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# Holomorphic Variations of Minimal Disks with Boundary on a Lagrangian Surface

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*Abstract.* Let $L$ be an oriented Lagrangian submanifold in an $n$-dimensional Kähler manifold $M$. Let $u\colon D \to M$ be a minimal immersion from a disk $D$ with $u(\partial D) \subset L$ such that $u(D)$ meets $L$ orthogonally along $u(\partial D)$. Then the real dimension of the space of admissible holomorphic variations is at least $n + \mu(E, F)$, where $\mu(E, F)$ is a boundary Maslov index; the minimal disk is holomorphic if there exist $n$ admissible holomorphic variations that are linearly independent over $\mathbb{R}$ at some point $p \in \partial D$; if $M = \mathbb{C}P^n$ and $u$ intersects $L$ positively, then $u$ is holomorphic if it is stable, and its Morse index is at least $n + \mu(E, F)$ if $u$ is unstable.

## 1 Introduction

Let $L$ be a submanifold in a Riemannian manifold $M \subset \mathbb{R}^K$ and let $\Sigma$ be a compact Riemann surface with nonempty boundary $\partial\Sigma$. Consider the space $\Omega$ of maps $u$ from $\Sigma$ to $M$ with $u(\partial\Sigma)$ in $L$ that are continuous on $\overline{\Sigma}$ and in the Sobolev space $H^1(\Sigma)$. The first variation of the Dirichlet energy at $u$ for an admissible variation field $V$, $V \in H^1(\Sigma, u^*(TM))$ with $V$ tangent to $L$ along $\partial\Sigma$, is

$$\delta E(u)(V) = -\int_\Sigma \langle \Delta u - A_u(du, du), V\rangle \ da + \int_{\partial\Sigma} \langle \nu, V\rangle \ ds,$$

where $\nu$ is the outward pointing unit normal of $\partial\Sigma$ along $u$ and $A$ is the second fundamental form of $M$ in $\mathbb{R}^K$. A critical point of the energy on $\Omega$ is a harmonic map $u$ such that $u(\Sigma)$ meets $L$ orthogonally along $u(\partial\Sigma)$; that is, satisfying the boundary condition $\nu(z) \perp T_{u(z)}L, z \in \partial\Sigma$. The Morse index of $E$ at a critical point $u$ is the maximal dimension of a subspace of admissible variations $X$ on which the second variation of $E$ is negative, *i.e.*, $\delta^2 E(u)(X, X) < 0$. A critical point is stable if its index is zero. In the case where $\Sigma = D$, the unit disk in $\mathbb{R}^2$, a critical point from the disk is a conformal branched minimal immersion, and it is well known that the Morse index of the energy $E$ at a critical point $u$ is equal to the Morse index of the area of the minimal disk $u(D)$.

An important free boundary problem that arises in complex and symplectic geometry is the problem of constructing holomorphic disks with boundary on a closed Lagrangian submanifold $L$ of $\mathbb{C}^n$, or in a Kähler manifold $M$. Because of the geometry, such a disk is necessarily a minimizer for the free boundary problem. It is natural to try to construct such disks by first constructing an area minimizing disk

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for the free boundary problem, and then using the second variation to show that the disk is holomorphic. There are a few cases where stable minimal surfaces have been shown to be holomorphic. Siu–Yau [7] proved that any stable minimal two-sphere in a Kähler manifold with positive holomorphic bisectional curvature is holomorphic. Micallef [4] proved that any stable minimal immersion of $\mathbb{C}$ into $\mathbb{R}^4$ is holomorphic with respect to an orthogonal complex structure on $\mathbb{R}^4$. Arezzo [1] has found a sufficient condition under which orientable complete stable minimal surfaces into a hyper-Kähler 4-manifold $M$ are holomorphic with respect to some orthogonal complex structure on $M$. All of these involve a complex formula for the second variation of energy or area. This complex second variation formula is also very important in obtaining index estimates for minimal surfaces. When the ambient space $M$ is of dimension four with positive isotropic curvature (see [5], [6] for a background on this curvature condition), any nontrivial minimal two-sphere in $M$ has index at least two [5], and any minimal two-disk that is a solution to the free boundary problem inside a domain in $M$ with two-convex boundary has index at least one [2]. These index estimates are useful in studying the relationship between the curvature and topology of manifolds.

A key step in applying the complex second variation formula is to produce holomorphic variation fields. In the case of minimal two-spheres, one can appeal to the classical Riemann–Roch formula for existence of holomorphic variations. In the case of the free boundary problem, it is necessary to construct holomorphic variations satisfying appropriate boundary conditions. A natural setting for finding these variations, when the constraint submanifold $L$ is Lagrangian in a Kähler manifold $M$, is to consider a pair of vector bundles $(E, F)$ over the immersed surface $(\Sigma, \partial\Sigma)$, where $E$ is the holomorphic tangent bundle of $M$ restricted to $\Sigma$ and $F$ is the subbundle of $E$ along $\partial\Sigma$ determined by the tangent spaces of $L$. By the Riemann–Roch theorem in [3], the real dimension of the space of admissible holomorphic variations is at least $n + \mu(E, F)$, where $\mu(E, F)$ is the boundary Maslov index.

