Can. J. Math., Vol. XXXI, No. 2, 1979, pp. 312–319

AN ALTERNATIVE APPROACH TO LAGUERRE POLYNOMIAL IDENTITIES IN COMBINATORICS

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1. In their paper "Permutation Problems and Special Functions," Askey and Ismail [**1**] give the following striking identity. Consider three boxes containing j, k, m distinguishable balls, and consider all possible rearrangements of these balls such that each box still has the same number of balls; i.e., j end up in the first, k in the second, m in the third. One disregards the order of the balls within a box so there are $(j + k + m)!/(j!k!m!)$ possible rearrangements. Let R_E be the number of rearrangements where an even number of balls change boxes and R_0 the number of rearrangements where an odd number change boxes. The identity is

$$(1.1) \quad R_E - R_0 = 2^{j+k+m+1} \int_0^\infty L_j(x)L_k(x)L_m(x)e^{-2x}dx$$

where

$$(1.2) \quad L_j(x) = \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{x^r}{r!}$$

is the jth Laguerre polynomial. These polynomials are orthonormal with respect to the weight function e^{-x}; i.e.

$$\int_0^\infty L_j(x)L_k(x)e^{-x}dx = \delta_{jk}.$$

In combinatorial theory Laguerre polynomials are called *Rook polynomials.*

Substituting for $L_j(x)$, $L_k(x)$ and $L_m(x)$ in (1.1) from (1.2) and integrating one obtains

$$(1.3) \quad R_E - R_0 = 2^{j+k+m} \sum_{r=0}^{j} \sum_{s=0}^{k} \sum_{t=0}^{m} (-\tfrac{1}{2})^{r+s+t} \frac{(r+s+t)!}{r!s!t!} \binom{j}{r}\binom{k}{s}\binom{m}{t}.$$

The method of proof for (1.1) in [**1**] was to first calculate the generating function for

$$\int_0^\infty L_j(u)L_k(u)L_m(u)e^{-2u}du$$

which is elementary. The result is the reciprocal of a simple cubic polynomial of three variables, x, y, z, in which no variable is squared or cubed. Thus the

Received October 31, 1977 and in revised form January 26, 1978. This work was supported in part by National Science Foundation Grant No. 144J854.

https://doi.org/10.4153/CJM-1979-034-7 Published online by Cambridge University Press

coefficient of $x^j y^k z^m$ in the power series expansion of this rational function is

$$\int_0^\infty L_j(u)L_k(u)L_m(u)e^{-2u}du.$$

Then by using a powerful theorem of MacMahon's called the Master Theorem [**8**, pp. 93–98] (also stated in [**1**]) this coefficient can be identified except for a constant multiple with the coefficient of $x^j y^k z^m$ in the expansion of $(x - y - z)^j(- x + y - z)^k(- x - y + z)^m$ and by examination one sees that the coefficient of $x^j y^k z^m$ in $(x - y - z)^j(- x + y - z)^k(- x - y + z)^m$ is $R_E - R_0$.

Our object is to prove the identity (1.1) by proving (1.3) directly. This not only avoids the Master Theorem, but also has some additional interest since when combined with the recent work of Ismail and Tamhankar [**5**] it provides an elementary and purely combinatorial proof of the positivity of

$$\int_0^\infty L_j(u)L_k(u)L_m(u)e^{-2u}du.$$

The idea for the proof can be discovered by doing a simple example. If $j = 2$, $k = 1$, $m = 1$, there are 12 rearrangements and it is easy to see in this case that $R_E = 8$ and $R_0 = 4$ so that $R_E - R_0 = 4$. If one evaluates the Laguerre integral one has

$$2^{2+1+1+1}\int_0^\infty (1 - 2x + \tfrac{1}{2}x^2)(1 - x)(1 - x)e^{-2x}dx$$
$$= 16 - 32 + 44 - 36 + 12 = 4,$$

and the identity holds. We see that the way the integral counts is in a sense uneconomical; the numbers alternate in sign and each overcompensates for the previous one. However this extravagance cancels out in the end giving the correct result. This reminds one of the derangement problem in probability where one counts the number of ways an event can happen by successive inclusion and exclusion, and this is the method of counting we use to establish (1.3).

