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THE LOCAL CLASS GROUP OF A KRULL DOMAIN

BY
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ABSTRACT. The local class group of a Krull domain A is the quotient group $G(A) = \mathrm{Cl}(A)/\mathrm{Pic}(A)$. A Krull domain A is locally factorial if and only if $G(A) = 0$. In this paper, we characterize the Krull domains for which $G(A)$ is a torsion group. We evaluate the local class group of several examples and finally, we explain why every abelian group is the local class group of a Krull domain.

Let A be a Krull domain, $\mathrm{Pic}(A)$ its Picard group, Cl(A) its class group and $G(A) = \mathrm{Cl}(A)/\mathrm{Pic}(A)$ the quotient group, called the *local class group of A*. As we pointed out in [8], *the Krull domain A is locally factorial if and only if* $G(A) = 0$. Since the size of $G(A)$ indicates how far a Krull domain is from being a locally factorial ring, it seemed natural to study the local class-group of Krull domains, namely:

Given A, how does one compute $G(A)$?

Given $G(A)$, what can one conclude about A?

What are the abelian groups which are the local class group of a Krull domain?

In this paper, we characterize the Krull domains for which $G(A)$ is a torsion group. We evaluate the local classgroup of $A(x_1, \ldots, x_n)$, $A[\![x]\!]$, $S^{-1}A$ (in some cases), $A[x_\alpha]_{\alpha \in \Lambda}$ and more generally those of some semigroup Krull domains. Finally we explain why any group is the local class-group of some Krull domain following recent results from D. F. Anderson and L. Chouinard.

§1. **Terminology and notations.** Terminology and notations are mainly from [7] and [10] for commutative algebra; [14] is the reference for graded domains.

Let A be a domain; we write $\mathrm{Spec}(A)$ for its prime spectrum, $\mathrm{Max}(A)$ for its maximal spectrum and $X^{(1)}(A)$ for the set of height one prime ideals of A. A nonzero (fractional) ideal is called *divisorial* if it is the intersection of a nonempty family of principal (fractional) ideals. Let $I(A)$ be the monoid of nonzero ideals of A and $D(A)$ the subset of divisorial ideals. We write $\mathrm{Div}(A)$ for the set of *divisors* of A; that is, the quotient of $I(A)$ by the Artin congruence defined by $I \equiv J$ if and only if $A : I = A : J$. Let div: $I(A) \to \mathrm{Div}(A)$

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be the canonical surjection; its restriction to $D(A)$ is a bijection. We write $I^{-1} = A : I$ and $(I^{-1})^{-1}$ is the divisorial ideal associated to I. If P is a prime ideal and m an integer, $P^{(m)}$ denotes the usual symbolic power of P; if A is a Krull domain, it is the divisorial ideal associated to P^m. Let $P(A)$ be the group of principal ideals and $\mathrm{Cart}(A)$ *the Cartier group* of invertible ideals of A; every invertible ideal is divisorial.

Let A be a Krull domain; then $\mathrm{Div}(A)$ is a free abelian group with $\{\mathrm{div}\, P : P \in X^{(1)}(A)\}$ as a basis. The subgroup of *principal divisors* is denoted by $\mathrm{Prin}(A)$. For the law $I * J = A : (A : IJ)$, $D(A)$ is a group isomorphic to $\mathrm{Div}(A)$. The quotient group $\mathrm{Cl}(A) = \mathrm{Div}(A)/\mathrm{Prin}(A) = D(A)/P(A)$ is called the *class group of* A and the canonical image of $\mathrm{Cart}(A)$ in $\mathrm{Cl}(A)$ is the *Picard group of* A, denoted by $\mathrm{Pic}(A)$. If I is a ideal, its image in $\mathrm{Cl}(A)$ is denoted by $[I]$.
Let A and B be two Krull domains and $A \to B$ be a homomorphism satisfying (NBU)—see [10, p. 30]. Then, there exists a canonical homomorphism $\psi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ defined by $\psi([I]) = [IB]$, such that $\psi(\mathrm{Pic}(A)) \subset \mathrm{Pic}(B)$.