In this paper we prove the following.

***Theorem 1.1*** *Let $L$ be an oriented Lagrangian submanifold in an $n$-dimensional Kähler manifold $M$. Let $u\colon D \to M$ be a minimal immersion from a disk $D$ with $u(\partial D) \subset L$, and such that $u(D)$ meets $L$ orthogonally along $u(\partial D)$. Then*

(1) *if there exist $n$ admissible holomorphic variations that are linearly independent over $\mathbb{R}$ at some point $p \in \partial D$, then $u$ is holomorphic.*
(2) *if $M = \mathbb{C}P^n$ and $u$ intersects $L$ positively, then*
    (a) *if $u$ is stable, then $u$ is holomorphic*
    (b) *if $u$ is unstable, then $u$ has index at least $n + \mu(E, F)$.*

Stable minimal disks that solve the free boundary problem, in general, need not be holomorphic. The example below illustrates the importance of understanding how minimal disks intersect the constraint Lagrangian submanifold. We say that a minimal immersion $u\colon D \to M$ intersects a Lagrangian submanifold $L$ *positively* if for all $z \in \partial D$ and $X \in T_{u(z)}L$,

$$\langle \nabla_X X, \nu + JT \rangle \leq 0,$$

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where $\nu$ is the outward pointing unit normal of $u(D)$ along $u(\partial D)$ and $T$ is the positively oriented unit tangent vector along $u(\partial D)$. The quantity $\langle \nabla_X X, \nu + JT \rangle$ is the second fundamental form of $L$ in the normal direction determined by the disk. The condition on positive intersection holds when $L$ is a closed convex hypersurface and the disk stays inside $L$.

In view of the second variation formula, there are at least two reasons why a stable disk may not be holomorphic: whether the disk intersects the constraint Lagrangian submanifold positively and whether the ambient Kähler manifold is positively curved. For example, the Clifford torus $\mathbb{T}^2$ in $\mathbb{C}P^2$, given by $[1{:}e^{ix}{:}e^{iy}]$ for $x, y \in \mathbb{R}$, in the homogeneous coordinates of $\mathbb{C}P^2$, is Lagrangian and minimal. The complex line $[1{:}z{:}z]$, $z \in \mathbb{C}$, intersects $\mathbb{T}^2$ along a circle which bounds a holomorphic disk $D$ meeting $\mathbb{T}^2$ orthogonally. Because $\mathbb{T}^2$ is Lagrangian and by Stokes' Formula, $D$ is area minimizing among all disks with boundary homotopic to the circle in $\mathbb{T}^2$. One can also check that $[1{:}z{:}\bar{z}]$ meets $\mathbb{T}^2$ orthogonally along a circle and that the disk $D'$ it bounds is area minimizing among all disks whose boundary is homotopic to the circle in $\mathbb{T}^2$, by noticing that any homotopy of $D'$ results in a homotopy of $D$ by taking the complex conjugate in the last coordinate component and that the area of $D$ is the same as that of $D'$. The stable disk $D'$ is not holomorphic, and it does not intersect $\mathbb{T}^2$ positively. On the other hand, if one considers the similar example in $\mathbb{R}^4$, instead of in $\mathbb{C}P^2$, the two corresponding minimizing disks meet the constraint Lagrangian torus positively, and so the failure to be holomorphic for one disk is due to the ambient curvature.

We break the proof down as follows. In Section 2, we observe that the space of admissible holomorphic variations has real dimension at least $n\chi(\Sigma) + \mu(E, F)$, for an immersed Riemann surface $\Sigma$ in a Kähler manifold with boundary $\partial\Sigma$ in an oriented Lagrangian submanifold $L$. This follows from a Riemann–Roch Theorem in [3]. In Section 3 we prove (1), which gives a criterion for a surface, whose boundary lies on a Lagrangian submanifold, to be holomorphic, in terms of linear relations of the admissible holomorphic variations at a single boundary point. In Section 4, we derive a real formulation of the complex second variation formula and prove (2). The existence of admissible holomorphic variations in Section 2 and the characterization of holomorphicity via admissible variations in Section 3 are needed.

We would like to thank the referee for pointing out a mistake in Section 2 of the earlier version of this paper.

