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REGULAR POLYGONS AND TRANSFINITE DIAMETER

MICHEL GRANDCOLAS

We study the behaviour of the transfinite diameter of regular polygons of fixed diameter, as a function of the number of their vertices.

1. INTRODUCTION

We recall that Favard's Problems involved studying the behaviour of the diameter of complete sets of conjugate algebraic integers. Two problems were solved in [**2**], where it was shown that

$$\inf_{X \in G} t_2(X) \geqslant \sqrt{3} \tag{1.1}$$

$$\lim_{d \to \infty} \inf_{X \in G_d} t_2(X) = 2 \tag{1.2}$$

where X is a set of conjugates of an algebraic integer belonging to G, (respectively G_d) the set of all sets of conjugates of algebraic integers (respectively, those of degree bigger or equal to d) and $t_2(X)$ is the diameter of X. Favard's second problem was solved by the inequality: $t(X) \leqslant t_2(X)/2$ where $t(X)$ is the transfinite diameter of X (see [**3**]) due to Bieberbach. These problems suggest a great number of still open questions on the links between the diameter, weighted diameter or transfinite diameter of a complete set of conjugate algebraic integers, or more generally a convex set in the plane (or particular convex sets, say regular polygons). We can also study the links between diameters and characteristic values of a convex set (see also [**1**]), for example the t_3 diameter and the length of a convex set (see [**6**] and the definition of Section 2.1). Minimal diameters or weighted diameters of complete sets of conjugate algebraic integers for small degrees have been determined by the author and Lloyd-Smith [**7**, **8**, **5**]).

With this paper we continue our contributions to Favard type problems. We prove a Theorem on the transfinite diameter of regular polygons: we study the behaviour of the transfinite diameter of a regular polygon of fixed diameter as a function of the number of its vertices. A surprising result is that the transfinite diameter of a regular polygon with fixed diameter does not increase as a function of the number of its vertices. We deduce

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a nice necessary and sufficient condition on the diameter and transfinite diameter to ensure equality between two regular polygons (isometrically).

In Section 2 we determine the transfinite diameter as a function of the diameter. In Sections 3 and 4, we study the behaviour of the transfinite diameter of a regular polygon of fixed diameter as a function of the number of its vertices. In Section 5, we obtain the Principal Theorem and we give also a conjecture which generalises this result.

2. Generalities on the diameters of regular polygons

Definition 2.1: Let X be a convex set in the plane.

(a) The diameter of X, denoted by $t_2(X)$, is defined to be

$$t_2(X) = \sup_{(\alpha_i,\alpha_j)\in X^2} |\alpha_i = \alpha_j| .$$

(b) The weighted diameter of X, called the t_n-diameter of X if $n > 2$, is defined to be

$$t_n(X) = \sup_{(\alpha_i)\in X^n} \left(\prod_{i\neq j} |\alpha_i - \alpha_j| \right)^{1/(n(n-1))} \tag{2.1}$$

(Note that this is just the diameter if $n = 2$.)

(c) The transfinite diameter of X is defined to be

$$t(X) = \lim_{n\to\infty} t_n(X) \tag{2.2}$$

We remark that the sequence $\left(t_n(X)\right)$ is decreasing (see [7]). The transfinite diameter is also called the capacity, and is generally difficult to compute. However, its value is known for regular polygons.

Proposition 2.2. [9] *The transfinite diameter of a regular polygon with n vertices and side length d is*

$$t(X) = \frac{d}{4\pi} \frac{\Gamma^2(1/n)}{\Gamma(2/n)}. \tag{2.3}$$

There are two formulae (depending on the parity of the number of vertices) which give the diameter of a regular polygon in terms of its side length d.

