AVERAGING OPERATORS IN NON COMMUTATIVE L^p SPACES I

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1.0. Introduction. The origin of the theory of averaging operators is explained in [**1**]. The theory has been developed on spaces of continuous functions that vanish at infinity by Kelley in [**3**] and on the L^p spaces of measure theory by Rota [**5**]. The motivation for this paper arose out of the latter paper. The aim of this paper is to prove a generalisation of Rota's main representation theorem (every average is a conditional expectation) in the context of a 'non commutative integration'. This context is as follows. Let $\mathcal{A}$ be a finite von Neumann algebra and ϕ a faithful normal finite trace on $\mathcal{A}$ such that $\phi(I)=1$, where I is the identity of $\mathcal{A}$. We can construct the Banach spaces $L^p(\mathcal{A}, \phi)$, where $1 \leq p < \infty$, with norm $\|x\|_p = \phi(|x|^p)^{1/p}$, of possibly unbounded operators affiliated with $\mathcal{A}$ as in [**9**]. We note that $\mathcal{A}$ is dense in $L^p(\mathcal{A}, \phi)$. These spaces share many of the features of the L^p spaces of measure theory; indeed if $\mathcal{A}$ is abelian then $L^p(\mathcal{A}, \phi)$ is isometrically isomorphic to L^p of some measure space.

We shall need to know a little about conditional expectations. Let $\mathcal{A}$ and $\mathcal{B}$ be finite von Neumann algebras with $\mathcal{B}$ a subalgebra of $\mathcal{A}$. The Radon Nikodym theorem of Segal [**6**] indicates that to each $x \in \mathcal{A}$ we can associate a unique $M(x)$ in $\mathcal{B}$ satisfying

$$\phi(xy) = \phi(M(x)y) \quad (y \in \mathcal{B}).$$

The map so defined is a positive linear idempotent that contracts $\| \ \|_p$ for $1 \leq p \leq \infty$. The (unique) extension of this map to a map of $L^p(\mathcal{A}, \phi)$ onto $L^p(\mathcal{B}, \phi)$ is called the *conditional expectation* of $L^p(\mathcal{A}, \phi)$ onto $L^p(\mathcal{B}, \phi)$. Umegaki has given sufficient conditions for a map of $\mathcal{A}$ into itself to coincide with the conditional expectation in Theorem 1 of [**8**].

1.1. Definition. We shall define an averaging operator as a linear mapping A of $L^p(\mathcal{A}, \phi)$, where $1 \leq p \leq \infty$, and p is fixed, into itself, that satisfies

(i) $\|A(x)\|_p \leq \|x\|_p$ $(x \in L^p(\mathcal{A}, \phi))$,
(ii) $A(x^*) = A(x)^*$, where * denotes the Hilbert Space adjoint,
(iii) $A(yA(x)) = A(y)A(x)$ $(y \in \mathcal{A}, x \in L^p(\mathcal{A}, \phi))$.

We shall often refer to an averaging operator as an average.

1.2. Remark. Condition (ii) is redundant in the context of Rota's paper. I have not, as he does, assumed the condition that A should preserve the identity, although the substantial portion of this paper will do so. I hope to deal with averages that do not preserve the identity in a subsequent paper.

1.3. Examples. The examples in Rota's paper are most instructive. For the present context we have the following results.

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(a) Any conditional expectation of $L^p(\mathscr{A}, \phi)$ onto $L^p(\mathscr{B}, \phi)$, where $\mathscr{B}$ is a von Neumann subalgebra of $\mathscr{A}$, is an average.

(b) Let P be a projection in $\mathscr{A}$. Then the map $X \to PXP$, where $X \in L^p(\mathscr{A}, \phi)$, is an average.

(c) Let $Z = Z^*$ be a central operator with $\|Z\|_\infty \leq 1$ in $\mathscr{A}$. Then the map $X \to ZX$, where $X \in L^p(\mathscr{A}, \phi)$, is an average.

