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RINGS WHOSE INDECOMPOSABLE INJECTIVE MODULES ARE UNISERIAL

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Introduction. A module is *uniserial* in case its submodules are linearly ordered by inclusion. A ring R is *left* (*right*) *serial* if it is a direct sum of uniserial left (right) R-modules. A ring R is *serial* if it is both left and right serial. It is well known that for artinian rings the property of being serial is equivalent to the finitely generated modules being a direct sum of uniserial modules [**8**]. Results along this line have been generalized to more arbitrary rings [**6**], [**13**].

This article is concerned with investigating rings whose indecomposable injective modules are uniserial. The following question is considered which was first posed in [**4**]. If an artinian ring R has all indecomposable injective modules uniserial, does this imply that R is serial? The answer is yes if R is a finite dimensional algebra over a field. In this paper it is shown, provided R modulo its radical is commutative, that R has every left indecomposable injective uniserial implies that R is right serial.

The following definitions and notation will be needed. All rings have an identity, and all modules are unital. The Jacobson radical will be denoted by J. A submodule K is *large* in a module M in case $K \cap L \neq 0$ for every non-zero submodule L of M. The *injective hull* of M, denoted by $E(M)$, is an injective module such that there exists a monomorphism $i: M \to E(M)$ with the property that $i(M)$ is large in $E(M)$.

The *socle* of M, denoted by $S(M)$, is the largest semi-simple submodule of M. If R is artinian and M is any R-module then M/JM, denoted by $T(M)$, is a direct sum of simples and is called the *top* of M.

If M is a semi-simple module the number of simple direct summands of M will be denoted by $C(M)$. The notation $M^{(n)}$ will be used to denote the direct sum of n-copies of M. The notation ${}_RM(M_R)$ will often be used to signify that M is a left (right) R-module. We shall say that M is an $R - S$ bimodule where R and S are rings in case ${}_RM_S$ and $(rm)s = r(ms)$ for $r \in R$, $s \in S$, $m \in M$.

1. Rings with indecomposable injectives uniserial.

1.1 PROPOSITION. *Let R be an artinian ring such that every indecomposable injective left R-module is uniserial. Then every factor ring of R has this*

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property. Conversely, if R/J^2 has every indecomposable injective left R-module uniserial, then so does R.

Proof. Let R/I be a factor ring of R and $E_{R/I}$ be an indecomposable injective over R/I. As $S(E_{R/I})$ is simple, the injective hull of $E_{R/I}$ is uniserial since it is a submodule of a uniserial module.

The final statement follows using an argument dual to the one in [**8**].

1.2 LEMMA. *Let R be a ring such that every indecomposable injective left R-module is uniserial. Then any ring Morita equivalent to R has this property.*

Proof. This result follows from the Morita theorems and will be left to the reader.

A ring is said to be *basic* in case $1 = e_1 + \cdots + e_n$ where $e_1, \ldots, e_n$ are primitive orthogonal idempotents and $Re_i \not\cong Re_j$ for $i \neq j$. Since any artinian ring is Morita equivalent to a basic artinian ring, 1.2 allows us to restrict our attention to basic rings.

The following lemma was essentially proved in [**4**].

1.3 LEMMA. *Let R be a basic artinian ring with $J^2 = 0$. Suppose that every indecomposable left injective R-module is uniserial. Let e and f be primitive idempotents of R such that $fJe \neq 0$, then $fJ \cong T(eR)^{(n)}$.*

Proof. This result follows from [**4**, Theorem 2.4].

Letting $e\bar{R}e = eRe/eJe$ and $f\bar{R}f = fRf/fJf$, it is clear that $fJ = fJe$ is a left $f\bar{R}f$ right $e\bar{R}e$ vector space. Likewise, it is not difficult to show that the left fRf action on fJe corresponds to the left R action, and the right eRe action on fJe corresponds to the right R action.

1.4 LEMMA. *Let R be as in* 1.3. *Consider the indecomposable projective left R-module Re where e is a primitive idempotent. Suppose $fJe \neq 0$ for some primitive idempotent f. Then fJe contains no proper non-zero left fRf right eRe submodules.*

Proof. Suppose ${}_{fRf}(Ie)_{eRe}$ is a non-zero left fRf right eRe submodule of fJe. Consider an indecomposable injective uniserial module $E = Re/Ke$, with Ke a maximal submodule of Je. Since

$$r_E(fI) = \{x \in E \mid fI \cdot x = 0\} = Je/Ke,$$

[**4**, Lemma 2.3] implies that $fIe = fJe$.