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Proposition 2.3. *Let R be the radius of the circle circumscribing a regular polygon X with n vertices and side length d.*

(a) *If n is even,*

$$t_2(X) = 2R = \frac{d}{\sin(\pi/n)}.$$

(b) *If n is odd*

$$t_2(X) = 2R\cos(\pi/2n) = \frac{d}{2\sin(\pi/2n)}.$$

Proof: (a) is obvious. For (b) note that $t_2(X) = 2R\sin(\pi/2 - \pi/2n)$, since $\pi - \pi/n$ is the angle $\widehat{AOC}$ where O is the centre of the circle circumscribing the polygon, and A and C two vertices of the polygon such that $AC = t_2(X)$. ∎

Using Proposition 2.3 and formula (2.3), we can express the transfinite diameter of a regular polygon with n vertices in terms of its diameter as follows.

Proposition 2.4.

(a) *If n is even,*

$$t(X) = \frac{t_2(X)\sin(\pi/n)}{4\pi}\frac{\Gamma^2(1/n)}{\Gamma(2/n)}.$$

(b) *If n is odd,*

$$t(X) = \frac{2t_2(X)\sin(\pi/2n)}{4\pi}\frac{\Gamma^2(1/n)}{\Gamma(2/n)}.$$

Definition 2.5: The transfinite diameter function tr is defined for $n \in \mathbb{N}$ by:

$$tr(n) = \frac{\sin(\pi/n)\Gamma^2(1/n)}{\Gamma(2/n)} \text{ if } n \text{ is even and } n \geqslant 2,$$
$$tr(n) = \frac{2\sin(\pi/2n)\Gamma^2(1/n)}{\Gamma(2/n)} \text{ if } n \text{ is odd and } n \geqslant 3.$$

Thus $tr(n)$ represents the transfinite diameter of a regular polygon with n vertices $(n \geqslant 2)$ of diameter 4π.

3. Technical lemmas

Put

$$f(x) = \frac{\Gamma^2(1/x)}{\Gamma(2/x)} \sin\left(\frac{\pi}{x}\right)$$

and

$$g(x) = 2\frac{\Gamma^2(1/x)}{\Gamma(2/x)} \sin\left(\frac{\pi}{2x}\right).$$

We obtain bounds for $f(x)$ and $g(x)$ for large x.

We have (see [10]) $\Gamma'(x) = \Gamma(x)\Big(-c-x+\sum_{k=1}^{\infty}(1/k-1/(k+x))\Big)$ where c is Euler's constant. By using Taylor's formula of order 4, we obtain that for $z \in [0, 0.04]$,

$$1 + cz + az^2 + bz^3 \leqslant \Gamma(1-z) \leqslant 1 + cz + az^2 + bz^3 + 2z^4$$

where

$$a = \frac{1}{2}\left(c^2 + \frac{\pi^2}{6}\right), \quad b = \frac{1}{6}\left(c^3 + \frac{c\pi^2}{2} + 2\sum_{k=1}^{\infty}\frac{1}{k^3}\right).$$

Hence, using the formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \tag{3.1}$$

we obtain bounds of $\Gamma(z)$ for $z \in [0, 0.04]$:

$$\frac{\pi}{\sin(\pi z)}\,\frac{1}{1+cz+az^2+bz^3+2z^4} \leqslant \Gamma(z) \leqslant \frac{\pi}{\sin(\pi z)}\,\frac{1}{1+cz+az^2+bz^3}. \tag{3.2}$$

We use (3.2) to estimate $f(x)$ and $g(x)$ for large x. If $x \geqslant 50$,

$$f(x) = \frac{\Gamma^2(1/x)}{\Gamma(2/x)} \sin\left(\frac{\pi}{x}\right) \leqslant 2\pi\left(1 - \frac{\pi^2}{3x^2} + \frac{6b-6ac+2c^3}{x^3} + \frac{30}{x^4}\right)$$

$$f(x) = \frac{\Gamma^2(1/x)}{\Gamma(2/x)} \sin\left(\frac{\pi}{x}\right) \geqslant 2\pi\left(1 - \frac{\pi^2}{3x^2} + \frac{6b-6ac+2c^3}{x^3} - \frac{30}{x^4}\right)$$

$$g(x) = \frac{2\Gamma^2(1/x)}{\Gamma(2/x)} \sin\left(\frac{\pi}{2x}\right) \leqslant 2\pi\left(1 - \frac{5\pi^2}{24x^2} + \frac{6b-6ac+2c^3}{x^3} + \frac{35}{x^4}\right)$$

$$g(x) = \frac{2\Gamma^2(1/x)}{\Gamma(2/x)} \sin\left(\frac{\pi}{2x}\right) \geqslant 2\pi\left(1 - \frac{5\pi^2}{24x^2} + \frac{6b-6ac+2c^3}{x^3} - \frac{32}{x^4}\right)$$

