Canad. J. Math. Vol. **56** (2), 2004 pp. 277–292

Spectral Properties of the Commutator of Bergman's Projection and the Operator of Multiplication by an Analytic Function

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Abstract. It is shown that the singular values of the operator $aP - Pa$, where P is Bergman's projection over a bounded domain Ω and a is a function analytic on $\bar{\Omega}$, detect the length of the boundary of $a(\Omega)$. Also we point out the relation of that operator and the spectral asymptotics of a Hankel operator with an anti-analytic symbol.

1 Introduction

Let Ω be a bounded domain in $\mathbb{C}$. Denote by $L^2(\Omega)$ the space of complex-valued functions on Ω such that the norm

$$\|f\| = \Big(\int_\Omega |f(\xi)|^2 \, dA(\xi) \Big)^{\frac{1}{2}}$$

is finite. Here dA denotes the Lebesgue measure on Ω.

Let $L_a^2(\Omega)$ denote the space of analytic functions on Ω such that

$$\int_\Omega |f(\xi)|^2 \, dA(\xi) < \infty.$$

Note that $L_a^2(\Omega)$ is a Hilbert subspace of $L^2(\Omega)$ and is called the *Bergman space*. Let P denote the orthogonal projector of $L^2(\Omega)$ on $L_a^2(\Omega)$ (Bergman's projection).

Let A be a compact operator on a separable Hilbert space $\mathcal{H}$. Denote by $s_1(A), s_2(A), \dots$ the sequence of eigenvalues of the positive operator $(A^*A)^{\frac{1}{2}}$ arranged in nondecreasing order taking account of multiplicity. We call $s_1(A), s_2(A), \dots$ the *singular values* of A. A detailed exposition of the properties of the singular values of compact operators can be found in [6].

Denote by c_p the set of all compact operators A on $\mathcal{H}$ such that

$$|A|_p \stackrel{\text{def}}{=} \Big(\sum_{n \geqslant 1} s_n^p(A) \Big)^{\frac{1}{p}} < \infty.$$

Received by the editors May 21, 2002; revised June 28, 2003.
AMS subject classification: 47B10.

https://doi.org/10.4153/CJM-2004-013-8 Published online by Cambridge University Press

It is known that c_p $(p \geq 1)$ are Banach spaces. In particular, c_1 is the trace class (nuclear operator).

In a series of papers N. L. Vasilevski (see [7]) studied the Banach algebra $\mathcal{R}$ generated by all the operators (acting on $L^2(\Omega)$) of the form $aI + bP + T$, where Ω is a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of closed smooth Jordan curves, a and b are continuous functions on $\bar{\Omega}$, T is compact operator and P is the Bergman projection.

The Bergman projection is a singular integral operator which has many properties similar to the singular operator of Cauchy,

$$C\varphi(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(s)}{t-s}\, ds.$$

In particular, P is not compact (because $L_a^2(\Omega)$ is not finite dimensional), $P^2 = P$ and for every function $a \in C(\bar{\Omega})$ the operator $aP - Pa$ is compact. This enables us to consider the algebra $\mathcal{R}$ as a two-dimensional analogue of the algebra of one-dimensional singular operators.

In this paper we study the spectral properties of $aP - Pa$, *i.e.*, the asymptotic behavior of its singular values and the connection with the geometric properties of the domain Ω. The spectral properties of Cauchy's operator and its product with Bergman's projection are studied in details in [5].

The notation $a_n \sim b_n$ will mean $\lim_{n\to\infty} a_n/b_n = 1$. Also, we will denote by $\int_\Omega T(z,\xi) \cdot dA(s)$ the integral operator acting on $L^2(\Omega)$ with kernel $T(\cdot,\cdot)$.