2. The case when $J^2 = 0$. Throughout this section let R be a basic artinian ring with $J^2 = 0$ and such that every left indecomposable injective module is uniserial. The purpose of this section will be to characterize those artinian rings with $J^2 = 0$ whose indecomposable injectives are uniserial. To prove the following lemmas, we shall need a number of facts and definitions.

For R an artinian ring with $J^2 = 0$ and such that every indecomposable left injective module is uniserial, let e be a primitive idempotent. We shall assume there exists a primitive idempotent f such that $fJe \neq 0$. Thus $fJ = fJe$ is a left $f\bar{R}f$ right $e\bar{R}e$ vector space. Let Ie be a maximal (possibly $Ie = 0$ if Je is simple) proper semi-simple left ideal contained in Je with the property that the complement of Ie in Je is isomorphic to a copy of $T(Rf)$. That is, $Ie \oplus Rx = Je$ with $Rx \cong T(Rf)$. The hypothesis that $fJe \neq 0$ guarantees the existence of such maximal left subideals.

Define

$$e\Phi e = \{e\varphi e \in Re \mid Ie\varphi e \subseteq Ie\}.$$

Then it is easily seen that $e\Phi e$ is a subring of eRe and $eJe \subseteq e\Phi e$. Likewise it is not difficult to show that $e\bar{\Phi}e = e\Phi e/eJe$ is a division subring of $e\bar{R}e$. Also it is clear that Ie, fIe, Je, fJe are left R right $e\Phi e$ modules and that fJe and fIe are left $f\bar{R}f$ right $e\bar{\Phi}e$ vector spaces. As Re/Ie is uniserial, it is the injective hull of $T(Rf)$. Likewise, we have that

$$\mathrm{End}_R\,(Re/Ie) \cong e\Phi e/eIe.$$

Let M be a left R-module and $S = \mathrm{End}_R\,(M)$. Then M is naturally an S-module in the obvious way. The module M is said to be *balanced* in case the natural ring homomorphism $R \rightarrow \mathrm{End}_S\,(M)$ is surjective. If M is injective then M is a *cogenerator* in case M contains an isomorphic copy of each simple left R-module.

2.1 LEMMA. *Let $e\Phi e$, e, f, Ie be as defined previously. Suppose $\gamma \in \mathrm{End}_{e\Phi e}$ (Re/Ie) satisfies $\mathrm{Im}\,(\gamma) \subseteq Je/Ie \subseteq \ker\,(\gamma)$. Then γ is given by a left multiplication of an element in R.*

Proof. Let $E = E_1 \oplus \cdots \oplus E_n$ be a minimal injective left cogenerator for R where each E_i is an indecomposable uniserial injective module. Set $S_i = \mathrm{End}_R\,(E_i)$ and $S = \mathrm{End}_R\,(E)$. Thus for some $\alpha \leqq n$,

$$E_\alpha \cong Re/Ie \quad \text{and} \quad e\Phi e/eIe \cong S_\alpha.$$

So we may assume that $\gamma \in \mathrm{End}_{S_\alpha}\,(E_\alpha)$. Since E is an injective cogenerator in an artinian ring R, E is balanced [**1**, page 218, exercise 32]. Extend γ to $\gamma' : E \rightarrow E$ by defining $\gamma'(E_j) = 0$ for $j \neq \alpha$ and $\gamma'(x) = \gamma(x)$, $(x \in E_\alpha)$. It need only be shown that γ' is an S-homomorphism in order to prove the lemma.

Each element $s \in S$ can be represented as an $n \times n$ matrix where the ij'th entry s_{ij} is an R-homomorphism from E_j to E_i. We make the following observations: For $j \neq i$, $s_{ij}(JE_j) = 0$. Otherwise s_{ij} is an isomorphism between E_i and E_j, a contradiction to E minimal. Since E_α has composi-

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tion length $= 2$, these remarks imply that

$$\mathrm{Im}\,(s_{\alpha j}) \subseteq JE_\alpha, \quad (j \neq \alpha).$$

Using these remarks and letting $s \in S$, $x \in E$ where $x_j = \pi_j(x)$, $(j = 1, \ldots, n)$, yields

$$\gamma'(s(x)) = \gamma'\left(\sum_{j=1}^{n} s_{1j}(x_j), \ldots, \sum s_{\alpha j}(x_j), \ldots\right) = \gamma\left(\sum s_{\alpha j}(x_j)\right) = \gamma(s_{\alpha\alpha}x_\alpha).$$

Also,

$$s\gamma'(x) = s \cdot (0, \ldots, \gamma(x_\alpha), \ldots, 0) = (s_{1\alpha}\gamma(x_\alpha), \ldots, s_{\alpha\alpha}\gamma(x_\alpha), \ldots, s_{n\alpha}\gamma(x_\alpha)) = s_{\alpha\alpha}\gamma(x_\alpha).$$

Since γ is an S_α homomorphism, the above two equations are equal. Thus γ' is an S-momomorphism. This means that γ' (and therefore γ) is given by a left multiplication of an element in R.

