta}} - \sigma_n\underline{\underline{\zeta}}$, $\mu^* - \sigma_n\mu^*$, $(\mu^*)' - \sigma_n(\mu^*)'$ all belong to L we may choose $\underline{\underline{\zeta}}$ so that

$$\langle \underline{\underline{\zeta}} + \mu_n + \mu^*, \alpha^\vee \rangle > 0 \quad \text{and} \quad \langle \underline{\underline{\zeta}} + \mu_n + (\mu^*)', \alpha^\vee \rangle > 0$$

for all roots $\alpha^\vee$ of ${}^LM_n{}^0 \cap {}^LB^0$ (cf. [**7**]). Then, if necessary, adding a σ_n-invariant element of $L \otimes \mathbf{C}$ to the chosen $\underline{\underline{\zeta}}$, we may assume that

$$\langle \underline{\underline{\zeta}} + \mu_n + \mu^*, \alpha^\vee \rangle > 0 \quad \text{and} \quad \langle \underline{\underline{\zeta}} + \mu_n + (\mu^*)', \alpha^\vee \rangle > 0$$

for all roots of ${}^LB^0$. This implies, in particular, that $\varphi_0(\mathbf{C}^\times)$ contains a $({}^LG^0-)$ regular element. Hence φ' must be of the form ad $g \circ \varphi$, where ad g normalizes ${}^LT^0$. By definition, the action of ad g on ${}^LT^0$ commutes with σ_n. Hence

$$\zeta + (\mu^*)' + \iota_n = \omega(\zeta + \mu^* + \mu_n)$$

and

$$\eta + (\lambda^*)' \equiv \omega(\eta + \lambda^*) \bmod (L + \{\nu - \sigma_n \nu : \nu \in L \otimes \mathbf{C}\}),$$

for some $\omega \in \Omega({}^L G^0, {}^L T^0)$ commuting with σ_n. Note that for the congruence an argument as in Proposition 3.3.2 is needed (cf. [**7**]). Our choice of ζ forces ω to be trivial and hence (7.0.3) is proved.

8. Main theorem. Suppose that $\{\chi(\mu^* + \iota_n - \iota_n', \lambda^*)\}$ is a set of correction characters for H. In this section we will show that there is an admissible embedding $\xi : {}^L H \hookrightarrow {}^L G$ such that $\xi = \xi(\mu^*, (\lambda^*)')$ where

$$(\lambda^*)' \equiv \lambda^* \bmod (L + \{\nu - \sigma_H \nu : \nu \in L \otimes \mathbf{C}\}).$$

Thus, by Theorem 7.0.2, we have:

THEOREM 8.0.1. *There is a one to one correspondence between Φ-equivalence classes of admissible embeddings of ${}^L H$ in ${}^L G$ and sets of correction characters for H.*

To begin the construction, choose $m \in {}^L M_N{}^0$ such that $m \times (1 \times \sigma)$ acts on ${}^L H^0$ as σ_H; this is possible because ${}^L H$ is in standard position. Suppose that

$$\lambda^\vee(m) = e^{2\pi i \langle \lambda_0^*, \lambda^\vee \rangle}$$

for all $\lambda^\vee \in L^\vee$ extending to a rational character on ${}^L M_N{}^0$. We claim that it is enough to show that there is $(\lambda^*)' \in \lambda^* + L + \{\nu - \sigma_H \nu\}$ such that

$$(8.0.2) \quad \langle \lambda_0^*, \alpha^\vee \rangle \equiv \langle (\lambda^*)', \alpha^\vee \rangle \bmod \mathbf{Z}$$

for all roots $\alpha^\vee$ of ${}^L H^0$. For, suppose that this has been shown. Choose $t \in {}^L T^0$ such that $\lambda^\vee(t) = e^{2\pi i \langle (\lambda^*)' - \lambda_0^*, \lambda^\vee \rangle}$ $\lambda^\vee \in L^\vee$. Then t lies in the center of ${}^L H^0$. Thus we may replace m by $n = tm$ without changing the action on ${}^L H^0$; $\lambda^\vee(n) = e^{2\pi i \langle (\lambda^*)', \lambda^\vee \rangle}$, for all $\lambda^\vee \in L^\vee$ extending to ${}^L M_N{}^0$. To show that $\xi(z \times 1) = t_z \times (z \times 1)$,

$$\lambda^\vee(t_z) = z^{\langle \mu^*, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_H \mu^*, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee, z \in \mathbf{C}^\times \quad \text{and}$$

$$\xi(1 \times \sigma) = n \times (1 \times \sigma)$$

defines an admissible embedding of ${}^L H$ in ${}^L G$ we just have to check that $n \sigma_G(n) = t_{-1}$. This is immediate because, at least, ξ defines an embedding ${}^L T_N \hookrightarrow {}^L M_N$ via the congruence

$$\tfrac{1}{2}(\mu^* - \sigma_N \mu^*) + \iota_N \equiv ((\lambda^*)' + \sigma_N (\lambda^*)') \bmod L$$

provided by our correction characters.

We now show (8.0.2). First we will find $(\lambda^*)'$ so that (8.0.2) holds for

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all ${}^LH^0$-simple roots $\alpha^\vee$ satisfying $\sigma_H\alpha^\vee \neq \alpha^\vee$. Afterwards we will show that (8.0.2) is then true for all simple, and hence all, roots $\alpha^\vee$ of ${}^LH^0$.

From $\langle \mu^*, \alpha^\vee \rangle = 0$ and

$$\tfrac{1}{2}(\mu^* - \sigma_N\mu^*) + \iota_N \equiv (\lambda^* + \sigma_N\lambda^*)(\text{mod } L)$$

we have

$$\langle \lambda^* + \sigma_H\lambda^*, \alpha^\vee \rangle \equiv \langle \iota_N, \alpha^\vee \rangle \text{ mod } \mathbf{Z}$$

for all roots $\alpha^\vee$ of ${}^LH^0$. On the other hand;

PROPOSITION 8.0.3. $\langle \lambda_0^* + \sigma_H\lambda_0^*, \alpha^\vee \rangle \equiv \langle \iota_N, \alpha^\vee \rangle \text{mod } \mathbf{Z}$ *for all roots* $\alpha^\vee$ *of* ${}^LH^0$.