§2. **The local class group of a Krull domain**

Definition. Let A be a Krull domain; the *local class group of* A is the group $G(A)$ defined by

$$G(A) = \mathrm{Cl}(A)/\mathrm{Pic}(A) = D(A)/\mathrm{Cart}(A).$$

We choose this terminology to recall that the local class group gives local information on A and the notation $G(A)$ because it has already been used in [6]. The local class group is generated by the $[P] + \mathrm{Pic}(A)$, where $P \in X^{(1)}(A)$.

Let A be a Krull domain; A is *factorial* (respectively *almost factorial* [15]) if $\mathrm{Cl}(A) = 0$ (respectively if $\mathrm{Cl}(A)$ is a torsion group). In the same way, we say A is *locally factorial* or *almost locally factorial* if $G(A) = 0$ or if $G(A)$ is a torsion group. Locally factorial domains (respectively almost locally factorial π-domains), are sometimes called π-domains (respectively almost π-domains), for instance in [2] or [11]. In [8], several characterizations of locally factorial Krull domains are given. For convenience, we recall some of them, without proof.

Lemma 1. *For a Krull domain A, the following assertions are equivalent*:

(i) *A is a locally factorial Krull domain*;
(ii) *A_M is factorial for each $M \in \mathrm{Max}(A)$*;
(iii) *A_p is factorial for each $p \in \mathrm{Spec}(A)$*;
(iv) *Every height one prime ideal is invertible.*

Factorial domains, Dedekind domains and all regular domains are locally factorial Krull domains; a locally factorial Krull domain A is factorial if and only if $\mathrm{Pic}(A) = 0$.

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As $0 \to \text{Pic}(A) \to \text{Cl}(A) \to G(A) \to 0$ is an exact sequence of groups, a Krull domain A is almost factorial if and only if it is almost locally factorial and $\text{Pic}(A)$ is a torsion group. So, if A is a Krull domain with $\text{Pic}(A)=0$, then A is almost factorial if and only if A is almost locally factorial. In particular, this holds whenever A is semilocal.

In [6] an example is given of a (noetherian) Krull domain A which is almost locally factorial but not almost factorial nor locally factorial (i.e. $G(A)$ is a nonzero torsion group while $\text{Cl}(A)$ is not a torsion group). Other examples appear in [12].

Let A and B be two Krull domains and $A \to B$ a homomorphism satisfying (NBU); the homomorphism from $G(A)$ to $G(B)$ defined by $[I]+\text{Pic}(A) \mapsto [IB]+\text{Pic}(B)$ is called the *canonical homorphism*. It is clear that if $\text{Cl}(A) \to \text{Cl}(B)$ is onto (for instance if $B = S^{-1}A$ or $B = A[X_\alpha]_{\alpha\in\Lambda}$) then, so is $G(A) \to G(B)$. If $\text{Cl}(A) \to \text{Cl}(B)$ is an isomorphism, then there exists an exact sequence $0 \to \text{Coker}(\text{Pic}(A) \to \text{Pic}(B)) \to G(A) \to G(B) \to 0$.

THEOREM 2. *A Krull domain A is almost locally factorial if and only if A_M is almost factorial for every $M \in \text{Max}(A)$.*

Proof. If A is an almost locally factorial Krull domain, then $G(A)$ is a torsion group and so is $\text{Cl}(A_M) = G(A_M)$. Conversely(*), let $P \in X^{(1)}(A)$ and $M \in \text{Max}(A)$. Because A_M is almost factorial, there exists an integer s_M such that $P^{(s_M)}A_M$ is invertible. For every $n \geq 1$, let $I_n = p^{(n)}(P^{(n)})^{-1}$. Because $I_{s_M}A_M$ is not contained in MA_M, one has $\sum_{n\geq 1} I_n = A$. So there exists $N \geq 1$ such that $A = \sum_{n=1}^{N} I_n$. Let $r = N!$; then $A = \sum_{n=1}^{r} I_n^{r/n} \subset P^{(r)}(P^{(r)})^{-1}$ and so $P^{(r)}(P^{(r)})^{-1} = A$. □

As a corollary, we get now a result from [6]:

COROLLARY 3. *A noetherian Krull domain A is almost locally factorial if and only if A_M is almost factorial for every $M \in \text{Max}(A)$*

Theorem 2 shows that almost locally factorial Krull domains are the same thing as locally almost factorial Krull domains.