Another version of the next proposition was proved by V. P. Camillo and K. R. Fuller when R is local, $J^2 = 0$, and $C({}_RJ) = 2$ [**2**].

2.2 PROPOSITION. *Let e, f, Ie, and $e\Phi e$ be as defined previously. Then*

$$\dim_{e\bar{\Phi}e} (\bar{R}e_{e\bar{\Phi}e}) \leqq C({}_RJe) < \infty.$$

Proof. Let $n = C({}_RJe)$. Suppose that

$$m = \dim_{e\bar{\Phi}e} (\bar{R}e_{e\bar{\Phi}e}) > n.$$

So consider an $e\bar{\Phi}e$ independent set $\{\tau_i\}_{i=0}^{n} \subset \bar{R}e = e\bar{R}e$ where $\tau_0 = e$. Also $Je = Ie \oplus Rt$ with $Rt \cong T(Rf)$. Define for i $(1 \leqq i \leqq n - 1)$ the $e\Phi e$-homomorphisms $\psi_i : Re/Je \to Je/Ie$ as follows:

$$\psi_i(\tau_i\varphi) = t\varphi + Ie, \ (\varphi \in e\Phi e) \quad \text{and}$$
$$\psi_i(\tau_j) = 0, \quad \text{for } j \neq i, \quad (0 \leqq j \leqq n).$$

It is routine to verify that ψ_i can be extended to all of Re/Je and defines a $e\bar{\Phi}e$ homomorphism. Consider,

$$Re/Ie \xrightarrow{\epsilon} Re/Je \xrightarrow{\psi_i} Je/Ie \xrightarrow{i} Re/Ie$$

where ϵ is the natural epimorphism of Re/Ie onto Re/Je and i is the natural monomorphism of Je/Ie into Re/Ie. Therefore, $\gamma_i = i\psi_i\epsilon$ defines a $e\bar{\Phi}e$-endomorphism of Re/Ie such that

$$\mathrm{Im}\,(\gamma_i) \subseteq Je/Ie \subseteq \ker\,(\gamma_i).$$

Applying 2.1 for each γ_i, $(1 \leqq i \leqq n - 1)$, there is a $\rho_i \in Re$ such that

$$\rho_i \neq 0, \ \rho_i \in Ie, \ \rho_i\tau_i \notin Ie, \ \rho_i\tau_j \in Ie \quad (j \neq i), \quad \text{where } (1 \leqq j \leqq n).$$

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Consider $\{R\rho_l, \ldots, R\rho_{n-1}\}$. We claim that

$$Ie = \sum_{i=1}^{n-1} R\rho_i.$$

Define

$$I^{(k)}e = \sum_{i=k}^{n-1} R\rho_i \quad \text{for each } 1 \leqq k \leqq n-1.$$

Suppose $\rho_k \in I^{(k)}e$. Then

$$R\rho_k \cdot \tau_k \subseteq I^{(k)}e \cdot \tau_k.$$

But $\rho_i\tau_k \in Ie$, $i \neq k$. This implies that

$$\rho_k\tau_k \in I^{(k)}e \cdot \tau_k \subseteq Ie$$

a contradiction. Thus $R\rho_k \cap I^{(k)} = 0$ for each $1 \leqq k \leqq n-1$. Therefore, the sum $\sum R\rho_k$ is direct and since $C({}_RIe) = n-1$, $Ie = \sum R\rho_k$.

Let $x \in Ie$. Then x can be written as $x = \sum \alpha_i\rho_i$, $(\alpha_i \in R)$. Using that $\rho_i\tau_n \in Ie$ $(1 \leqq i \leqq n-1)$, yields

$$x \cdot \tau_n = \sum \alpha_i\rho_i\tau_n \in Ie, \quad \forall x \in Ie.$$

As $Ie \cdot \tau_n \subseteq Ie$, we have that $\tau_n \in e\bar{\Phi}e$, a contradiction.