Proof. We have that $m \times (1 \times \sigma)$ acts on ${}^LH^0$ as σ_H and

$$\lambda^\vee(m) = e^{2\pi i \langle \lambda_0^*, \lambda^\vee \rangle}$$

for all $\lambda^\vee \in L^\vee$ extending to ${}^LM_N{}^0$. Let $m = tm_1$, where t lies in the connected center of ${}^LM_N{}^0$ and m_1 in $\mathscr{M}^0 = ({}^LM_N{}^0)_{\text{der}}$. Since $(m \times (1 \times \sigma))^2$ centralizes ${}^LH^0$ we have that

$$t\sigma_H(t)m_1\sigma_G(m_1)$$

lies in the center of ${}^LH^0$, and so

$$\alpha^\vee(t\sigma_H(t))\alpha^\vee(m_1\sigma_G(m_1)) = e^{2\pi i \langle \lambda_0^* + \sigma_H\lambda_0^*, \alpha^\vee \rangle}\alpha^\vee(m_1\sigma_G(m_1)) = 1$$

for each root $\alpha^\vee$ of ${}^LH^0$, since $\alpha^\vee + \sigma_H\alpha^\vee$ extends to ${}^LM_N{}^0$. We have thus to show that

$$(8.0.4) \quad (m_1 \times (1 \times \sigma))^2 Y_{\alpha^\vee} = e^{2\pi i \langle \iota_N, \alpha^\vee \rangle} Y_{\alpha^\vee}$$

for each simple root $\alpha^\vee$ of ${}^LH^0$. If we replace $m_1 \times (1 \times \sigma)$ by any element of ${}^L\mathscr{M} (= {}^L\mathscr{M}^0 \rtimes W$ with the inherited action of W) which normalizes ${}^LT^0 \cap {}^L\mathscr{M}^0$ and acts on the torus as σ_N then $(m_1 \times (1 \times \sigma))^2$ does not change. Recall that ${}^L\mathscr{M}^0$ is of type $A_1 \times \ldots \times A_1$; if $\alpha_1^\vee, \ldots, \alpha_d^\vee$ are the positive roots of ${}^L\mathscr{M}^0$ in ${}^LT^0 \cap {}^L\mathscr{M}^0$ then we may take for m_1 any element of ${}^L\mathscr{M}^0$ which realizes $\omega_{\alpha_1^\vee} \ldots \omega_{\alpha_d^\vee}$; recall that σ_G acts trivially on ${}^L\mathscr{M}^0$. Thus

$$m_1\sigma_G(m_1) = \exp i\pi(\alpha_1 + \ldots + \alpha_d) = \exp 2\pi i\, \iota_N.$$

Here we have identified the Lie algebra of ${}^LT^0$ with $L \otimes \mathbf{C}$. Hence (8.0.4) is true, and the proposition proved.

From the proposition we conclude that

$$\langle \lambda^* - \lambda_0^*, \sigma_H\alpha^\vee \rangle \equiv -\langle \lambda^* - \lambda_0^*, \alpha^\vee \rangle \text{ mod } \mathbf{Z}$$

for all roots $\alpha^\vee$ of ${}^LH^0$. An elementary argument then shows that we may

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add to λ^* an element of $\{\mu - \sigma_H \nu : \nu \in L \otimes \mathbf{C}\}$ to obtain $(\lambda^*)'$ such that

$$\langle (\lambda^*)', \alpha^\vee \rangle \equiv \langle \lambda_0^*, \alpha^\vee \rangle \bmod \mathbf{Z}$$

for all simple roots $\alpha^\vee$ satisfying $\sigma_H \alpha^\vee \neq \alpha^\vee$.

Suppose now that $\alpha^\vee$ is simple in ${}^L H^0$ and that $\sigma_H \alpha^\vee = \alpha^\vee$. Let α' be the coroot of $\alpha^\vee$ in $\underset{\sim}{H}$; α' is a root of $\underset{\sim}{T}_N'$. Let α denote the image of α' in the roots of $\underset{\sim}{T}_N$. We can find $\underset{\sim}{T}_n$ such that $\underset{\sim}{S}_n$ is of codimension 1 in $\underset{\sim}{S}_N$ and there exists a Cayley transform $s : \underset{\sim}{T}_n \to \underset{\sim}{T}_N$ mapping some root β coming from $\underset{\sim}{M}_n'$ to α. Note that $\underset{\sim}{M}_n'$ is a quasi-split group of **R**-split rank one. The simply-connected covering of the derived group of $\underset{\sim}{M}_n'$ is therefore SL_2 or $SU(2, 1)$. The group $SU(2, 1)$ is excluded because $\alpha^\vee$ is simple. Thus $\langle \iota_n', \beta^\vee \rangle = 1$ for any I_n^+ adapted to β. Property (iv) of correction characters then implies that

$$\langle (\lambda^*)', \alpha^\vee \rangle \equiv \tfrac{1}{2}(\langle \iota_n, \beta^\vee \rangle - 1) \bmod \mathbf{Z}$$

for any I_n^+ adapted to β.

On the other hand, we may use a lemma of Langlands reported in [**1**] as Lemma 2.3 to compute $\langle \lambda_0^*, \alpha^\vee \rangle$. Indeed, $\langle \iota_n, \beta^\vee \rangle$ is easily seen to be the term "$\langle \rho_p, \alpha_0^\vee \rangle$" if we substitute $\alpha^\vee$ for "$\alpha_0^\vee$" and so the lemma says that

$$\langle \lambda_0^*, \alpha^\vee \rangle \equiv \tfrac{1}{2}(\langle \iota_n, \beta^\vee \rangle - 1) \bmod \mathbf{Z}.$$

Therefore (8.0.2) is proved, and our construction completed.