Let S_n be the set of all permutations of the integers $1, 2, \ldots, n$. S_n has $n!$ elements. Here $n = j + k + m$. Let

$$A_i = \{\pi \in S_n | 1 \leqq \pi(i) \leqq j\}, i = 1, \ldots, j;$$
$$A_{j+i} = \{\pi \in S_n | j + 1 \leqq \pi(j + i) \leqq j + k\} \text{ and, } i = 1, \ldots, k; \quad \text{and}$$
$$A_{j+k+i} = \{\pi \in S_n | j + k + 1 \leqq \pi(j + k + i) \leqq n\}, \quad i = 1, \ldots, m.$$

Then $\bigcup_1^n A_i$ represents the event "one ball remains in the same box" and $\bigcap_1^n A_i$ the event "all balls remain in the same box". When counting we have to divide by $j!k!m!$ since we are disregarding order within a box.

Let $p_r = P\{A_r\}$, the probability of A_r, $p_{rs} = P\{A_r \cap A_s\}$,

$$p_{rst} = P\{A_r \cap A_s \cap A_t\}, \ldots \quad r, s, t \ldots = 1, \ldots, n.$$

Finally let $S_0 = 1$, $S_1 = \sum_r p_r$, $S_2 = \sum_{r<s} p_{rs}$, $S_3 = \sum_{r<s<t} p_{rst}, \ldots$. Then from [**4**, p. 106] we have

THEOREM 1. *For any integer M with $1 \leqq M \leqq n$ the probability P_M that exactly M among the n events $A_1, \ldots, A_n$ occur simultaneously is given by*

$$P_M = S_M - \binom{M+1}{M} S_{M+1} + \binom{M+2}{M} S_{M+2} - \ldots \pm \binom{n}{M} S_n.$$

Multiplying both sides by $n!$ changes all probabilities into numbers of ways an event can occur.

The number of ways that r of the events $A_1, \ldots, A_j$, s of the events $A_{j+1}, \ldots, A_{j+k}$, and t of the events $A_{j+k+1}, \ldots, A_n$ can occur is

$$_jC_r\, {}_nC_s\, {}_mC_t\, j^{(r)} k^{(s)} m^{(t)} (n - (r + s + t))!/j!k!m!$$

Here $j^{(r)} = j(j-1) \ldots (j-r+1)$, $_jC_r = \binom{j}{r}$, and we have divided by $j!k!m!$ for the reason noted above.

Therefore, applying the theorem above, the number of rearrangements leaving exactly M balls fixed is

$$(1.4) \quad N_M = \sum_{r=0}^{j} \sum_{s=0}^{k} \sum_{t=0}^{m} (-1)^{r+s+t-M}\, {}_{(r+s+t)}C_M\, {}_jC_r\, {}_kC_s\, {}_mC_t \times j^{(r)} k^{(s)} m^{(t)} (n - (r+s+t))!/j!k!m!$$

with $r + s + t \geqq M$. Hence the number of rearrangements leaving an even number of balls *fixed* minus the number of rearrangements leaving an odd number of balls *fixed* is

$$\sum_{M=0}^{n} (-1)^M N_M.$$

Substituting from (1.4) in this last equation, and noting that

$$\sum_{M=0}^{r+s+t} {}_{(r+s+t)}C_M = 2^{r+s+t}$$

and $j^{(r)}/j! = 1/(j-r)!$ it is easily seen that

$$\sum_0^n (-1)^M N_M = \sum_{r=0}^{j} \sum_{s=0}^{k} \sum_{t=0}^{m} (-2)^{r+s+t}\, {}_jC_r\, {}_kC_s\, {}_mC_t \times (n - (r+s+t))!/(j-r)!(k-s)!(m-t)!.$$

Replacing r by $j - r$, s by $k - s$, t by $m - t$ and using $n = j + k + m$ gives

$$\sum_0^n (-1)^M N_M = (-1)^n 2^{j+k+m} \sum_{r=0}^{j} \sum_{s=0}^{k} \sum_{t=0}^{m} \left(-\tfrac{1}{2}\right)^{r+s+t} \times \frac{(r+s+t)!}{r!s!t!} \binom{j}{r} \binom{k}{s} \binom{m}{t}.$$

To change this to the number of rearrangements where an even number of balls *move* minus the number of rearrangements where an odd number of balls *move* we multiply by $(-1)^n$ and this is the right hand side of (1.3) which completes the proof.

It is apparent that there is nothing special about three boxes, and the proof above generalizes to N boxes with the corresponding integral of N Laguerre polynomials on the right hand side of (1.1).

It is only fair to say that although the above proof is simpler than the original, the identity would not easily have been discovered this way since no combinatorial interpretation of the right hand side of (1.3) comes readily to mind. Hence the discovery of (1.1) had to follow the method of Askey and Ismail or some other means.