## 2 Construction of Admissible Holomorphic Variations

Let $\Sigma$ be a compact oriented Riemann surface with disjoint smooth boundary curves $c_1, \ldots, c_l$. Let $L$ be an oriented submanifold in an $n$-dimensional Kähler manifold $M$ which is Lagrangian with respect to the Kähler form $\omega$. Consider a smooth map

$$u\colon \Sigma \to M$$

such that $u(\Sigma)$ and $L$ intersect along the smooth curves $u(c_1), \ldots, u(c_l)$ where the outward normal derivative satisfies

$$\frac{\partial u}{\partial \nu}(z) \perp T_{u(z)}L, \quad z \in c_j.$$

The holomorphic tangent bundle $T^{1,0}M$ of $M$ pulls back to a smooth complex vector bundle $E = u^*(T^{1,0}M)$ over $\Sigma$. The complex linear extension of the Riemannian metric of $M$ to $TM \otimes \mathbb{C}$ pulls back to a complex bilinear form $\langle \cdot\, , \cdot \rangle$ on $E$. The complex linear extension of the Levi–Civita connection of $M$ to $TM \otimes \mathbb{C}$ preserves vectors of type $(1, 0)$ and hence pulls back to a connection $\nabla$ on $E$.

Let $\mathcal{A}^{p,q}(E)$ denote the space of $(p, q)$-forms on $\Sigma$ with values in $E$. The connection $\nabla$ divides into two components:

$$\nabla' : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{1,0}(E) \quad \nabla'' : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,1}(E)$$

and it is well known that there exists a unique holomorphic structure on $E$ with respect to which $\nabla'' = \bar{\partial}$, the $\bar{\partial}$-operator on $E$. With respect to this holomorphic structure, a section $V$ of $E$ is holomorphic

$$\Leftrightarrow \nabla'' V = 0 \Leftrightarrow \nabla_{\frac{\partial}{\partial \bar{z}}} V = 0$$

where $z = x + iy$ are local complex coordinates on $\Sigma$.

***Definition 2.1*** We call a section $s = X - iJX$ of $E$ an *admissible holomorphic variation* if

$$\begin{cases} \bar{\partial} s = 0 & \text{on } \Sigma \\ X(z) \in T_{u(z)}L & \text{for } z \in \partial\Sigma. \end{cases}$$

Let $F \subset E|_{\partial\Sigma}$ be the totally real subbundle whose fiber at the point $z \in \partial\Sigma$ is

$$F_z = \{X - iJX \mid X \in T_{u(z)}L\}$$

Then the Riemann–Roch Theorem of [3, Theorem C.1.10] implies the following:

***Theorem 2.2*** *Let $L$ be an oriented Lagrangian submanifold in an $n$-dimensional Kähler manifold $M$. Let $\Sigma$ be a Riemann surface with boundary $\partial\Sigma$ and let $u \colon \Sigma \to M$ be an immersion with $u(\partial\Sigma) \subset L$. Then the real dimension of the space of admissible holomorphic variations*

$$\mathcal{H} = \{V \in \Gamma(E) : \bar{\partial} V = 0 \textit{ on } \Sigma, \operatorname{Re} V(z) \in T_{u(z)}L \textit{ on } \partial\Sigma\}$$

*is at least $n\chi(\Sigma) + \mu(E, F)$, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$, and $\mu(E, F)$ is the boundary Maslov index (see [3, Section C.3]). In particular, the dimension is at least $n + \mu(E, F)$ if $\Sigma$ is of disk type.*

## 3 A Criterion for Being Holomorphic

As we have observed in Theorem 2.2, there are at least $n+\mu(E, F)$ independent admissible holomorphic variations for a disk type surface with boundary in a Lagrangian submanifold. In this section, we will prove in Theorem 3.2, without using the second variation formula, that a disk type solution to the free boundary problem must be holomorphic if there exist $n$ admissible holomorphic variations $V_1, \dots, V_n$ that are linearly independent at some point on the boundary of the disk. Theorem 3.2 will be used in the next section when we apply the second variation formula.

***Lemma 3.1*** *Let $L$ be an oriented Lagrangian submanifold in an $n$-dimensional Kähler manifold $M$. Let $\Sigma$ be a Riemann surface with boundary $\partial\Sigma$ and let $u\colon \Sigma \to M$ be an immersion with $u(\partial\Sigma) \subset L$. Assume that there exist admissible holomorphic variations $V_1, \dots, V_n$ that are linearly independent over $\mathbb{R}$ at a point $p \in \partial\Sigma$. If $W$ is an admissible anti-holomorphic variation with a zero, then $W(p) = 0$.*

**Proof** For $j = 1, \dots, n$ the functions

$$F_j = \langle V_j, \overline{W} \rangle$$

are holomorphic on $\Sigma$. In fact,

$$\frac{\partial F_j}{\partial \bar{z}} = \langle \nabla_{\frac{\partial}{\partial \bar{z}}} V_j, \overline{W} \rangle + \langle V_j, \overline{\nabla_{\frac{\partial}{\partial z}} W} \rangle = 0.$$

Since $V_j$ and $W$ are admissible and $L$ is Lagrangian, $F_j$ is real on $\partial\Sigma$. In fact, setting $V_j = X_j - iJX_j$ and $W = Y - iJY$, then along $\partial\Sigma$

$$F_j = \left\langle X_j - iJX_j, Y + iJY \right\rangle = 2 \left\langle X_j, Y \right\rangle .$$

Therefore $F_j$, $j = 1, \dots, n$, are constant functions on $\Sigma$.