The relationship between locally factorial Krull domains and almost factorial Krull domains can be made more precise; for a Krull domain A, the following statements are equivalent:

(i) A is locally factorial;

(ii) A is almost locally factorial and for every $P \in X^{(1)}(A)$ and every $n > 0$, one has $P^n = P^{(n)}$.

Indeed, (i)⇒(ii) obviously. (ii)⇒(i) because if $M \in \text{Max}(A)$ and $PA_M \in$

(*) This proof has been communicated to us by M. Chamarie. Another proof will appear in "Globalization of some local properties in Krull domains" by D. D. Anderson.

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$X^{(1)}(A_M)$, then, there exists $s>0$ such that $P^sA_M = P^{(s)}A_M$ is invertible; so PA_M is invertible and thus principal.

Other characterizations of *noetherian* locally almost factorial Krull domains can be found in [12].

Remark 4.(*) Let A be a Krull domain and S a multiplicatively closed subset. There exists a canonical homomorphism $\mathrm{Cl}(A) \to G(S^{-1}A)$.

Suppose $(M_i)_{i\in I}$ is the set of maximal ideals such that A_{M_i} is not factorial, $S = A - \bigcup M_i$ and $\varphi : \mathrm{Cl}(A) \to G(S^{-1}A)$ the canonical homomorphism.

If $[I] \in \operatorname{Ker} \varphi$ and I is divisorial, then I is locally principal. If locally principal ideals are finitely generated then $G(S^{-1}A) = G(A)$.

Suppose now that $A \to B$ is a faithfully flat homomorphism between two Krull domains and let us consider the following commutative diagram with exact rows, where all the homomorphisms are canonical:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Pic}(A) & \to & \mathrm{Cl}(A) & \to & G(A) & \to & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \to & \mathrm{Pic}(B) & \to & \mathrm{Cl}(B) & \to & G(B) & \to & 0 \end{array}$$

Because $\operatorname{Ker} u \to \operatorname{Ker} v$ is an isomorphism, [10; 6–10], using the snake lemma, one has:

Lemma 5. *With the previous notations*:

(a) *If u and v are onto, then* $G(A) \simeq G(B)$;

(b) *If u is an isomorphism, there exists an exact sequence* $0 \to G(A) \to G(B) \to \mathrm{Coker}(\mathrm{Cl}(A) \to \mathrm{Cl}(B)) \to 0$.

Given a domain A, we denote by $A(x_1, \ldots, x_n)$ the domain $S^{-1}A[x_1, \ldots, x_n]$ where S is the multiplicatively closed set of polynomials whose coefficients generate the unit ideal in A.

Corollary 6. *Let A be a Krull domain*;

(a) $G(A(x_1, \ldots, x_n)) = \mathrm{Cl}(A(x_1, \ldots, x_n)) = G(A)$.

(b) *If A is noetherian, there exists an exact sequence* $0 \to G(A) \to G(A[\![x]\!]) \to \mathrm{Coker}(\mathrm{Cl}(A) \to \mathrm{Cl}(A[\![x]\!]) \to 0$.

Proof. (a) $A \to A(x_1, \ldots, x_n)$ is a faithfully flat extension; by [1], $\mathrm{Pic}(A(x_1, \ldots, x_n)) = 0$ and $\mathrm{Cl}(A) \to \mathrm{Cl}(A(x_1, \ldots, x_n))$, which is composed of two surjective homomorphisms, is onto.