2. Earlier we mentioned that the above argument provides an elementary proof of the positivity of the integral

$$\int_0^\infty L_j(x)L_k(x)L_m(x)e^{-2x}dx.$$

By using the binomial theorem Ismail and Tamhankar [**5**] proved that the coefficient of $r^js^kt^m$ in $(r - s - t)^j(- r + s - t)^k(- r - s + t)^m$ is positive. As noted before this coefficient is $R_E - R_0$ which is

$$2^{j+k+m+1}\int_0^\infty L_j(x)L_k(x)L_m(x)e^{-2x}dx$$

by (1.1) which completes the proof.

Ismail and Tamhankar first proved the positivity of the coefficient of $r^js^kt^m$ in $(r - s - t)^j(- r + s - t)^k(- r - s)^m$ from which the positivity of the corresponding coefficient in $(r - s - t)^j(- r + s - t)^k(- r - s + t)^m$ follows immediately from the binomial theorem

$$(r - s - t)^j(- r + s - t)^k(- r - s + t)^m = \sum_{i=0}^{m} {}_mC_i t^{m-i} \times (r - s - t)^j(- r + s - t)^k(- r - s)^i.$$

The coefficient of $r^js^kt^m$ in $(r - s - t)^j(- r + s - t)^k(- r - s)^m$ also has a combinatorial meaning [**5**]. Consider again three boxes with j, k, and m balls and consider all possible rearrangements such that no ball remains in the last box; i.e., the box with m balls is a derangement box. Then the coefficient of $r^js^kt^m$ equals $D_E - D_0$ where D_E is the number of rearrangements where an even number of balls change boxes and D_0 the number where an odd number change. Notice that at least $2m$ balls must always change boxes and the total number of rearrangements is

$${}_{(n-m)}C_m m!\,(n - m)!/j!k!m! = {}_{(n-m)}C_m(n - m)!/j!k!\,.$$

Here $n = j + k + m$.

It follows from Askey, Ismail and Koornwinder [**2**], again via the Master Theorem, that this number equals an integral of Laguerre polynomials:

$$\text{(2.1)} \quad D_E - D_0 = 2^{j+k}\int_0^\infty e^{-x}L_j(\tfrac{1}{2}x)L_k(\tfrac{1}{2}x)L_m(x)dx.$$

They actually prove a more general result:

Letting $B(j, k, m)$ be the coefficient of $r^j s^k t^m$ in

$$[(1-\lambda)r - \sqrt{\lambda(1-\lambda)}s - \sqrt{\lambda}t]^j[-\sqrt{\lambda(1-\lambda)}r + \lambda s - \sqrt{1-\lambda}t]^k \times [-\sqrt{\lambda}r - \sqrt{1-\lambda}s]^m,$$

(2.2) $$B(j,k,m) = \int_0^\infty e^{-x}L_j(\lambda x)L_k((1-\lambda)x)L_m(x)dx.$$

We now prove (2.1) by extending the method in section 1. First we make a trivial change of variables

(2.3) $$D_E - D_0 = 2^{j+k+1}\int_0^\infty L_j(x)L_k(x)L_m(2x)e^{-2x}dx.$$

Substituting from (1.2) and integrating:

(2.4) $$D_E - D_0 = 2^{j+k}\sum_{r=0}^{j}\sum_{s=0}^{k}\sum_{t=0}^{m}(-1)^{r+s+t}(r+s+t)!\,{}_jC_r\,{}_kC_s\,{}_mC_t/2^{r+s}r!s!t!$$

which is the same as the right hand side of (1.3) except a factor of 2^t is missing in the denominator. Because of this we can perform the t sum,

$$\sum_{t=0}^{m}(-1)^t\,{}_mC_t(r+s+t)!/t! = \sum_{t=0}^{m}(-1)^t\,{}_mC_t(t+1)_{r+s}$$

where $(a)_n \equiv a(a+1)\ldots(a+n-1)$. Replacing t by $m-t$, this is

$$(-1)^m\sum_{t=0}^{m}(-1)^t\,{}_mC_t(m-t+1)_{r+s}.$$

This is a Vandermonde sum. Using that summation formula or the binomial coefficient identity (12.18) in [**4**, p. 65] this equals $(-1)^m\,{}_{r+s}C_m(r+s)!$. Substituting back in (2.4),

(2.5) $$D_E - D_0 = (-1)^m 2^{j+k}\sum_{r=0}^{j}\sum_{s=0}^{k}(-\tfrac{1}{2})^{r+s}(r+s)!\,{}_jC_r\,{}_kC_s\,{}_{(r+s)}C_m/r!s!$$