2.3 Lemma. *Suppose $fJe \neq 0$ for primitive idempotents e and f, and let $e\Phi e$ and Ie be as defined previously. Then*

$$\dim_{e\bar{\Phi}e}(fIe) = \dim_{e\bar{\Phi}e}(fJe) - 1.$$

Proof. Let $0 \neq x = fx \in Je, x \notin Ie$. It will be shown that $x \cdot e\Phi e \oplus Ie = Je$ as $e\bar{\Phi}e$ vector spaces. Clearly $Ie \oplus Rx = Je$. For each $\alpha \in R$, define an R-homomorphism $\sigma : Je \rightarrow Re/Ie$ by

$$\sigma(rx) = r\alpha x + Ie, \ \sigma(Ie) = 0.$$

By the injectivity of Re/Ie, σ can be extended to a right multiplication by an element $\varphi \in Re$ such that $Ie\varphi \subseteq Ie$ and $x\varphi - \alpha x \in Ie$. This implies that

$$Ie + x \cdot e\bar{\Phi}e = Ie + Rx = Je.$$

It is straightforward to check that the above sum is a direct sum as $e\bar{\Phi}e$ subspaces. Thus $Rx \cong T(Rf)$ implies

$$fIe + x \cdot e\bar{\Phi}e = fJe.$$

This yields the result.

The problem of determining the rings whose left indecomposable injectives are uniserial can be cast into the framework of linear algebra. Let F and K be skew fields and ${}_FV_K$ a bi-vector space over F and K. Then ${}_FV_K$ is said to be *simple* in case ${}_FV_K$ contains no proper non-zero

$F - K$ bi-vector subspaces. Given an artinian ring whose left indecomposable injective modules are uniserial, by 1.4, 2.2 and 2.3 one can always construct from these rings skew fields F, K, K', $F \subseteq K$, and $K' - F$, $K' - K$ vector spaces ${}_{K'}W_F \subseteq {}_{K'}V_K$, such that ${}_{K'}V_K$ is simple and

$$\dim_F (W) = \dim_F (V) - 1 < \infty$$
$$\dim_{K'} (W) = \dim_{K'} (V) - 1 < \infty .$$

In fact the existence of such a construction allows us to determine when such a ring is right serial. This discussion is summed up in the next theorem.

In order to prove 2.4, the following definition will be needed: A module N is said to be *injective relative to* M in case for every submodule $K \subseteq M$ and homomorphism $\delta : K \to N$, there is an extension of δ to M. When $M = N$, N is said to be *quasi-injective.*

2.4 THEOREM. *The following two statements are equivalent:*

(1) *Every artinian ring whose indecomposable injective left modules are uniserial is a right serial ring.*

(2) *Every simple bi-vector space* ${}_FV_K$ *over skew fields* $F \subset K$ *that possesses a subspace* ${}_FW_F \subseteq {}_FV_F$ *such that*

$$\dim_F W = \dim_F V - 1 < \infty$$
$$\dim (W)_F = \dim (V)_F - 1 < \infty$$

satisfies $\dim (V)_K = 1$.

Proof. Using 1.1 and a theorem of Nakayama, it suffices to prove the theorem when $J^2 = 0$.

(1) implies (2): Consider a simple bi-vector space ${}_FV_K$ over skew fields $F \subset K$ with subspace ${}_FW_F \subseteq {}_FV_F$ such that $\dim_F W = \dim_F V - 1 < \infty$ and $\dim (W)_F = \dim (V)_F - 1 < \infty$. Let R be the ring of matrices of the form

$$R = \begin{bmatrix} F & V \\ 0 & K \end{bmatrix}.$$

So R has primitive idempotents

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By [**4**, Proposition 1.2], Re_2/Je_2 is injective. So it will suffice to show that Re_2/We_2 is injective. Suppose φ: $Ve_2/We_2 \to Re_2/We_2$ is an R-homomorphism. Since Ve_2/We_2 is simple as a left R-module, φ is defined by its action on an element $ve_2 + We_2$ via

$$\varphi(ve_2 + We_2) = \alpha ve_2 + We_2 \ (ve_2 \notin We_2), \ (\alpha \in K).$$

By hypothesis $W \oplus vF = V$ as F vector spaces, so $\alpha v - v\beta \in W$ for some

$\beta \in F$. Thus φ can be extended to Re_2/We_2 via right multiplication by βe_2. Therefore, Re_2/We_2 is quasi-injective.