9. The number of embeddings of ${}^L H$ in ${}^L G$.

(9.1) *Uniqueness.* Suppose that $\xi, \xi' : {}^L H \hookrightarrow {}^L G$ are admissible embeddings. Then, clearly $\xi'(1 \times w) = x(w)\xi(1 \times w)$, $w \in W$, where $x(\ )$ is a continuous 1-cocycle of W in $Z({}^L H^0)$, the center of ${}^L H^0$. We write $\xi' = x\xi$. Define $\mu_0, \lambda_0 \in L \otimes \mathbf{C}$ by

$$\lambda^\vee(x(z)) = z^{\langle \mu_0, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_H \mu_0, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee, z \in \mathbf{C}^\times,$$
$$\lambda^\vee(x(1 \times \sigma)) = e^{2\pi i \langle \lambda_0, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee,$$

and write $x = x(\mu_0, \lambda_0)$. Then $\mu_0 - \sigma_H \mu_0 \in L$ and

$$\tfrac{1}{2}(\mu_0 - \sigma_H \mu_0) \equiv (\lambda_0 + \sigma_H \lambda_0) \bmod L,$$

so that we obtain a quasicharacter $\chi(\mu_0, \lambda_0)$ on $T_H = T_N'$ (cf. (4.1)). The following is just a restatement of some material in § 2 of [**7**], for real groups.

PROPOSITION 9.1.1. (i) *The correspondence* $(x(\mu_0, \lambda_0), \chi(\mu_0, \lambda_0))$ *induces a one to one correspondence between* $H^1(W, Z({}^L H^0))$ *and the set of quasi-characters* $\chi(\mu, \lambda)$ *on* T_H *such that*:

(ia) $\langle \mu, \alpha^\vee \rangle = 0$ *for all roots* $\alpha^\vee$ *of* ${}^L H^0$, *and*

(ib) $\langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}$ *for all simple roots* $\alpha^\vee$ *of* ${}^L H^0$ *fixed by* σ_H.

(ii) *Each quasicharacter on* T_H *as in* (i) *extends uniquely to a quasi-*

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character on H and conversely the restriction to T_H of a quasicharacter on H is as in (i).

We denote the one to one correspondence between $H^1(W, Z(^LH^0))$ and quasicharacters on H, thus established, by $x \to \chi_x$.

Proof. (i) The only part that requires an argument is recovering $x(\mu_0, \lambda_0')$ from $\chi(\mu_0, \lambda_0)$ for some

$$\lambda_0' \equiv \lambda_0 \bmod (L + \{\nu - \sigma_H\nu : \nu \in L \otimes \mathbf{C}\}).$$

For this we just have to choose λ_0' so that $\langle \lambda_0', \alpha^\vee \rangle \in \mathbf{Z}$ for all roots $\alpha^\vee$ of $^LH^0$; since, clearly, $\langle \lambda_0 + \sigma_H\lambda_0, \alpha^\vee \rangle \in \mathbf{Z}$ for all roots $\alpha^\vee$ of $^LH^0$, this is possible (see § 8).

(ii) We can use the fact that T_H meets every component of H (cf. [**6**]) to obtain $H = T_H(H, H)$, so that

$$H/(H, H) = T_H/T_H \cap (H, H),$$

where (H, H) denotes the derived group of H. The only problem is to check that every quasicharacter $\chi(\mu_0, \lambda_0)$ as in (i) is trivial on $T_H \cap (H, H)$. We can avoid this by quoting the argument of [**7**]. Take a $\chi(\mu_0, \lambda_0)$ and attach a 1-cocycle as in (i). Then in [**7**] there is constructed a quasicharacter on H whose restriction to T_H is $\chi(\mu_0, \lambda_0)$, and it is noted why this restriction determines the quasicharacter. Thus the proposition is proved.

Let $\xi = \xi(\mu^*, \lambda^*)$ and $x = x(\mu_0, \lambda_0)$. Then

$$x\xi = x\xi(\mu^* + \mu_0, \lambda^* + \lambda_0).$$

We conclude:

PROPOSITION 9.1.2. *$H^1(W, Z(^LH^0))$ acts simply transitively on the set of Φ-equivalence classes of embeddings of ^LH in ^LG; the action of $x \in H^1(W, Z(^LH^0))$ corresponds to multiplying a set of correction characters by the quasicharacter χ_x on H (that is, for each n, we multiply the correction character by the transfer to T_n of the restriction to T_n' of χ_x).*

(9.2) *Existence.* (examples and counterexamples). From now on, we do not require ^LH to be in standard position. Thus ^LH is any one of the isomorphic L-groups attached to given pair (T, κ) as in (2.1). For $w \in W$, pick $g(w) \in {}^LG^0$ such that $g(w) \times w$ acts on $^LH^0$ as $1 \times w \in {}^LH$, and define $x(\ ,\ )$ by

$$g(w_1)g(w_2) = x(w_1, w_2)g(w_1w_2), \quad w_i \in W.$$

Then $x(\ ,\ )$ is a continuous 2-cocycle of W in $Z(^LH^0)$. [**8**] shows that if $Z(^LG^0)$ is connected then this cocycle splits so that there is an embedding of ^LH in ^LG; if $Z(^LG^0)$ is not connected then an example in $E_7 \times A_n$ shows that the cocycle need not split.

For the rest of this section we assume that G^* contains a Cartan subgroup T_0 compact modulo the center of G^*, and that $H = H(T_0, \kappa_0)$, for some κ_0 attached to T_0.

