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SHORT PROOF OF AN INTERNAL CHARACTERIZATION OF COMPLETE REGULARITY

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ABSTRACT. A short proof is given of an internal characterization of completely regular spaces due to J. Kerstan.

Complete regularity as introduced by P. Urysohn differs from other separation axioms which can be imposed on a topological space in that it is defined externally by means of continuous real-valued functions. Therefore several authors have been interested in finding internal characterizations of completely regular spaces (e.g. see [2], [4], [5], [9], [10]). Among the early contributions to this problem is the following nice theorem of J. Kerstan [7], which, however, seems to be less known. Following [7] let us call a family $\mathcal{U}$ of open subsets of a topological space X *completely regular* if for every $U \in \mathcal{U}$ there exist two sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $U=\bigcup\{U_n \mid n\in\mathbb{N}\}$ and $U_n \subset X\setminus V_n \subset U$ for each $n\in\mathbb{N}$.

THEOREM. *A T_1-space is completely regular if and only if it has a completely regular* (*sub*)*base.*

Quite recently a variant of this theorem was rediscovered by G. Reynolds [8]. It is also worth mentioning that a similar (though not identical) characterization of complete regularity is already contained in the work of A. D. Alexandroff [1].

In order to prove the non-trivial implication of the theorem one has to produce enough real-valued continuous functions to guarantee complete regularity provided that a completely regular subbase is given. In the proofs presented in [7] or [8] this is achieved by introducing suitable, but somewhat lengthy, modifications of the classical technique used by P. Urysohn to prove his famous characterization of normal spaces. The basic idea of this method was isolated in [6]. Our aim here is to point out that the theorem can be proved much shorter in an elementary way by directly applying the Urysohn metrization theorem. To this end it suffices (see [3], Chapter 3) to give a brief argument for the following proposition, which is of independent interest.

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PROPOSITION [7]. *A subset of a topological space is a cozero-set if and only if it belongs to a completely regular family.*

Proof. Clearly the family of all cozero-sets of a topological space is completely regular. Conversely, if a subset U of a topological space X belongs to a completely regular family $\mathcal{U}$ of open subsets of X, one may assume without loss of generality that $\mathcal{U}$ is countable. Let $X_{\mathcal{U}}$ be the topological space obtained by supplying X with the topology generated by $\mathcal{U}$ as a subbase. Since $\mathcal{U}$ is completely regular, $X_{\mathcal{U}}$ is regular, but not necessarily T_1. However, identifying points in $X_{\mathcal{U}}$ which have equal closures yields a second countable regular T_1-space Y. If $f: X_{\mathcal{U}} \to Y$ denotes the corresponding quotient map and $V = f[U]$, then $f: X \to Y$ is continuous, V is open in Y, and $U = f^{-1}[V]$. By virtue of the Urysohn metrization theorem, Y is metrizable. Therefore V, as an open subset of a metrizable space, is a cozero-set. Consequently U is a cozero-set in X.

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