We now show (2.5) by counting $D_E - D_0$ using Theorem 1. But now we only need the events $A_1, \ldots, A_{j+k}$ from before. The number of ways that r of the events $A_1, \ldots, A_j$ and s of the events $A_{j+1}, \ldots, A_{j+k}$ can occur is

$${}_jC_r j^{(r)}\,{}_kC_s k^{(s)}\,{}_{n-m-r-s}C_m m!(n-m-r-s)!/j!k!m!.$$

Then by Theorem 1, the number of rearrangements leaving M balls fixed is

$$N_M = \sum_{r=0}^{j}\sum_{s=0}^{k}(-1)^{r+s-M}\,{}_{r+s}C_M\,{}_jC_r\,{}_kC_s\,{}_{(n-m-r-s)}C_m(n-m-r-s)!/(j-r)!(k-s)!$$

with the restriction that $r+s \leqq n-2m$. Therefore the number of rearrangements leaving an even number of balls *fixed* minus the number of rearrangements leaving an odd number of balls *fixed* is

$$\sum_{M=0}^{n-2m}(-1)^M N_M.$$

Substituting for N_M and bringing the summation over M inside,

$$\sum_0^{n-2m}(-1)^M N_M = \sum_{r=0}^{j}\sum_{s=0}^{k}(-2)^{r+s}(n-m-r-s)! \times {}_jC_r\,{}_kC_s\,{}_{(n-m-r-s)}C_m/(j-r)!(k-s)!$$

with $r + s \leqq n - 2m$. Replacing r by $j - r$ and s by $k - s$ and using $n = j + k + m$ gives

$$\sum (-1)^M N_M = (-1)^n(-1)^m 2^{j+k} \sum_{r=0}^{j} \sum_{s=0}^{k} (-\tfrac{1}{2})^{r+s}(r+s)! \times {}_jC_r\, {}_kC_s\, {}_{(r+s)}C_m/r!s!$$

with $r + s \geqq m$. To change this to $D_E - D_0$, the number of rearrangements where an even number *move* minus the number where an odd number *move*, we multiply by $(-1)^n$. Noting also that $r + s \geqq m$ is now superfluous we have (2.5) which ends the proof.

Using the result of Ismail and Tamhankar [**5**] that the coefficient of $r^j s^k t^m$ in $(r - s - t)^j(-r + s - t)^k(-r - s)^m$ is positive we have a new proof of the positivity of

$$\int_0^\infty e^{-x} L_j(\tfrac{1}{2}x) L_k(\tfrac{1}{2}x) L_m(x)dx.$$

3. It now is natural to try to extend this argument to prove (2.2). Ismail and Tamhankar [**5**] proved the positivity of $B(j, k, m)$ as before, by an elementary application of the binomial theorem, so the positivity of the integral in (2.2) (Koornwinder's inequality [**2**], [**7**]) follows from the identity (2.2). However, now the weights assigned to different rearrangements are distinctly inhomogeneous due to the inhomogeneity of

$$[(1-\lambda)x - \sqrt{\lambda(1-\lambda)}y - \sqrt{\lambda}z]^j \times [-\sqrt{\lambda(1-\lambda)}x + \lambda y - \sqrt{1-\lambda}z]^k[-\sqrt{\lambda}x - \sqrt{1-\lambda}y]^m$$

and for each M several different weights are assigned to the rearrangements leaving M balls fixed. Hence a more refined argument is needed.

Evaluating the integral in (2.2) as before

$$(3.1)\quad \int_0^\infty e^{-x} L_j(\lambda x) L_k((1-\lambda)x) L_m(x)dx = \sum_{r=0}^{j}\sum_{s=0}^{k}\sum_{t=0}^{m} (-1)^{r+s+t}\lambda^r(1-\lambda)^s(r+s+t)!\,{}_jC_r\,{}_kC_s\,{}_mC_t/r!s!t!.$$

Now by the binomial theorem

$$B(j,k,m) = \sum_{r=0}^{j}\sum_{s=0}^{k} (1-\lambda)^r\lambda^s\,{}_jC_r\,{}_kC_s(-1)^{n-r-s} \times (\text{coefficient of } x^{j-r}y^{k-s}z^m \text{ in } \sqrt{\lambda(1-\lambda)}y + \sqrt{\lambda}z]^{j-r}[\sqrt{\lambda(1-\lambda)}x + \sqrt{1-\lambda}z]^{k-s} \times [\sqrt{\lambda}x + \sqrt{1-\lambda}y]^m).$$