First we attach a pair $(\mu^\dagger, \lambda^\dagger)$ to an admissible embedding $\xi : {}^LH \hookrightarrow {}^LG$ (as always, extending the inclusion of ${}^LH^0$ in ${}^LG^0$). Recall the properties of ξ listed in (3.1). We set

$$\lambda^\vee(\xi_0(z \times 1)) = z^{\langle \mu^\dagger, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_H \mu^\dagger, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee, z \in \mathbf{C}^\times,$$
$$\lambda^\vee(\xi_0(1 \times \sigma)) = e^{2\pi i \langle \lambda^\dagger, \lambda^\vee \rangle}$$

for all rational characters $\lambda^\vee$ on ${}^LT^0$ which extend to ${}^LG^0$, where $\xi_0(w)$ is defined by $\xi(w) = \xi_0(w) \times w$, $w \in W$. From what we have already done, it follows easily that $\mu^\dagger - \sigma_H \mu^\dagger \in L$, $\langle \mu^\dagger, \alpha^\vee \rangle = 0$ for all roots $\alpha^\vee$ of ${}^LH^0$, and

$$\tfrac{1}{2}(\mu^\dagger - \sigma_0 \mu^\dagger) + \iota_0 - \iota_0' \equiv (\lambda^\dagger + \sigma_0 \lambda^\dagger) \bmod L.$$

Conversely, given such a pair $(\mu^\dagger, \lambda^\dagger)$ we construct an admissible embedding of ${}^LH$ in ${}^LG$, as follows. It is clear how to define $\xi_0(\mathbf{C}^\times)$. Then pick $n_H \in {}^LH^0$, $n_G \in {}^LG^0$ such that $n_H \times (1 \times \sigma) \in {}^LH$ acts on ${}^LT^0$ as σ_0, $n_G \times (1 \times \sigma) \in {}^LG$ acts on ${}^LT^0$ as σ_0, and $n_H^{-1} n_G \times (1 \times \sigma) \in {}^LG$ acts on ${}^LH^0$ as σ_H. Then if $n = n_H^{-1} n_G$ we have

$$n \sigma_G(n) = (n_H \sigma_H(n_H))^{-1} n_G \sigma_G(n_G).$$

By adjusting our choice of n_H, n_G we can arrange that $n \sigma_G(n) = \xi_0(-1)$ (cf. Proposition 1.3.5, or Lemma 3.2 of [**7**]), and then $\xi_0(1 \times \sigma) = n$ completes the definition of ξ.

Note that while the datum $(\mu^\dagger, \lambda^\dagger)$ determines the existence of an embedding of ${}^LH$ in ${}^LG$, it is not adequate for attaching correction characters, that is, for determining the Φ-equivalence class of an embedding. This is illustrated very simply by the following:

Example 9.2.1. Let $\underset{\sim}{G}^* = PGL_2 = \underset{\sim}{H}$. There are two ($\Phi$-inequivalent) admissible embeddings of ${}^LG = SL_2(\mathbf{C}) \times W$ in itself, which extend the identity map on ${}^LG^0$. One is the identity and the other is ξ, defined by

$$\xi(g \times w) = g \begin{bmatrix} \epsilon(w) & 0 \\ 0 & \epsilon(w) \end{bmatrix} \times w, g \in SL_2(\mathbf{C}), w \in W,$$

where ϵ is the nontrivial character on $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ lifted to W. For either embedding, $\mu^\dagger = 0$ and $\lambda^\dagger$ is an arbitrary element of $L \otimes \mathbf{C}$.

Since any ${}^LG$ is in standard position with respect to itself and $\underset{\sim}{T}^*$, our earlier datum (μ^*, λ^*) is well-defined. We obtain:

(i) $\mu^* = 0$ and λ^* an element of L, pointing to the trivial character on H, in the case of the identity embedding, and

(ii) $\mu^* = 0$ and λ^* an element of $\frac{1}{2}L$ not in L, pointing to the nontrivial character (sgn det, appropriately defined) on H, in the case of ξ.

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We denote by $(\underset{\sim}{G}^*)_{\text{der}}$ the derived group of $\underset{\sim}{G}^*$ and by L_{der} the group of rational characters on $\underset{\sim}{T}^* \cap (\underset{\sim}{G}^*)_{\text{der}}$. Any element of L_{der} extends to a rational character on $\underset{\sim}{T}^*$ (cf. [**2**]); that is, the natural map $L \to L_{\text{der}}$ is surjective. There is a natural inclusion of $(L_{\text{der}})^\vee$ in $L^\vee$ and so we may regard κ_0 as attached to $T_0 \cap (\underset{\sim}{G}^*)_{\text{der}}$. We write $H_{(\text{der})}$ for the attached group. The following is immediate.

PROPOSITION 9.2.2. (i) *Suppose that* $\lambda \in L_{\text{der}}$ *satisfies* $\langle \lambda, \alpha^\vee \rangle = \langle \iota_0 - \iota_0', \alpha^\vee \rangle$ *for each root* $\alpha^\vee$ *of* ${}^LH^0$. *Then for any* $\lambda' \in L$ *extending* λ *we have that* $\mu^\dagger = \lambda' - (\iota_0 - \iota_0')$, $\lambda^\dagger = -\frac{1}{2}\lambda'$ *defines an embedding of* ${}^LH$ *in* ${}^LG$.

(ii) *If* G^* *is semisimple then* ${}^LH$ *embeds (admissibly) in* ${}^LG$ *if and only if there exists* $\lambda \in L$ *such that*

(9.2.3) $\quad \langle \lambda, \alpha^\vee \rangle = \langle \iota_0 - \iota_0', \alpha^\vee \rangle$ *for all roots* $\alpha^\vee$ *of* ${}^LH^0$.

(iii) *If* ${}^LH_{(\text{der})}$ *embeds in* ${}^L(G_{\text{der}}{}^*)$ *then* ${}^LH$ *embeds in* ${}^LG$.