(b) $A \to A[\![x]\!]$ is faithfully flat and by [10, 18–21], $\mathrm{Pic}\, A \to \mathrm{Pic}\, A[\![x]\!]$ is an isomorphism. □

From the corollary, it follows:

(a) A is locally factorial (resp. almost locally factorial) if and only if $A(x_1, \ldots, x_n)$ is factorial (resp. almost factorial).

(*) Basically in [6] and [10] when A is noetherian.

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(b) If $A[\![x]\!]$ is locally factorial or almost locally factorial, so is A.

(c) $G(A[\![x]\!]) = G(A)$ if and only if $\mathrm{Cl}(A[\![x]\!]) = \mathrm{Cl}(A)$.

(d) If A is a factorial domain such that $A[\![x]\!]$ is not factorial—see [10; § 19]—then, $G(A) = \mathrm{Cl}(A) = 0$, but $G(A[\![x]\!]) = \mathrm{Cl}(A[\![x]\!]) \neq 0$.

Question. When is $G(A[\![x]\!])$ isomorphic to $G(A)$?

COROLLARY 7. *Let A be a Krull domain; then*

$$G(A[x_\alpha]_{\alpha\in\Lambda}) = G(A).$$

So, $A[x_\alpha]_{\alpha\in\Lambda}$ is locally factorial—resp. almost locally factorial—if and only if so is A.

As in [14], we call a *grading semigroup* any commutative cancellative semigroup with zero; such a semigroup Γ is called *torsionless* if its total group of quotients $\langle\Gamma\rangle$ is without torsion. Any torsionless grading semigroup can be endowed with a total order compatible with its structure as a semigroup. Let $-\Gamma$ be the set of inverses of elements of Γ.

Let Γ be a torsionless grading semigroup; we say $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is a Γ-*graded domain* (or a graded domain) if each A_γ is an additive subgroup of A and if $A_\gamma A_\delta \subset A_{\gamma+\delta}$. The elements of $\bigcup A_\gamma$ are the *homogeneous elements* and γ is called the degree of the nonzero elements of A_γ.

An ideal I of A is called a *homogeneous ideal* provided I is generated by homogeneous elements or equivalently if $I = \bigoplus (I \cap A_\gamma)$. Let $I \subset A$ be a nonzero homogeneous ideal; then $A:(A:I)$ is homogeneous; if I and J are two homogeneous ideals so is $A:(A:IJ)$. Then, the set $D_h(A)$, of divisorial homogeneous ideals, is a subgroup of $D(A)$. Let $P_h(A) = D_h(A) \cap P(A)$. When A is a Krull domain, Anderson has showed [3]: $\mathrm{Cl}(A)$ is isomorphic to $D_h(A)/P_h(A)$. The following lemma is a slight refinement of [3; 6.2].

LEMMA 8. *Let Γ be a semigroup isomorphic to a sub-semigroup of some $\mathbb{N}^{(I)}$ and let $A = \bigoplus A_\gamma$ be a graded Krull domain. Then*

$$\mathrm{Pic}(A) = \mathrm{Pic}(A_0).$$

Proof. For any $i \in I$, let $pr_i : \mathbb{N}^{(I)} \to \mathbb{N}$ be the i-th projection; for any $n \in \mathbb{N}$, let $n(\Gamma) = \{\gamma \in \Gamma / \sum \mathrm{pr}_i(\gamma) = n\}$ and let $B_n = \bigoplus_{\gamma\in n(\Gamma)} A_\gamma$. Then $A = \bigoplus_{n\geqslant o} B_n$ and $B_0 = A_0$. Now, by [3–6.2] one has

$$\mathrm{Pic}(A) = \mathrm{Pic}(B_0) = \mathrm{Pic}(A_0). \quad \square$$

So, with the above hypothesis, if A_0 is a field, then $\mathrm{Pic}(A) = 0$.

Now we can state our first result on the local class group of a graded Krull domain:

THEOREM 9. Let Γ be a semigroup isomorphic to a sub-semigroup of some $\mathbb{N}^{(I)}$ and let $A = \bigoplus A_\gamma$ be a graded Krull domain. Then, there exists an injection

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$G(A_0) \to G(A)$ and if $A_0 \to A$ satisfies (NBU), then the map is a monomorphism.