Since $W$ has a zero by assumption, the functions $F_j$ are identically zero. But the complex dimension of $T^{1,0}_{u(z)}M$ is $n$ and the $n$ vectors $V_1(z), \dots, V_n(z)$ are each orthogonal to $W(z)$, therefore they must be linearly dependent over $\mathbb{C}$ whenever $W(z) \neq 0$. Since $V_1, \dots, V_n$ are linearly independent over $\mathbb{R}$ at a point $p \in \partial\Sigma$, it follows in particular that $V_1, \dots, V_n$ are nonzero at $p$. Suppose $W(p) \neq 0$. Then $V_1(p), \dots, V_n(p)$ are linearly dependent over $\mathbb{C}$,

$$\sum_{j=1}^{n} c_j V_j(p) = 0$$

for some $c_j = a_j + ib_j \in \mathbb{C}$, not all zero. Equivalently,

$$\sum_{j=1}^{n} \big( a_j X_j(p) + b_j JX_j(p) \big) = 0.$$

Since $V_1, \dots, V_n$ are admissible and $L$ is Lagrangian, this implies that

$$\sum_{j=1}^{n} a_j X_j(p) = 0 \quad \text{and} \quad \sum_{j=1}^{n} b_j JX_j(p) = 0.$$

But some $a_j$ or $b_j$ is nonzero, and this contradicts the fact that $V_1, \dots, V_n$ are linearly independent over $\mathbb{R}$ at $p$. Therefore we must have $W(p) = 0$. ■

We now introduce some notation that will be used in the rest of the paper. Let $x, y$ be local isothermal coordinates on $\Sigma$. Define

$$\bar{\partial}_J u = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + J\frac{\partial u}{\partial y}\Big) \quad \text{and} \quad \partial_J u = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - J\frac{\partial u}{\partial y}\Big)$$

and define

$$\overline{\nabla}_J = \frac{1}{2}(\nabla_{\frac{\partial}{\partial x}} + J\nabla_{\frac{\partial}{\partial y}}) \quad \text{and} \quad \nabla_J = \frac{1}{2}(\nabla_{\frac{\partial}{\partial x}} - J\nabla_{\frac{\partial}{\partial y}}).$$

Observe that $V = X - iJX \in \Gamma(E)$ is holomorphic, *i.e.*, $\nabla_{\frac{\partial}{\partial \bar{z}}} V = 0$, if and only if $\overline{\nabla}_J X = 0$. Similarly, $W = Y - iJY \in \Gamma(E)$ is anti-holomorphic, *i.e.*, $\nabla_{\frac{\partial}{\partial z}} W = 0$, if and only if $\nabla_J Y = 0$. Also, $\bar{\partial}_J u = 0$ is equivalent to $u$ is holomorphic, *i.e.*, satisfying the Cauchy–Riemann equations.

***Theorem 3.2*** *Let $L$ be an oriented Lagrangian submanifold in an $n$-dimensional Kähler manifold $M$. Let $u\colon D \to M$ be a minimal immersion from a disk $D$ with $u(\partial D) \subset L$, and such that $u(D)$ meets $L$ orthogonally along $u(\partial D)$. If there exist admissible holomorphic variations $V_1, \dots, V_n$ that are linearly independent over $\mathbb{R}$ at some point $p \in \partial D$, then $u$ is holomorphic.*

**Proof** We may assume that $D$ is the unit disk in $\mathbb{C}$. For $z = x + iy \in D$, let

$$X(z) = J(x - Jy)\bar{\partial}_J u(z).$$

Then since $u$ is harmonic,

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial u}{\partial x} + \nabla_{\frac{\partial}{\partial y}} \frac{\partial u}{\partial y} = 0,$$

and it is straightforward to check that $\nabla_J X = 0$. Also, observe that

$$2X = J\Big(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\Big) + y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = Jr\frac{\partial u}{\partial r} - \frac{\partial u}{\partial \theta}$$

where $(r, \theta)$ are the polar coordinates. Since $u(D)$ meets $L$ orthogonally along $\partial D$, $J\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ are tangent to $L$ along $\partial D$. Therefore,