Because of (iii), we will assume that G^* is semisimple.

Note that the choice of positive system (for all roots of ${}^LG^0$, respectively, all roots of ${}^LH^0$) in the definition of ι_0, ι_0' is of no consequence to (9.2.2). Thus we will use the "diagram of (T_0, κ_0)" from [**8**] to make convenient choices. We may assume ${}^LG^0$ simple (cf. (5.1.7)). Fix some simple system $\alpha_1{}^\vee, \ldots, \alpha_r{}^\vee$ for the roots of ${}^LH^0$. Consider also the roots $\beta_1{}^\vee, \ldots, \beta_s{}^\vee$, minimal for the ordering $\leqq$ on the roots outside ${}^LH^0$, given by $\gamma^\vee \leqq \beta^\vee$ if and only if $\beta^\vee = \gamma^\vee + \sum_{i=1}^r n_i\alpha_i{}^\vee$, for some non-negative integers n_i. Note that, by our assumptions, κ_0 is of order two and so $\beta^\vee$ lies outside ${}^LH^0$ if and only if $\kappa_0(\beta^\vee) = -1$. According to [**8**], $\{\alpha_1{}^\vee, \ldots, \alpha_r{}^\vee, \beta_1{}^\vee, \ldots, \beta_s{}^\vee\}$ is either a simple system for the roots of ${}^LG^0$ or an extended simple system (that is, a simple system together with the negative of the top root for that system).

PROPOSITION 9.2.4. *Suppose that* $\{\alpha_2{}^\vee, \ldots, \alpha_r{}^\vee, \beta_1{}^\vee, \ldots, \beta_s{}^\vee\}$ *is a simple system for* ${}^LG^0$ *and that* $-\alpha_1{}^\vee$ *is the top root of that system. Then, either*

(i) $s = 1$, $\alpha_1{}^\vee \equiv -2\beta_1{}^\vee \bmod \langle \alpha_2{}^\vee, \ldots, \alpha_r{}^\vee \rangle$, *and* ${}^LG^0$ *is not of type* A_n, *or*

(ii) $s = 2$, $\alpha_1{}^\vee = -(\beta_1{}^\vee + \beta_2{}^\vee) \bmod \langle \alpha_2{}^\vee, \ldots, \alpha_r{}^\vee \rangle$, *and* ${}^LG^0$ *is of type* A_n, D_n *or* E_6.

The proof is an easy calculation and examination of types (cf. [**4**]); we omit the details.

We return to our semisimple group G^*. In each factor (= factor of the simply-connected covering of) of ${}^LG^0$ we use simple systems as above to define ι_0, ι_0'. Suppose that the $\alpha_i{}^\vee$ of some factor are all simple in ${}^LG^0$. Then $\langle \iota_0, \alpha_i{}^\vee \rangle = \langle \iota_0', \alpha_i \rangle = 1$ so that $\langle \iota_0 - \iota_0', \alpha_i{}^\vee \rangle = 0$. On the other hand, if the $\alpha_i^\vee$ are as in (9.2.4) then

$$\langle \iota_0 - \iota_0', \alpha_i^\vee \rangle = 0, \quad i = 2, \ldots, r, \quad \text{and}$$
$$\langle \iota_0 - \iota_0', \alpha_1^\vee \rangle = -(m+1),$$

where, in the terminology of [**5**], m denotes the altitude of the top root $-\alpha_1^\vee$.

We first seek the element λ of (9.2.3) in the span of the roots of $\underset{\sim}{G}^*$. For this, we may work one factor at a time. Thus in this paragraph we assume $\underset{\sim}{G}^*$ or ${}^LG^0$ simple. If $\alpha_1^\vee, \ldots, \alpha_r^\vee$ are simple in ${}^LG^0$ then we take $\lambda = 0$. Suppose that $\alpha_1^\vee, \ldots, \alpha_r^\vee$ are as in (9.2.4). Consider the case $s = 2$. Our computation of $\langle \iota_0 - \iota_0', \alpha_i^\vee \rangle$ shows that

$$\langle \iota_0 - \iota_0', \beta_1^\vee + \beta_2^\vee \rangle = m + 1.$$

Define a weight λ of $\underset{\sim}{G}^*$ by $\langle \lambda, \alpha_i^\vee \rangle = 0$, $i = 2, \ldots, r$, $\langle \lambda, \beta_1^\vee \rangle = 0$, $\langle \lambda, \beta_2^\vee \rangle = m + 1$; clearly λ satisfies (9.2.3). To prove that λ lies in the span of the roots, that is, that $\langle \lambda, \lambda^\vee \rangle \in \mathbf{Z}$ for all weights $\lambda^\vee$ of ${}^LG^0$, it is enough to show that the order of any element of the center of ${}^LG^0$ divides $m + 1$. Since ${}^LG^0$ is of type A_n, D_n or E_6 this is easily verified from the tables in [**5**]. Consider now the case $s = 1$. Here we obtain

$$\langle \iota_0 - \iota_0', \beta_1^\vee \rangle = -\tfrac{1}{2}(m+1).$$

The element λ of (9.2.3) can only be $\iota_0 - \iota_0'$; we have

$$\langle \lambda, \alpha_i^\vee \rangle = 0, \quad i = 2, \ldots, r, \quad \text{and}$$
$$\langle \lambda, \beta_1^\vee \rangle = -\tfrac{1}{2}(m+1).$$

Note that because type A_n is excluded ((9.2.4)) we have that $-\frac{1}{2}(m+1) \in \mathbf{Z}$ (cf. [**5**]). To place λ in the span of the roots it would be enough to show that the order of any element of the center of ${}^LG^0$ divides $\frac{1}{2}(m+1)$. Inspection shows that this is true unless ${}^LG^0$ is of type B_{2n+1}, C_{2n+1}, D_{2n}, D_{4n+3} or E_7. If ${}^LG^0$ is of type C_{2n+1} or D_4 then further inspection shows that there is no simple root of ${}^LG^0$ appearing with coefficient 2 in the top root and half-integer coefficient in some weight. Also if ${}^LG^0$ is of type D_{4n+3} there is no simple root of ${}^LG^0$ appearing with coefficient 2 in the top root and quarter-integer coefficient in some weight. Thus for ${}^LG^0$ of type C_{2n+1}, D_4 or D_{4n+3} we still obtain λ in the span of the roots of $\underset{\sim}{G}^*$.