When this is expanded, each term $x^{j-r}y^{k-s}z^m$ has the same coefficient,

https://doi.org/10.4153/CJM-1979-034-7 Published online by Cambridge University Press

$\lambda^{j-r}(1-\lambda)^{k-s}$. Thus,

$$\text{(3.2)}\quad B(j,k,m) = (-1)^n \sum_{r=0}^{j} \sum_{s=0}^{k} (-1)^{r+s}(1-\lambda)^r\lambda^s \, {}_jC_r\, {}_kC_s \lambda^{j-r}(1-\lambda)^{k-s} \times (\text{coefficient of } x^{j-r}y^{k-s}z^m \text{ in } [y+z]^{j-r}[x+z]^{k-s}[x+y]^m).$$

It is not difficult to see that the coefficient of $x^jy^kz^m$ in

$$(y+z)^j(x+z)^k(x+y)^m$$

is $D(j, k, m)$, the number of derangements of three boxes containing j, k, and m balls respectively; i.e., the number of rearrangements such that the number of balls in each box remains unchanged and no ball remains in its original box. From Askey, Ismail, and Rashed [**3**], or Jackson [**6**],

$$\text{(3.3)}\quad D(j,k,m) = (-1)^n \int_0^\infty e^{-x} L_j(x) L_k(x) L_m(x) dx$$

where $n = j + k + m$. Actually they give the corresponding formula for an arbitrary number of boxes. In both references a proof of (3.3) is given using the principle of inclusion and exclusion, the main tool we have been using here.

Evaluating the integral in (3.3)

$$\text{(3.4)}\quad D(j,k,m) = (-1)^n \sum_{r=0}^{j} \sum_{s=0}^{k} \sum_{t=0}^{m} (-1)^{r+s+t} {}_jC_r\, {}_kC_s\, {}_mC_t \times (r+s+t)!/r!s!t!.$$

Since $D(j, k, m)$ is also given by formula (1.4) with $M = 0$, by making the substitutions $r \to j - r$, $s \to k - s$, $t \to m - t$ we can arrive at (3.4) directly here in our notation.

Writing (3.2) as

$$B(j,k,m) = (-1)^n\lambda^j(1-\lambda)^k \sum_{r=0}^{j} \sum_{s=0}^{k} (-1)^{r+s}\lambda^{s-r}(1-\lambda)^{r-s} \times {}_jC_r\, {}_kC_s D(j-r, k-s, m)$$

and substituting for $D(j - r, k - s, m)$ from (3.4),

$$B(j,k,m) = (-1)^n\lambda^j(1-\lambda)^k \sum_{r=0}^{j} \sum_{s=0}^{k} (-1)^{r+s}\lambda^{s-r}(1-\lambda)^{r-s} \times {}_jC_r\, {}_kC_s (-1)^{j-r+k-s+m} \sum_{u=0}^{j-r} \sum_{v=0}^{k-s} \sum_{t=0}^{m} (-1)^{u+v+t} \times (u+v+t)!\, {}_{(j-r)}C_u\, {}_{(k-s)}C_v\, {}_mC_t/u!v!t!.$$

The upper limits in the u and v sums can be changed to j and k respectively since this only introduces zero terms. Therefore

$$B(j,k,m) = \lambda^j(1-\lambda)^k \sum_{r=0}^{j} \sum_{s=0}^{k} ((1-\lambda)/\lambda)^{r-s}\, {}_jC_r\, {}_kC_s \times \sum_{u=0}^{j} \sum_{v=0}^{k} \sum_{t=0}^{m} (-1)^{u+v+t}\, {}_{(j-r)}C_u\, {}_{(k-s)}C_v\, {}_mC_t (u+v+t)!/u!v!t!.$$

Interchanging the order of summation, performing the sums and then relabeling u, v as r and s,

$$B(j,k,m) = \sum_{r=0}^{j} \sum_{s=0}^{k} \sum_{t=0}^{m} (-1)^{r+s+t}\lambda^r(1-\lambda)^s\, {}_jC_r\, {}_kC_s\, {}_mC_t \times (r+s+t)!/r!s!t!$$

which is the right hand side of (3.1). This completes the proof of the identity (2.2).

It appears that the elementary approach used in this paper is a general method for proving Laguerre polynomial identities in combinatorics.

Acknowledgements. The author wishes to thank Richard Askey who first related the above problem to him and Mourad Ismail for many valuable suggestions regarding the content of the paper.

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