$$W = X - iJX$$

is an admissible anti-holomorphic variation on $D$ which has a zero at $z = 0$. Since $V_1, \dots, V_n$ are linearly independent over $\mathbb{R}$ at $p \in \partial D$, there is an arc $C \subset \partial D$ containing $p$ such that $V_1, \dots, V_n$ are linearly independent over $\mathbb{R}$ at each point in $C$. Therefore, by Lemma 3.1, $W = 0$ on $C$. Next, we take a local anti-holomorphic frame $\{\sigma_1, \dots, \sigma_n\}$ of $u^*(T^{1,0}M)$ on a neighborhood $U$ in $D$ containing a portion of $C$. Then $W = f^1\sigma_1 + \cdots + f^n\sigma_n$ for some anti-holomorphic functions $f^1, \dots, f^n$. But $f^1, \dots, f^n$ must be zero, since they vanish along $C$. It follows that $W = 0$ on $U$ hence on $D$ as $W$ is anti-holomorphic. Therefore $\bar{\partial}_J u = 0$ on $D$ and $u$ is holomorphic. ■

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## 4 Second Variation Formula

The following second variation formula is a real formulation, considered by Y. G. Oh, of the complex second variation formulas in [7] and [5].

***Lemma 4.1*** *Let $u\colon \Sigma \to M$ with $u(\partial\Sigma) \subset L$ be a harmonic map, meeting $L$ orthogonally along $\partial\Sigma$,* i.e.,

$$\frac{\partial u}{\partial \nu}(z) \perp T_{u(z)}L$$

*for all $z \in \partial\Sigma$. Then, the second variation of energy for any admissible real variation $X \in \Gamma\big(u^*(TM)\big)$ is given by*

$$\begin{aligned}\delta^2 E(X,X) = 2\int_\Sigma \big[2|\overline{\nabla}_J X|^2 + \big\langle R(X,\partial_J u)X, \bar{\partial}_J u\big\rangle \\ - \big\langle R(X, J\partial_J u)X, J\bar{\partial}_J u\big\rangle - \big\langle R(\bar{\partial}_J u, J\bar{\partial}_J u)\, JX, X\big\rangle\big]\, dx\, dy \\ + \int_{\partial\Sigma} \langle \nabla_X X, \nu + JT\rangle\, ds,\end{aligned}$$

*where $\nu$ is the outward pointing unit normal of $\partial\Sigma$ along $u$, and $T$ is the positively oriented unit tangent vector along $\partial\Sigma$.*

**Proof** Recall the standard formula for the second variation of energy of $u$ for an admissible real variation $X$, *i.e.*, $X \in \Gamma\big(u^*(TM)\big)$, with $X(z) \in T_{u(z)}L$ for all $z \in \partial\Sigma$:

$$\begin{aligned}\delta^2 E(X,X) = \int_\Sigma \Big[ |\nabla_{\frac{\partial}{\partial x}} X|^2 + |\nabla_{\frac{\partial}{\partial y}} X|^2 - \Big\langle R\Big(X, \frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x}, X\Big\rangle \\ - \Big\langle R\Big(X, \frac{\partial u}{\partial y}\Big)\frac{\partial u}{\partial y}, X\Big\rangle\Big]\, dx\, dy + \int_{\partial\Sigma} \langle \nabla_X X, \nu\rangle\, ds.\end{aligned}$$

Now,

$$4|\overline{\nabla}_J X|^2 = |\nabla_{\frac{\partial}{\partial x}} X|^2 + |\nabla_{\frac{\partial}{\partial y}} X|^2 + 2\Big\langle \nabla_{\frac{\partial}{\partial x}} X, J\nabla_{\frac{\partial}{\partial y}} X\Big\rangle$$

and

$$\begin{aligned}&2\Big\langle \nabla_{\frac{\partial}{\partial x}} X, J\nabla_{\frac{\partial}{\partial y}} X\Big\rangle \\ &\quad = \frac{\partial}{\partial x}\Big\langle X, J\nabla_{\frac{\partial}{\partial y}} X\Big\rangle - \Big\langle X, J\nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}} X\Big\rangle - \frac{\partial}{\partial y}\Big\langle X, J\nabla_{\frac{\partial}{\partial x}} X\Big\rangle + \Big\langle X, J\nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}} X\Big\rangle \\ &\quad = -\Big\langle JR\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)X, X\Big\rangle + \frac{\partial}{\partial x}\Big\langle X, J\nabla_{\frac{\partial}{\partial y}} X\Big\rangle - \frac{\partial}{\partial y}\Big\langle X, J\nabla_{\frac{\partial}{\partial x}} X\Big\rangle.\end{aligned}$$