In summary, we have:

PROPOSITION 9.2.5. *If $\underset{\sim}{G}^*$ has no simple factors of type C_{2n+1}, $D_{2n} (n \geqq 3)$, or E_7 then each ${}^LH$ embeds in ${}^LG$.*

We examine the excluded cases more carefully. First suppose that ${}^LG^0$ is simple. If ${}^LG^0$ is adjoint then $\underset{\sim}{G}^*$ is simply-connected and so λ of the last paragraph, while not necessarily in the span of the roots, lies in L. If ${}^LG^0$ is not adjoint, and not of type D_{2n}, then ${}^LG^0$ is simply-connected. Thus any weight $\lambda^\vee$ of ${}^LG^0$ lies in $L^\vee$. Also $2\lambda^\vee$ lies in the span of the roots of ${}^LG^0$. Suppose that $2\lambda^\vee = n\beta_1^\vee + \sum_{i=2}^r n_i\alpha_i^\vee$. Note that

$$\kappa_0(2\lambda^\vee) = \kappa_0(\lambda^\vee - \sigma_0\lambda^\vee) = 1,$$

by definition. Since $\kappa_0(\alpha_i^\vee) = 1$ and $\kappa_0(\beta^\vee) = -1$ we conclude that n is even. Hence $\langle\lambda, \lambda^\vee\rangle \in \mathbf{Z}$ for all weights of ${}^LG^0$ and so λ lies in the span of the roots of $\underset{\sim}{G}^*$. For the case D_{2n} and ${}^LG^0$ not adjoint, we assume instead that $\lambda^\vee \in L^\vee$ in the argument above, and so obtain that λ lies in L, if not the span of the roots. Hence:

PROPOSITION 9.2.6. *If $\underset{\sim}{G}^*$ is simple then each ${}^LH$ embeds in ${}^LG$.*

However, in general, the amalgamation of the centers of simple factors may cause problems:

Example 9.2.7. Let

$$\underset{\sim}{G}^* = Sp_6 \times SL_2/\{1, (-1, -1)\}$$

and

$$\underset{\sim}{T}^* = \underset{\sim}{D}_1 \times \underset{\sim}{D}_2/\{1, (-1, -1)\},$$

where

$$\underset{\sim}{D}_1 = \{\mathrm{diag}(x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1})\} \subset Sp_6$$

and

$$\underset{\sim}{D}_2 = \{\mathrm{diag}(y, y^{-1})\} \subset SL_2.$$

Then

$$L = \left\{ny + \sum_{i=1}^{3} m_i x_i : m_i, n \in \mathbf{Z}, n + \sum_{i=1}^{3} m_i \text{ even}\right\}.$$

The roots are $\pm(x_1 \pm x_2)$, $\pm(x_2 \pm x_3)$, $\pm(x_1 \pm x_3)$, $\pm 2x_1$, $\pm 2x_2$, $\pm 2x_3$, $\pm 2y$; the coroots may be identified, respectively, as $\pm(x_1 \pm x_2)$, $\pm(x_2 \pm x_3)$, $\pm(x_1 \pm x_3)$, $\pm x_1$, $\pm x_2$, $\pm x_3$, $\pm 2y$. We fix a compact Cartan subgroup T_0 and some diagonalization of T_0. We then choose κ_0 so that, on transferring to $\underset{\sim}{T}^*$, we get

$$\kappa_0(x_1 - x_2) = \kappa_0(x_2 - x_3) = 1, \quad \kappa_0(x_3) = -1, \quad \kappa_0(2y) = 1.$$

Thus $\pm 2x_1$, $\pm 2x_2$, $\pm 2x_3$ are the only roots not from $\underset{\sim}{H}$. For the usual choice of positive system we obtain $\iota_0 - \iota_0' = x_1 + x_2 + x_3$. Clearly the element λ of (9.2.3) can only be $\iota_0 - \iota_0'$. Since $\iota_0 - \iota_0' \notin L$ we conclude that there is no admissible embedding of ${}^LH$ in ${}^LG$.

There are similar examples for groups of type $D_{2n} \times \ldots$ $(n \geqq 3)$ or $E_7 \times \ldots$.

10. Appendix. We continue with the notation of § 1. Thus $\underset{\sim}{G}$ is a connected reductive group over $\mathbf{R}$, $\underset{\sim}{G}^*$ a quasi-split inner form of $\underset{\sim}{G}$, $\psi : \underset{\sim}{G} \to \underset{\sim}{G}^*$ an inner twist, etc.

Suppose that $\underset{\sim}{T}$ is a maximal torus in $\underset{\sim}{G}$, anisotropic modulo the center of $\underset{\sim}{G}$. Choose $y \in \underset{\sim}{G}^*$ such that $\psi_y = \operatorname{ad} y \circ \psi$ maps $\underset{\sim}{T}$ to $\underset{\sim}{T}^*$, the distinguished maximal torus in $\underset{\sim}{G}^*$. We transfer the Galois action on $\underset{\sim}{T}$ to $\underset{\sim}{T}^*$ via ψ_y, and thence to $L = L(\underset{\sim}{T}^*)$ and $L^\vee = L({}^LT^0)$ in the natural way, denoting the result by σ_T. Then σ_T maps each root of ${}^LT^0$ in ${}^LG^0$ to its negative, and is realized by (conjugation with respect to) an element $m \times (1 \times \sigma)$ of ${}^LG$, where m lies in ${}^LG^0$, normalizes ${}^LT^0$ and maps positive roots of ${}^LT^0$ to negative ones. In particular, the choice for y has no effect on σ_T.