1.4. Elementary Properties. We note that (c) above shows that an average need not be a projection (i.e. $A^2 = A$). If $A(I) = I$ then $A^2 = A$, as Proposition 1 of [**5**] shows; however (b) above shows that A can be a projection without mapping I to I. Since $\mathscr{A}$ is $\| \|_p$ dense in $L_p(\mathscr{A}, \phi)$ it follows that the range of the restriction of A to $\mathscr{A}$ $\| \|_p$ dense in the range of A; a knowledge of how A behaves on $\mathscr{A}$ will be useful in characterising A. We note further that the range of the restriction of A to $\mathscr{A}$ is a ring, and that the bounded elements of the range of A form a ring in case $A^2 = A$.

2.0. Identity preserving averages. Throughout this section we assume that $A(I) = I$. Our first result shows that A contracts $\| \|_\infty$ as well as $\| \|_p$, but first, we require the following lemma. See. Theorem 14 F of [**4**, p. 39].

2.1. Lemma. *Let $x \in L^p(\mathscr{A}, \phi)$ for some fixed p. with $1 \leq p < \infty$. Then*

$$\lim_{n \to \infty} \|x\|_n = \|x\|_\infty.$$

Proof. If $\|x\|_\infty = 0$, then $x = 0$ and the result is true. If $\|x\|_\infty > 0$, then choosing $0 < \delta < \|x\|_\infty$, and noting that if $|x| = \int_0^\infty \lambda \, dE_\lambda$ then $\delta(I - E_\delta) \leq (I - E_\delta)\,|x|$, we have by the change of measure principle $\delta^n(I - E_\delta) \leq (I - E_\delta)\,|x|^n$, so that

$$\delta\phi(I - E_\delta)^{1/n} \leq \phi(|x|^n)^{1/n} = \|x\|_n.$$

Now $\delta < \|x\|_\infty$ implies $\phi(I - E_\delta) > 0$; thus $\phi(I - E_\delta)^{1/n} \to 1$ as $n \to \infty$ and we have $\delta \leq \liminf_{n \to \infty} \|x\|_n$ for each $\delta < \|x\|_\infty$.

If $\|x\|_\infty = \infty$, then using the relation just proved we deduce that the lemma is true. If $\|x\|_\infty < \infty$, then using the functional calculus we have $|x|^n \leq \|x\|_\infty^n I$. It follows that

$$\limsup_{n \to \infty} \|x\|_n \leq \|x\|_\infty.$$

2.2. Proposition. *Let A be an average on $L^p(\mathscr{A}, \phi)$; then $\|A(x)\|_\infty \leq \|x\|_\infty$ for $x \in \mathscr{A}$.*

Proof. Let $x \in \mathscr{A}$ with $\|x\|_\infty \leq 1$. Then $\|x\|_p \leq 1$, by 2.5 (iii) of [**9**]. Suppose that for some natural number k,

(i) $|A(x)|^{2(K-1)} = A(H)$ for $H \in L^p(\mathscr{A}, \phi)$ and for such H,

(ii) $\|A(H)\|_p \leq \|x\|_p$;

then

$$\begin{aligned} |A(x)|^{2K} &= |A(x)|^2 \, . \, |A(x)|^{2(K-1)} = A(x)^* A(x) A(H) \\ &= A(x^*) A(xA(H)) = A(x^* A(xA(H))) \end{aligned}$$

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and $x^*A(xA(H)) \in L^p(\mathscr{A}, \phi)$. Also,

$$\|A(x^*A(xA(H)))\|_p \leq \|x^*A(xA(H))\|_p \leq \|x^*\|_\infty \|xA(H)\|_p$$
$$\leq \|x\|_\infty^2 \|A(H)\|_p \leq \|x\|_p,$$

using 2.5 (iii) of **[9]** and 1.1 (i) repeatedly. These relations clearly hold for $K = 0, 1$, and hence for all natural numbers. Now we use Lemma 2.1,

$$\|A(x)\|_\infty \leq \||A(x)|\|_\infty \leq \liminf_{K\to\infty} \||A(x)|\|_{2Kp}$$

$$= \liminf_{K\to\infty} \||A(x)|^{2K}\|_p^K \leq 1$$

2.3. Corollary. *Let $A\dagger$ denote the adjoint of A. Then both A and $A\dagger$ map positive operators to positive operators and $A(x^*x) \geq A(x)^*A(x)$ for all x in $\mathscr{A}$.*