Therefore, we may rewrite the second variation formula as

$$\begin{aligned}\delta^2 E(X,X) = \int_\Sigma \Big[4|\overline{\nabla}_J X|^2 - \Big\langle R\Big(X, \frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x}, X\Big\rangle \\ - \Big\langle R\Big(X, \frac{\partial u}{\partial y}\Big)\frac{\partial u}{\partial y}, X\Big\rangle + \Big\langle JR\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)X, X\Big\rangle\Big]\, dx\, dy \\ + \int_{\partial\Sigma} [\langle \nabla_X X, \nu\rangle + \langle \nabla_T X, JX\rangle]\, ds\end{aligned}$$

where $T$ is the positively oriented unit tangent vector around $\partial\Sigma$. Since $L$ is Lagrangian and $X$ is an admissible variation, we have

$$\langle \nabla_T X, JX\rangle = \langle \nabla_X T, JX\rangle = -\langle T, \nabla_X JX\rangle = \langle JT, \nabla_X X\rangle .$$

Also, substituting

$$\frac{\partial u}{\partial x} = \partial_J u + \bar\partial_J u, \quad \frac{\partial u}{\partial y} = -J(\bar\partial_J u - \partial_J u)$$

and simplifying, the curvature terms can be rewritten as

$$\begin{aligned}
&\Big\langle R\Big(X, \frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x}, X\Big\rangle - \Big\langle R\Big(X, \frac{\partial u}{\partial y}\Big)\frac{\partial u}{\partial y}, X\Big\rangle + \Big\langle JR\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)X, X\Big\rangle \\
&\quad= \langle R(X, \partial_J u)\partial_J u, X\rangle + \langle R(X, \bar\partial_J u)\bar\partial_J u, X\rangle \\
&\qquad + 2\langle R(X, \partial_J u)\bar\partial_J, X\rangle + \langle R(X, J\bar\partial_J u)J\bar\partial_J u, X\rangle \\
&\qquad + \langle R(X, J\partial_J u)J\partial_J u, X\rangle - 2\langle R(X, J\bar\partial_J u)J\partial_J u, X\rangle \\
&\qquad + \langle JR(\bar\partial_J u, J\bar\partial_J u)X, X\rangle - \langle JR(\partial_J u, J\partial_J u)X, X\rangle \\
&\quad= 2\Big[\langle R(X, \partial_J u)\bar\partial_J, X\rangle - \langle R(X, J\bar\partial_J u)J\partial_J u, X\rangle + \langle JR(\bar\partial_J u, J\bar\partial_J u)X, X\rangle\Big]
\end{aligned}$$

where we have used

$$R(JV, JW) = R(V, W) \quad \text{for all } V, W \in \Gamma\big(u^*(TM)\big),$$

and in the last equality we have used the identity

$$\langle R(V, W)W, V\rangle + \langle R(V, JW)JW, V\rangle = \langle R(W, JW)JV, V\rangle$$

which follows from the symmetries of the curvature tensor. Therefore,

$$\begin{aligned}
\delta^2 E(X, X) = 2\int_\Sigma \big[2|\overline{\nabla}_J X|^2 &+ \langle R(X, \partial_J u)X, \bar\partial_J u\rangle \\
&- \langle R(X, J\partial_J u)X, J\bar\partial_J u\rangle - \langle R(\bar\partial_J u, J\bar\partial_J u)JX, X\rangle\big]\, dx\, dy \\
&+ \int_{\partial\Sigma} \langle \nabla_X X, \nu + JT\rangle\, ds.
\end{aligned}$$

This completes the proof. ■

***Theorem 4.2*** *Let $L$ be an oriented Lagrangian submanifold in $\mathbb{C}P^n$ with the Fubini–Study metric. Let $u\colon D \to M$ be a harmonic map from a disk $D$ with $u(\partial D) \subset L$ and meeting $L$ orthogonally along $\partial D$. If $u$ intersects $L$ positively, then*

(1) *if $u$ stable, then $u$ is holomorphic;*
(2) *if $u$ is unstable, then $u$ has index at least $n + \mu(E, F)$.*

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**Proof** We prove (1) first. Let $X = J(x + Jy)\partial_J u$. Then, since $u$ is harmonic and by similar arguments to those in the proof of Theorem 3.2, $\overline{\nabla}_J X = 0$ and $X$ is an admissible real variation, and so $V = X - iJX$ is an admissible holomorphic variation. Note that

$$\begin{aligned} 2X &= J\Big(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\Big) - y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} \\ &= Jr\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}, \end{aligned}$$