Conversely, suppose that ${}^LG$ contains an element $m \times (1 \times \sigma)$ mapping ${}^LT^0$ to itself, and each root of ${}^LT^0$ to its negative. Then, according to [**7**], $\underset{\sim}{G}$ has a maximal torus $\underset{\sim}{T}$ which is anisotropic modulo the center of $\underset{\sim}{G}$, and $m \times (1 \times \sigma)$ acts on ${}^LT^0$ as σ_T. The proof is as follows. First, we use Theorem 1.7 of [**12**] to conclude that there is a torus $\underset{\sim}{T}_1$ in $\underset{\sim}{G}^*$ such that $m \times (1 \times \sigma)$ acts on L, and hence on ${}^LT^0$, as the Galois action of $\underset{\sim}{T}_1$ transferred by some p-d. (cf. proof of Lemma 7.0.9). This torus $\underset{\sim}{T}_1$ is anisotropic modulo the center of $\underset{\sim}{G}^*$, and hence fundamental in $\underset{\sim}{G}^*$. Lemma 2.8 of [**9**] then shows that there is a maximal torus T in G defined over $\mathbf{R}$ and $x \in \underset{\sim}{G}^*$ such that $\operatorname{ad} x \circ \psi$ maps $\underset{\sim}{T}$ to $\underset{\sim}{T}_1$ over $\mathbf{R}$. Clearly $\underset{\sim}{T}$ is as desired.

We assume still that $m \times (1 \times \sigma)$ maps each positive root of ${}^LT^0$ to its negative. Lemma 3.2 of [**7**] computes explicitly the square $m\sigma_G(m) \times (-1 \times 1)$ of such an element. Note that $m\sigma_G(m)$ lies in ${}^LT^0$. Also, if $\lambda^\vee \in L^\vee$ then

$$\mu^\vee = \lambda^\vee + (m \times (1 \times \sigma))\lambda^\vee$$

extends to a rational character on ${}^LG^0$.

LEMMA (Langlands).

$$\lambda^\vee(m\sigma_G(m)) = (-1)^{\langle 2\iota, \lambda^\vee \rangle}\mu^\vee(m), \quad \lambda^\vee \in L^\vee,$$

where ι is one half the sum of the roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{B}^*$.

Proof. If $m = tn$, where t lies in the connected center of ${}^LG^0$ and n in the derived group then calculation shows that

$$\begin{aligned}\lambda^\vee(m\sigma_G(m)) &= \mu^\vee(t)\lambda^\vee(n\sigma_G(n)) \\ &= \mu^\vee(m)\lambda^\vee(n\sigma_G(n)).\end{aligned}$$

Thus we have to show

$$(*) \qquad \lambda^\vee(n\sigma_G(n)) = (-1)^{\langle 2\iota, \lambda^\vee \rangle}, \quad \lambda^\vee \in L^\vee,$$

for each $n \in ({}^LG^0)_{\text{der}}$ such that $n \times (1 \times \sigma)$ maps each root of ${}^LT^0$ to its negative. Clearly $n\sigma_G(n)$ does not depend on the choice for n. Thus for the proof of (*) we may replace ${}^LG^0$ by the simply-connected covering of its derived group and argue separately in each simple factor of the covering. We therefore assume ${}^LG^0$ simple and simply-connected.

We will prove (*) by induction. Thus, suppose that (*) has been proved for all groups $\underset{\sim}{G}$ for which the dimension of $({}^LG^0)_{\text{der}}$ is less than that for our given group. Note that (*) is trivially true for the case of dimension zero.

Let $\beta^\vee$ be the largest (top) root for the ordering on the roots of ${}^LT^0$ induced by the choice of ${}^LB^0$. Then $\sigma_G\beta^\vee = \beta^\vee$. Each root of ${}^LT^0$ perpendicular to $\beta^\vee$ (under the canonical bilinear form (,) on $L^\vee$) is an integral linear combination of simple roots perpendicular to $\beta^\vee$. Hence if ${}^LH^0$ is the group generated by ${}^LT^0$ and the 1-parameter subgroups for the roots perpendicular to $\beta^\vee$, then ${}^LH^0$ is invariant under the action of W and ${}^LH = {}^LH^0 \rtimes W$ is an L-group (that is, an object in $\mathfrak{G}^\vee(R)$ ([**7**])). Let ${}^LJ^0$ be the subgroup of ${}^LG^0$ generated by ${}^LT^0$ and the 1-parameter subgroup for $\beta^\vee$. Then ${}^LJ^0$ is also W-invariant, but ${}^LJ^0 \rtimes W$ is not, in general, an L-group since $\sigma_G X_{\beta^\vee} = (-1)^l X_{\beta^\vee}$, where $X_{\beta^\vee}$ is some root vector for $\beta^\vee$ and l is one half the sum of the coefficients in the simple expansion of $\beta^\vee$ of those simple roots $\alpha^\vee$ satisfying $\sigma_G\alpha^\vee \neq \alpha^\vee$ and $(\sigma_G\alpha^\vee, \alpha^\vee) \neq 0$ (cf. [**8**, Lemma 3]). Nevertheless, we will be able to deal with ${}^LJ^0 \rtimes W$, by explicit computation. Note that ${}^LH^0$ and ${}^LJ^0$ commute.

Choose n_1 in the derived group of ${}^LH^0$, normalizing ${}^LT^0$ and taking the positive roots of ${}^LT^0$ in ${}^LH^0$ to negative ones. Choose n_2 in the derived group of ${}^LJ^0$ normalizing ${}^LT^0$ and mapping $\beta^\vee$ to $-\beta^\vee$.