Consider $l_R(E) = \{x \in R | x \cdot E = 0\}$, $E = Re_2/We_2$. As the $F - K$ action corresponds to the $R - R$ action on ideals of R, ${}_F V_K$ simple implies that $l_R(E)e_2 \subseteq Je_2$, or $l_R(E)e_2 = 0$. The first case can not occur as E is not semi-simple. Thus $l_R(E)e_2 = 0$. This implies that $l_R(E)e_1 = 0$ and so $l_R(E) = 0$. Hence E is faithful. By a theorem of K. R. Fuller [**5**], $Re_2/We_2 = E$ is injective. Therefore, using (1), R is right serial. This means that $\dim (V)_K = 1$.

(2) implies (1): Let f and e be primitive idempotents with $fJe \neq 0$. Setting $V = fJe$, $W = fIe$, $K = e\bar{R}e$, $K' = f\bar{R}f$, and $F = e\bar{\Phi}e$ and applying 1.4, 2.2, 2.3 yields a simple bi-vector space ${}_{K'}V_K$ with

$$\dim_{K'} (W) = \dim_{K'} (V) - 1 < \infty \quad \text{and}$$
$$\dim (W)_F = \dim (V)_F - 1 < \infty .$$

Therefore, $\dim_{K'} (V/W) = \dim (V/W)_F = 1$. Hence, the division rings K' and F are isomorphic via $\bar{x}a = \gamma(a)\bar{x}$, $(\bar{x} \in V/W)$. Applying the hypothesis, $\dim (V)_K = 1$. Thus $V = fJe = fJ$ is simple, and so fR is uniserial.

3. Rings with R/J commutative.

Since any artinian ring with R/J commutative is basic, the results of Section 2 can be applied directly.

3.1 LEMMA. *Suppose F, K, and K' are fields with $F \subseteq K$, $\dim_F (K) < \infty$, and ${}_{K'}V_K$, ${}_{K'}W_K$ are $K' - F$, $K' - K$ vector spaces such that ${}_{K'}W_F \subseteq {}_{K'}V_F$. If ${}_{K'}V_K$ is simple and ${}_{K'}V_K$, ${}_{K'}W_F$ satisfy*

$$\dim (W_F) = \dim (V)_F - 1 < \infty$$
$$\dim_{K'} (W) = \dim_{K'}(V) - 1 < \infty$$

then $\dim (V)_K = 1$.

Proof. The following notation will be used: $\dim_F (K_F) = n$, $\dim_K (V_K) = k$. Thus $\dim_F (V_F) = kn$, $\dim_F (W_F) = kn - 1$. We need only show that $\dim_F (W_F) = n - 1$.

Let $\{1, \tau_1, \ldots, \tau_{n-1}\}$ be a basis for K over F. Suppose that there exists $\{w_1, \ldots, w_n\}$ a set of non-zero F-linear independent vectors with $w_1, \ldots, w_n \in W$. We will show that there must exist a vector $0 \neq w \in W$ such that $w\tau_i \in W$, $(1 \leqq i \leqq n - 1)$ as follows: We will first show the existence of a set of non-zero F-linear independent vectors $\{\tilde{w}_1, \ldots, \tilde{w}_{n-s}\} = S$ such that for each $\tilde{w}_i \in S$,

(1) $\{\tilde{w}_i, \tilde{w}_i\tau_1, \ldots, \tilde{w}_i\tau_s\} \subseteq W$.

When $s = 0$ and let $\tau_0 = 1$, $\{w_1, \ldots, w_n\}$ constitute such a set. So apply induction assuming the existence of a set of $n - s$ non-zero, F-linear independent vectors $\{\tilde{w}_1, \ldots, \tilde{w}_{n-s}\}$ satisfying (1). Suppose there exists at least one $\tilde{w}_k$ with $\tilde{w}_k \cdot \tau_{s+1} \notin W$. If not, any subset of $n - s - 1$ F-independent vectors will do for the next step in the induction. So we may assume, re-indexing if necessary, that $\tilde{w}_{n-s} \cdot \tau_{s+1} \notin W$. Using $\tilde{w}_{n-s}$ we shall construct a set of $n - s - 1$ linearly independent non-zero vectors $\{\bar{w}_1, \ldots, \bar{w}_{n-s-1}\}$ such that $\bar{w}_i \cdot \tau_k \in W$, $(1 \leqq i \leqq n - s - 1)$, $0 \leqq k \leqq s + 1$. So suppose that for some $\tilde{w}_i$, $1 \leqq i < n - s, \tilde{w}_i \cdot \tau_{s+1} \notin W$. Observe that $\tilde{w}_{n-s} \cdot \tau_{s+1} \notin W$ implies that