Proof. Because $\Gamma \cap (-\Gamma) = 0$, it is a consequence of [3; 6.4] and Lemma 8. The map $G(A_0) \to G(A)$ is defined by

$$[I] + \mathrm{Pic}(A_0) \mapsto [A : (A : IA)] + \mathrm{Pic}(A). \quad \square$$

Let A be a domain and Γ a torsionless grading semigroup. We denote by $A[\Gamma]$ the semigroup domain whose elements are of the form $\sum a_\gamma X^\gamma$ with $X^\gamma X^\delta = X^{\gamma+\delta}$. It is proved in [9] that the domain $A[\Gamma]$ is a Krull domain if and only if A is a Krull domain and Γ is isomorphic to a direct product $\Gamma = G_x \Gamma^*$ where

(1) G is a torsion free commutative group satisfying the ascending chain condition on cyclic subgroups;

(2) Γ^* is a subsemigroup of some free group $F = \mathbb{Z}^{(I)}$ and $\langle \Gamma^* \rangle \cap \mathbb{N}^{(I)} = \Gamma^*$. Furthermore, for such a Γ, [9, Th. 2] claims there exists such a canonical decomposition; we use it to state the next two results.

THEOREM 10. *Let $A[\Gamma]$ be a Krull semigroup domain. With the previous notations*:

$$G(A[\Gamma]) = G(A) \oplus (F/\langle \Gamma^* \rangle)$$

Proof. By [9], we have $\mathrm{Cl}(A[\Gamma]) = \mathrm{Cl}(A) \oplus (F/\langle \Gamma^* \rangle)$. Following [3–6.2], if A is a Krull domain and F a free abelian group finitely generated, then $\mathrm{Pic}(A[F]) = \mathrm{Pic}(A)$. So, by a direct limit argument and Lemma 8, we obtain

$$\mathrm{Pic}(A[\Gamma]) = \mathrm{Pic}(A[\Gamma^*][G]) = \mathrm{Pic}\,(A[\Gamma^*]) = \mathrm{Pic}(A)$$

which concludes the proof of the proposition. $\square$

COROLLARY 11.

(a) *$A[\Gamma]$ is a locally factorial Krull domain if and only if A is a locally factorial Krull domain and $\langle \Gamma^* \rangle = \mathbb{Z}^{(I)}$.*

(b) *$A[\Gamma]$ is an almost locally factorial Krull domain if and only if A is an almost locally factorial Krull domain and $\mathbb{Z}^{(I)}/\langle \Gamma \rangle$ is a torsion group.*

Part (a) of this corollary will apear also in [5] with a different proof.

If A is a locally factorial Krull domain and G a group satisfying the ascending chain condition on cyclic subgroups, then $B = A[G][x_i, x_i^{-1}]_{i \in I}$ is a locally factorial Krull domain and $G(B) = G(A)$.

If A is an almost locally factorial Krull domain, so is $B = A[x^n, xy, y^n]$ and $G(B) = G(A) \oplus \mathbb{Z}/n\mathbb{Z}$. Furthermore, it is known [3] that $\mathbb{Z}^{(I)}/\langle \Gamma^* \rangle$ is a torsion group if and only if $k[\Gamma^*] \to k[x_i]_{i \in I}$ is integral, where k is a field.

COROLLARY 12. *Let A be a Krull domain*; *then*

$$G(A[x_\alpha, x_\alpha^{-1}]_{\alpha \in \Lambda}] = G(A).$$

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In [9], Chouinard has proved that any abelian group is the class group of some local Krull domain A; so:

THEOREM 13. *Every abelian group is the local class-group of a Krull domain.*

As a Krull domain A of dimension 1 is a Dedekind domain, one has $G(A)=0$. In other words, for a Krull domain A, $G(A)\neq 0$ implies dim $A\geq 2$.

Question. Is any group the local class-group of a Krull domain of dimension 2?

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