PROPOSITION. (i) *$n_1n_2 \times (1 \times \sigma)$ maps each root of ${}^LT^0$ in ${}^LG^0$ to its negative and*

(ii) *$n_1 \times (1 \times \sigma)$ maps each root of ${}^LT^0$ in ${}^LH^0$ to its negative.*

Proof. For (i) we just have to show that n_1n_2 maps each positive root of ${}^LG^0$ to a negative one, since we have assumed the existence of some $m \times (1 \times \sigma)$ mapping each root to its negative. Since n_2 fixes each root in ${}^LH^0$ and n_1 fixes $\beta^\vee$ it is clear that n_1n_2 maps $\beta^\vee$ and each positive root in ${}^LH^0$ to negative roots. Suppose that $\alpha^\vee$ is a root, not in ${}^LH^0$ and not equal to $\pm\beta^\vee$. Then $\alpha^\vee$ is positive if and only if $(\alpha^\vee, \beta^\vee) > 0$. But

$$(n_1n_2\alpha^\vee, \beta^\vee) = (\alpha^\vee, n_2^{-1}n_1^{-1}\beta^\vee) = -(\alpha^\vee, \beta^\vee).$$

Thus (i) is proved.

(ii) follows from (i) and the fact that n_2 fixes each root of ${}^LH^0$.

To prove the lemma, we can take $n = n_1n_2$. Then

$$n\sigma_G(n) = n_1\sigma_G(n_1)n_2\sigma_G(n_2).$$

We may apply the inductive hypothesis to ${}^LH$ to obtain

$$\lambda^\vee(n_1\sigma_G(n_1)) = (-1)^{\langle 2\iota^*, \lambda^\vee\rangle}, \quad \lambda^\vee \in L^\vee,$$

where ι_* is one half the sum of the positive roots of $\underset{\sim}{T}^*$ in $\underset{\sim}{G}^*$ which are perpendicular to β.

We now compute $n_2\sigma_G(n_2)$. The simply-connected covering of the derived group of ${}^LJ^0$ is $SL_2(\mathbf{C})$. We map $SL_2(\mathbf{C})$ to ${}^LJ^0$ in the usual way. Take for n_2 the image of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Recall that $\sigma_G X_{\beta^\vee} = (-1)^l X_{\beta^\vee}$. Hence $\sigma_G(n_2)$ is the image of

$$\begin{bmatrix} 0 & (-1)^l \\ -(-1)^l & 0 \end{bmatrix},$$

and $n_2\sigma_G(n_2)$ the image of

$$\begin{bmatrix} (-1)^{l+1} & 0 \\ 0 & (-1)^{l+1} \end{bmatrix}.$$

We conclude that

$$\lambda^\vee(n_2\sigma_G(n_2)) = (-1)^{(l+1)\langle\beta,\lambda^\vee\rangle}, \quad \lambda^\vee \in L^\vee.$$

Thus to prove the lemma we have to show that

$$(**) \qquad l\langle\beta, \lambda^\vee\rangle \equiv 2\langle\iota_{**}, \lambda^\vee\rangle \bmod 2\mathbf{Z}, \quad \lambda^\vee \in L^\vee,$$

where $\iota_{**} = \iota - \iota_* - \frac{1}{2}\beta$. If α is positive, $(\alpha, \beta) \neq 0$ and $\alpha \neq \beta$ then $-\omega_\beta(\alpha)$ has these same properties as α; that is, is positive, etc. Hence

$$2\langle\iota_{**}, \lambda^\vee\rangle = l'\langle\beta, \lambda^\vee\rangle$$

where $l' = \langle\iota_{**}, \beta^\vee\rangle = \langle\iota, \beta^\vee\rangle - 1$. Thus $l' + 1$ is the sum of the coefficients in the simple expansion of $\beta^\vee$. For (**) it would be sufficient to prove that $l' \equiv l \bmod 2\mathbf{Z}$. Recall that l is one half the sum of the coefficients, in the expansion of $\beta^\vee$, of those simple roots $\alpha^\vee$ such that $\alpha^\vee \neq \sigma_G\alpha^\vee$ and $(\alpha^\vee, \sigma_G\alpha^\vee) \neq 0$.

Since we have done so in similar situations (cf. § 9), we now appeal directly to classification. If ${}^LG^0$ is of type A_{2n} then $l = 1$; otherwise $l = 0$. On the other hand, if ${}^LG^0$ is of type A_{2n} then $l' = 2n - 1$; otherwise l' is even (cf. [**5**]). Hence the lemma is proved.

References

1. J. Arthur, *On the invariant distributions associated to weighted orbital integrals*, preprint.
2. A. Borel, *Linear algebraic groups* (Benjamin, 1969).
3. ——— *Automorphic L-functions*, Proc. Sympos. Pure Math. *33* Amer. Math. Soc. (1979), 27–61.
4. N. Bourbaki, *Groupes et algèbres de Lie*, Chs. 4, 5, 6 (Hermann, 1968).
5. H. Freudenthal and H. de Vries, *Linear Lie groups* (Academic Press, 1969).
6. Harish-Chandra, *Harmonic analysis on real reductive Lie groups I*, J. Funct. Analysis *19* (1975), 104–204.

7. R. Langlands, *On the classification of irreducible representations of real algebraic groups*, Notes, IAS.

8. ——— *Stable conjugacy; definition and lemmas*, Can. J. Math. *31* (1979), 700–725.

9. D. Shelstad, *Characters and inner forms of a quasi-split group over R*, Compositio Math. *39* (1979), 11–45.

10. ——— *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Scient. Ec. Norm. Sup., 4e série, t. *12* (1979), 1–31.

11. ——— *Notes on L-indistinguishability*, Proc. Sympos. Pure Math. *33* Amer. Math. Soc. (1979), 193–203.

12. R. Steinberg, *Regular elements of semi-simple algebraic groups*, Publ. Math. I.H.E.S. *25* (1965), 49–80.

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