$$V_F = W \oplus \tilde{w}_{n-s} \cdot \tau_{s+1} F$$

using that $\dim_F (W) = kn - 1$. Therefore,

$$\tilde{w}_i \tau_{s+1} = \tilde{w}_{n-s} \cdot \tau_{s+1} f + w$$

where $f \in F$, $w \in W$. Set $\bar{w}_i = \tilde{w}_i - \tilde{w}_{n-s} \cdot f$, and observe that $\bar{w}_i \neq 0$ since $\tilde{w}_i$, $\tilde{w}_{n-s}$ are linearly independent. Having made this selection for all $\tilde{w}_i$ (or leaving $\bar{w}_i = \tilde{w}_i$ in case $\tilde{w}_i \cdot \tau_{s+1} \in W$), it is straightforward to show that the set $\{\bar{w}_1, \ldots, \bar{w}_{n-s-1}\}$ is F-linearly independent. Also it is clear using the commutivity of K that $\bar{w}_i \cdot \tau_k \in W$, $(1 \leqq i \leqq n - s - 1)$, $(0 \leqq k \leqq s + 1)$. Therefore by induction, we may assume that there exists $0 \neq w \in W$ such that $w \cdot \tau_i \in W$, $(1 \leqq i \leqq n - 1)$.

Let $k \in K$. Thus $k = f_0 + \tau_1 f_1 + \ldots + \tau_{n-1} f_{n-1}$. So

$$w \cdot k = w \cdot (f_0 + \tau_1 f_1 + \ldots + \tau_{n-1} f_{n-1}) = \sum_{i=1}^{n-1} w \cdot \tau_i f_i + w f_0 \in W.$$

Therefore $w \cdot K \subseteq W$ which implies that $K'w \cdot K \subseteq W \subseteq V$, a contradiction. So $\dim_F (W_F) = n - 1 = kn - 1$. Thus, $k = 1$.

3.2 THEOREM. *Let R be an artinian ring with R/J commutative. Then every indecomposable injective left R-module is uniserial if and only if R is right serial.*

Proof. By 1.1 and a theorem of Nakayama [**9**], it suffices to consider the case when $J^2 = 0$. Suppose every indecomposable injective left R-module is uniserial. Let f be a primitive idempotent such that $fJ \neq 0$. So there exists a primitive idempotent e such that $fJe \neq 0$ and such that by 1.3 $fJ \cong T(eR)^{(n)}$. Applying 1.4, 2.2, and 2.3 there exist fields $K = e\bar{R}e$, $K' = f\bar{R}f$, $F = e\bar{\Phi}e$, and vector spaces ${}_{K'}V_K = fJe$, ${}_{K'}W_F = fIe$ satisfying the hypothesis of 3.1. Thus $\dim (V)_K = 1$. Since $fJ = fJe$, fJ is a one dimensional $K(= e\bar{R}e)$ vector space, and so $C(fJ_R) = 1$.

This means that fR is right uniserial for all primitive idempotents f such that $fJ \neq 0$. Since $fJ = 0$ implies that fR is simple, we must have that R is right serial.

Suppose that R is right serial. Since R is basic, let $\{f_i\}$ be a basic set of primitive idempotents with $1 = f_1 + \cdots + f_n$. Since $J^2 = 0$,

$$0 \subseteq f_1 J \subseteq (f_1 + f_2) J \subseteq \cdots \subseteq J$$

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is a sequence of [**12**, 2.6]. Applying [**12**, Theorem 2.7] and [**12**, Lemma 3.1] yields every indecomposable left injective module uniserial.

3.3 COROLLARY. *Let R be an artinian ring such that R/J is commutative. Then R is serial if and only if every indecomposable injective R-module is uniserial.*

Proof. Apply 3.2.

3.4. PROPOSITION. *Let R be an artinian ring which is Morita equivalent to a ring S with $S/J(S)$ commutative. Then every indecomposable injective left R-module is uniserial if and only if R is right serial.*

Proof. Apply 1.2 and 3.2.

Remark. Can 3.2 and 3.3 be extended to arbitrary rings, or for that matter to rings R such that R/J is a finite dimensional division algebra over a field? The author knows of no counter examples.

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