Now we show both $f(x)$ and $g(x)$ are increasing on $[1, \infty)$. Note

$$g'(x) = \frac{2\Gamma^2(1/x)}{\Gamma(2/x)}\,\frac{\cos(\pi/2x)}{x}\left[\frac{-\pi}{2x} + \left(\tan\left(\frac{x}{2x}\right)\right)\left(1 + \sum_{k=1}^{\infty}\frac{2}{(kx+1)(kx+2)}\right)\right].$$

The sign of $g'(x)$ is the same on $[1,\infty)$ as that of

$$p(x) = \frac{-\pi}{2x} + \left(\tan\left(\frac{\pi}{2x}\right)\right)\left(1\sum_{k=1}^{\infty}\frac{2}{(kx+1)(kx+2)}\right),$$

but $p(x) > q(x)$ where

$$q(x) = \frac{-x}{2x} + \left(\tan\left(\frac{\pi}{2x}\right)\right)\left(1+\sum_{k=1}^{\infty}\frac{2}{(kx+2)^2}\right).$$

Furthermore, for $x > 0$, $q'(x) < 0$ and $\lim_{x\to+\infty} q(x) = 0$, so $g'(x) > q(x) > 0$ if $x > 0$ and $g'(x) > 0$ on $[1,\infty)$. Hence g is an increasing function on $[1,\infty)$.

Similarly, the sign of $f'(x)$ on $[1,\infty)$ is the same as that of

$$r(x) = \frac{-\pi}{x} + \left(\tan\left(\frac{\pi}{x}\right)\right)\left(1+\sum_{k=1}^{\infty}\frac{2}{(kx+1)(kx+2)}\right),$$

but $r(x) > s(x)$ where

$$s(x) = \frac{-\pi}{x} + \left(\tan\left(\frac{\pi}{x}\right)\right)\left(1+\sum_{k=1}^{\infty}\frac{2}{(kx+2)^2}\right).$$

Furthermore, for $x > 0$, $s'(x) < 0$ and $\lim_{x\to+\infty} s(x) = 0$, so $s(x) > 0$ if $x > 0$ and $f'(x) > 0$ on $[1,\infty)$.

Hence f is an increasing function on $[1,\infty)$.

4. Study of tr

We apply the observation of the preceeding section to the transfinite diameter function tr.

Proposition 4.1.

(a) $tr(2n+2) > tr(2n)$, *for all* $n \in \mathbb{N}^*$;
(b) $tr(2n+1) > tr(2n-1)$, *for all* $n \in \mathbb{N}^*$;
(c) $tr(2n-1) > tr(2n)$, *for all* $n \in \mathbb{N} - \{0,1\}$;
(d) $tr(2n+1) > tr(2n)$, *for all* $n \in \mathbb{N}^*$.

Proof: (a) and (b) are immediate, since the functions f and g are increasing on $[1,\infty)$.

(c) Note $tr(2n-1) - tr(2n) = g(2n-1) - f(2n)$. For $n \geqslant 26$, if we use the bounds of the preceeding section we have:

$$tr(2n-1) - tr(2n) > \frac{1}{n^2}\left(\frac{3\pi^2}{96} - \frac{3\pi^2}{96n} - \frac{2}{n^2} - \frac{30}{16n^2}\right) > 0$$

if $n \geqslant 26$. From $n = 1$ to 25, we compute the values of $tr(2n-1)$ and $tr(2n)$.