and in particular, on $\partial D$,

$$2X = J\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}.$$

Since $u$ is stable, by the second variation formula in Lemma 4.1 we have

$$\begin{aligned} 0 &\le \delta^2 E(X,X) \\ &= \int_D \big[ 4|\overline{\nabla}_J X|^2 + \big\langle R(X,\partial_J u)X, \bar{\partial}_J u\big\rangle - \big\langle R(X, J\partial_J u)X, J\bar{\partial}_J u\big\rangle \\ &\qquad - \big\langle R(\bar{\partial}_J u, J\bar{\partial}_J u) JX, X\big\rangle\big]\, dx\, dy + \int_{\partial D} \Big\langle \nabla_X X, \frac{\partial u}{\partial r} + J\frac{\partial u}{\partial \theta}\Big\rangle\, d\theta \\ &\le \int_D \big[ \big\langle R(X,\partial_J u)X, \bar{\partial}_J u\big\rangle - \big\langle R(X, J\partial_J u)X, J\bar{\partial}_J u\big\rangle - \big\langle R(\bar{\partial}_J u, J\bar{\partial}_J u) JX, X\big\rangle\big]\, dx\, dy, \end{aligned}$$

where the last inequality follows since $\overline{\nabla}_J X = 0$ and $u$ intersects $L$ positively.

Using the curvature formula for the Fubini–Study metric on $\mathbb{C}P^n$,

$$R(X,W)Z = \langle W, Z\rangle X - \langle X, Z\rangle W + \langle JW, Z\rangle JX - \langle JX, Z\rangle JW + 2\langle X, JW\rangle JZ$$

we have

$$\begin{aligned} &\big\langle R(X,\partial_J u)X, \bar{\partial}_J u\big\rangle - \big\langle R(X, J\partial_J u)X, J\bar{\partial}_J u\big\rangle \\ &\qquad = 4\Big[ \langle \partial_J u, X\rangle \big\langle \bar{\partial}_J u, X\big\rangle + \langle J\partial_J u, X\rangle \big\langle \bar{\partial}_J u, JX\big\rangle\Big]. \end{aligned}$$

This expression is equal to zero, as observed by Y. G. Oh. In order to prove this, we will show that

$$\big\langle \bar{\partial}_J u, X\big\rangle = 0 \quad \text{and} \quad \big\langle \bar{\partial}_J u, JX\big\rangle = 0.$$

Define the function

$$\begin{aligned} F(z) &= \big\langle zJ(\bar{\partial}_J u + iJ\bar{\partial}_J u), V\big\rangle \\ &= 2\big\langle zJ(\bar{\partial}_J u + iJ\bar{\partial}_J u), X\big\rangle. \end{aligned}$$

By a straightforward calculation, we have

$$4\nabla_{\frac{\partial}{\partial \bar{z}}} zJ(\bar{\partial}_J u + iJ\bar{\partial}_J u) = \Big(\nabla_{\frac{\partial}{\partial x}}\frac{\partial u}{\partial x} + \nabla_{\frac{\partial}{\partial y}}\frac{\partial u}{\partial y}\Big) + iJ\Big(\nabla_{\frac{\partial}{\partial x}}\frac{\partial u}{\partial x} + \nabla_{\frac{\partial}{\partial y}}\frac{\partial u}{\partial y}\Big) = 0,$$

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since $u$ is harmonic. Since $V$ is also holomorphic, $F$ is a holomorphic function on the disk. By expanding, we have

$$zJ(\bar{\partial}_J u + iJ\bar{\partial}_J u) = Y + iJY$$

where

$$\begin{aligned} 2Y &= J\Big(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\Big) + y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} \\ &= Jr\frac{\partial u}{\partial r} - \frac{\partial u}{\partial \theta}. \end{aligned}$$

From this it follows that

$$\begin{aligned} \operatorname{Im} F|_{\partial D} &= \operatorname{Im}\langle Y + iJY, X - iJX\rangle \\ &= 2\langle JY, X\rangle \\ &= -\Big\langle \frac{\partial u}{\partial r} + J\frac{\partial u}{\partial \theta}, X\Big\rangle \\ &= 0, \end{aligned}$$

since $\frac{\partial u}{\partial r}$ and $J\frac{\partial u}{\partial \theta}$ are orthogonal to $L$, and $X$ is tangent to $L$. Therefore, $F$ is a holomorphic function on $D$ that is real on the boundary, and so $F$ must be a constant on $D$. Hence $F$ is identically 0 because $F(0) = 0$. Therefore the sum of the first two terms in the integral is zero. The third term is the holomorphic bisectional curvature of the planes $\partial_J u \wedge J\partial_J u$ and $X \wedge JX$, and so

$$\big\langle R(\bar{\partial}_J u,\, J\bar{\partial}_J u)JX, X\big\rangle \geq 0$$

and equality holds at a point $z \in D$ if and only if $V(z) = 0$ or $\bar{\partial}_J u(z) = 0$. Since $V$ is holomorphic and not identically zero, $V$ can only vanish at a discrete set $K$ of points on $D$. Then,

$$0 \leq \delta^2 E(X, X) = \int_{D\setminus K} -\big\langle R(\bar{\partial}_J u,\, J\bar{\partial}_J u)JX, X\big\rangle\, dx\, dy \leq 0,$$

and it follows that $\bar{\partial}_J u = 0$ on $D \setminus K$, thus $\bar{\partial}_J u = 0$ everywhere on $D$ since $u$ is smooth.