Proof. Consider A restricted to $\mathscr{A}$ and let this be denoted by A too. It is a projection of norm one onto its range which is a C^* algebra. It follows from Theorem 3.4 of [**7**, p. 131] that A and $A\dagger$ enjoy the properties stated.

2.4. Proposition. *$A\dagger(I) = I$ and hence $\phi(A(x)) = \phi(x)$.*

Proof. The duality between $L^p(\mathscr{A}, \phi)$ and $L^q(\mathscr{A}, \phi)$, where $1/p + 1/q = 1$ and $p > 1$, means that $\phi(A(x)y) = \phi(xA\dagger(y))$ $(x \in L^p, y \in L^q)$. If we have $A\dagger(I) = I$, then putting $y = I$ gives the second conclusion. For the first we argue as follows. Since

$$1 = \phi(IA\dagger(I)) \leq \|I\|_p \,.\, \|A\dagger(I)\|_q \leq 1,$$

we have

$$1 = \phi(A\dagger(I)) = \phi(A\dagger(I)^q)^{1/q} \quad (q > 1).$$

By considering the spectral representation of $A\dagger(I)$, it follows that there is a probability measure on $\mathbb{R}^+$, μ say, such that

$$\phi(A\dagger(I)^s) = \int_0^\infty \lambda^s \, d\mu(\lambda) \quad (s > 0).$$

Thus

$$1 = \int_0^\infty \lambda \, d\mu(\lambda) = \int_0^\infty \lambda^q \, d\mu(\lambda)$$

and, by Hölder's inequality, $1 = \lambda$ (μ − a.e.): i.e. μ is the point mass at 1. Thus $\phi(A\dagger(I)^2) = \|A\dagger(I)\|_2^2 = 1$. By considering $\|A\dagger(I) - I\|_2^2$ it follows that $A\dagger(I) = I$.

For $p = 1$ we note that since A contracts $\| \;\|_1$ and $\| \;\|_\infty$ it satisfies the conditions of Proposition 1 of **[10]**, and hence maps each L^p into itself for $1 < p < \infty$. We can now use the appropriate analogue of the Riesz convexity theorem, (VI.10.11 of **[2]**), to show that A contracts $\| \;\|_p$ for $1 < p < \infty$, and hence we can use the results above.

2.5. Corollary. *Let $\mathscr{B}=A(\mathscr{A})$; then $\mathscr{B}$ is a von Neumann algebra and A is the conditional expectation of $L^p(\mathscr{A},\phi)$ onto $L^p(\mathscr{B},\phi)$.*

Proof. Consider A restricted to $\mathscr{A}$; then A is a linear map of $\mathscr{A}$ into itself which satisfies $A^2=A$, $A(x)^*A(x)\leqslant A(x^*x)$, $A(x)\geqslant 0$ whenever $x\geqslant 0$, and $A(I)\leqslant I$, by Corollary 2.3. Also, by Proposition 2.4,

$$\phi(xA(y))=\phi(A(xA(y)))=\phi(A(x)A(y))=\phi(A(A(x)y))=\phi(A(x)y).$$

These are the conditions required by Theorem 1 of [**8**], which shows that $\mathscr{B}$ is a von Neumann algebra and A agrees with the conditional expectation from $\mathscr{A}$ onto $\mathscr{B}$. It follows that A is the conditional expectation from $L^p(\mathscr{A},\phi)$ onto $L^p(\mathscr{B},\phi)$.

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