(d) Note $tr(2n+1) > tr(2n-1) > tr(2n)$ if $n \geqslant 2$ and $tr(3) > tr(4)$. ∎

Finally, for large n, we observe that

$$\lim_{n \to +\infty} tr(n) = t(C) = 2\pi$$

where C is a circle of radius 2π.

5. Principal theorem

Theorem 5.1. *If X and Y are two regular polygons with the same diameter and the same transfinite diameter, then $X = Y$ (isometrically).*

Proof: Since X and Y have the same diameter, it is enough to show that the tr function is one to one. We prove that if $n \neq n'$ then $tr(n) \neq tr(n')$.

(1) If n and n' are even, this comes from Proposition 4.1(a).

(2) If n and n' are odd, this comes from Proposition 4.1(b).

(3) If n is odd and n' is even, $n > n'$, this comes from $tr(n) > tr(n-1) \geqslant tr(n')$ by Proposition 4.1(d) and (a).

(4) If n is odd and n' is even, $n < n'$, we can set $n = 2l - 1$ and $n' = 2p$ with $l \leqslant p$.

There are 3 possible cases.

Case (a). $l \leqslant \sqrt{5/8}p$:
From the bounds of Section 3, if $p \geqslant 25$ and $l \geqslant 26$

$$tr(2p) - tr(2l-1) \geqslant -1 + \frac{5\pi^2}{96l^2} + \left(\frac{5\pi^2}{96} - \frac{h}{8}\right)\frac{1}{l^3} + 1 - \frac{\pi^2}{12p^2} + \frac{h}{8p^3} > 0$$

with $h = 6b - 6ac + 2c^3$, where a, b, c are the values of Section 3, because $5\pi^2/96l^2 \geqslant \pi^2/12p^2$ and the other terms are strictly positive.

Case (b). $1 + \sqrt{5/8}p \leqslant l \leqslant 0.8p$:
if $p \geqslant 25$ and $l \geqslant 26$, using the bound of $1/p$ as a function of $1/l$, $\sqrt{5/8}/(l-1) \leqslant 1/p \leqslant 4/5l$, we get $tr(2p) - tr(2l - l) < -5\pi^2/96l^3 + 5/l^4 < 0$.

Case (c). $0.8p \leqslant l$:
if $p \geqslant 25$ and $l \geqslant 26$,

$$tr(2p) - tr(2l-1) \leqslant \frac{-\pi^2}{12p^2} + \frac{h}{8p^3} + \frac{30}{16p^4} + \frac{5\pi^2}{96l^2} + \frac{\left(\frac{5\pi^2}{96} - \frac{h}{8}\right)}{l^3} + \frac{2}{l^4}.$$

If we replace l by p, $tr(2p) - tr(2l-1) \leqslant \left(-1.44\pi^2 + 0.5\right)/p^2 < 0$.

Conclusion of (4): $tr(n) \neq tr(n')$ if n, n' are bigger than 50. Furthermore if $n \geqslant 66$, $tr(n) \geqslant tr(66) = 6.278492\ldots > tr(51) = 6.278330\ldots \geqslant tr(n')$ if n' is between 1 and 51. So for all $n' \in \mathbb{N}$, $n' \geqslant 2$, $tr(n) \neq tr(n')$ if n is bigger than 66. We also check that the values of $tr(n)$ are distinct from 2 to 65. ∎

Conjecture 5.2: If X and Y are two compact convex sets in the plane, such that

$$t_n(X) = t_n(Y) \quad \text{for all } n \in \mathbb{N}^*\backslash\{1\}$$

then

$$X = Y$$

isometrically.

Remark. The result holds for regular polygons. Furthermore, a convex set X in the plane is specified by a discrete number of points. By a Theorem of Riemann, there is a meromorphic function which maps the exterior of the unit disk onto the exterior of the convex set X. The coefficients of this map probably have a link with the weighted diameters of X. It should be interesting to prove this result for two triangles. Indeed this conjecture seems very difficult to prove.

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UFR MIM, Département de Mathématiques
Université de Metz
Ile du Saulcy
57045 Metz Cédex 01
France
e-mail: grandcol@poncelet.univ-metz.fr