Now we prove (2). If $u$ is not stable, then $\bar{\partial}_J u$ is not identically 0. In fact, if $u$ is holomorphic and $u_t$ is a continuous family of maps from $D$ to $M$ with $u_t(\partial D) \subset L$ and $u_0 = u$, then by the well-known formula (*cf.* [7])

$$E(u_t) = \frac{1}{2}\int_D |\nabla u_t|^2 = 2\int_D |\bar{\partial}_J u_t|^2 + \int_D u_t^* \omega$$

and by Stokes's theorem and the fact that $L$ is Lagrangian

$$\begin{aligned} 0 &= \int_{\bigcup_{0\le s\le t} u_s(D)} d\omega \\ &= \int_D u_t^*\omega - \int_D u_0^*\omega + \int_{\bigcup_{0\le s\le t} u_s(\partial D)} \omega \\ &= \int_D u_t^*\omega - \int_D u_0^*\omega. \end{aligned}$$

Hence,

$$E(u_t) = E(u_0) + 2\int_D |\bar\partial_J u_t|^2.$$

We see that $u$ minimizes the energy among $u_t$, therefore $u$ must be minimizing.

Let $m = n + \mu(E, F)$. Theorem 2.2 asserts the existence of an $m$-dimensional space $\mathcal{H}$ of admissible holomorphic variations. Let $V = Y - iJY \in \mathcal{H}$. Away from the zeros of $Y$, which is a discrete set of points in $D$, we have

$$\left\langle R(\bar\partial_J u, J\bar\partial_J u)JY, Y\right\rangle > 0.$$

Then by arguments as above, we have

$$\delta^2 E(Y, Y) < \int_{\partial D} \left\langle \nabla_Y Y, \frac{\partial u}{\partial r} + J\frac{\partial u}{\partial \theta}\right\rangle d\theta \le 0,$$

since $u$ intersects $L$ positively. Clearly, the space of all the real parts $Y$ of all $V = Y - iJY \in \mathcal{H}$ is a real $m$-dimensional space of real admissible variations. Thus the Morse index of $u$ is at least $m = n + \mu(E, F)$. ■

In the two-dimensional case, using Theorem 3.2, we obtain an index estimate under a weaker intersection assumption.

***Theorem 4.3*** *Let $L$ be an oriented Lagrangian surface in $\mathbb{C}P^2$ with the Fubini–Study metric. Let $u\colon D \to M$ be a harmonic map from a disk $D$ with $u(\partial D) \subset L$ and meeting $L$ orthogonally along $\partial D$. Assume*

$$\langle \nabla_{J\nu+T}(J\nu + T), \nu + JT\rangle \le 0$$

*along $\partial D$, and $\mu(E, F) \ge 0$. If $u$ is unstable, then $u$ has index at least $2 + \mu(E, F)$.*

**Proof** Let $m = 2 + \mu(E, F)$. By Theorem 2.2 there exists an $m$-dimensional space $\mathcal{H}$ of admissible holomorphic variations. As above, the real parts form a real $m$-dimensional space of real admissible variations on which we will show the index form is negative definite. Let $V = Y - iJY \in \mathcal{H}$. Since $u$ is not stable, then as in the previous proof, away from a discrete set of points on $D$, we have

$$\left\langle R(\bar\partial_J u, J\bar\partial_J u)JY, Y\right\rangle > 0,$$

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and hence

$$\delta^2 E(Y,Y) < \int_{\partial D} \left\langle \nabla_Y Y, \frac{\partial u}{\partial r} + J\frac{\partial u}{\partial \theta} \right\rangle d\theta.$$

Now consider the admissible holomorphic variation $X - iJX$ with

$$X = J(x + Jy)\partial_J u,$$

used in the proof of Theorem 4.2 (1). Recall $2X = J\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}$ on $\partial D$. By Theorem 3.2, if there exist two admissible holomorphic variations that are linearly independent over $\mathbb{R}$ at some point $p \in \partial D$, then $u$ is holomorphic. Since $u$ is not holomorphic, $V$ and $X - iJX$ must pointwise linearly dependent over $\mathbb{R}$ along $\partial D$. Therefore,

$$Y = f\Big( J\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}\Big)$$

for some smooth real valued function $f$ along $\partial D$, and

$$\delta^2 E(Y,Y) < \int_{\partial D} f^2 \left\langle \nabla_{J\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}} J\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial r} + J\frac{\partial u}{\partial \theta} \right\rangle d\theta \leq 0,$$

by the assumption along $\partial D$. Thus the Morse index of $u$ is at least $2 + \mu(E, F)$. ■

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