2 Main result

Theorem 1 *If Ω is a bounded, simply connected domain with the analytic boundary and $z \mapsto a(z)$ an analytic (or anti-analytic) function in some neighborhood of $\bar{\Omega}$, then $aP - Pa$ is a Volterra operator and there holds the following asymptotic formula:*

$$s_n(aP - Pa) \sim \frac{1}{2\pi n} \int_{\partial\Omega} |a'(z)|\, |dz|, \quad n \to \infty.$$

Corollaries

1. *If a is an analytic (or anti-analytic) function on a neighborhood of $\bar{\Omega}$, then $aP - Pa$ is a nuclear operator if and only if $a =$* const. *Also $aP - Pa \in c_p$ for every $p > 1$.*
2. *The singular values of the operator $aP - Pa$ detect the length of the boundary of the domain $a(\Omega)$. In particular, if $a(z) = z$, then*

$$s_n(zP - Pz) \sim \frac{|\partial\Omega|}{2\pi n}, \quad n \to \infty.$$

($|\partial\Omega|$ is the length of $\partial\Omega$).

For the proof of our result a few lemmas will be needed.

If K is a compact operator on a separable Hilbert space $\mathcal{H}$, then by

$$\mathcal{N}_t(K) = \sum_{s_n(K)\geq t} 1 \quad (t > 0)$$

we denote the singular value distribution function of K.

Lemma 2 ([5]) *Let T be a compact operator and suppose that for every $\varepsilon > 0$ there exists a decomposition $T = T'_\varepsilon + T''_\varepsilon$ where $T'_\varepsilon, T''_\varepsilon$ are compact operators such that:*

(I) *There exists $\lim_{t\to 0^+} t^\gamma \mathcal{N}_t(T'_\varepsilon) = C(T'_\varepsilon)$, and $C(T'_\varepsilon)$ is a bounded function in the neighborhood of $\varepsilon = 0$.*
(II) $\overline{\lim}_{n\to\infty} n^{\frac{1}{\gamma}} s_n(T''_\varepsilon) \leq \varepsilon$.

Then there exists $\lim_{\varepsilon\to 0^+} C(T'_\varepsilon) = C(T)$ and

$$\lim_{t\to 0^+} t^\gamma \mathcal{N}_t(T) = C(T). \qquad \blacksquare$$

Lemma 3 *Let $\{s_n\}_{n=1}^\infty$ be a non-increasing sequence ($s_n \geq 0$) and let the series $\sum_1^\infty a_n$ ($a_n \geq 0$) be convergent. If for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$ the inequality*

$$s_{kn} \leqslant a_n + \varepsilon s_n$$

holds, where $0 < \varepsilon < \frac{1}{k}$, then $\sum_{n=1}^\infty s_n < +\infty$ and hence $\lim_{n\to\infty} n s_n = 0$. $\blacksquare$

Since $s_k \leqslant a_1 + \varepsilon s_1$, we have

$$s_{k+1} + \cdots + s_{2k} \leqslant k s_k \leqslant k a_1 + \varepsilon k s_1$$

From $s_{2k} \leqslant a_2 + \varepsilon s_2$, we get

$$s_{2k+1} + \cdots + s_{3k} \leqslant k s_{2k} \leqslant k a_2 + \varepsilon k s_2$$

Continuing this procedure we get

$$s_{(n-1)k+1} + \cdots + s_{nk} \leqslant k a_{n-1} + \varepsilon k s_{n-1}.$$

By summation, we get

$$\sum_{\nu=k+1}^{nk} s_\nu \leqslant k \sum_{\nu=1}^{n-1} a_\nu + \varepsilon k \sum_{\nu=1}^{n-1} s_\nu$$

which implies

$$\sum_{\nu=1}^{nk} s_\nu \leqslant k \sum_{\nu=1}^{\infty} a_\nu + \varepsilon k \sum_{\nu=1}^{nk} s_\nu + s_1 + \cdots + s_k$$

and, since $1 - \varepsilon k > 0$, we have

$$\sum_{\nu=1}^{nk} s_\nu \leqslant \frac{k}{1-\varepsilon k} \sum_{\nu=1}^{\infty} a_\nu + \frac{s_1 + \cdots + s_k}{1-\varepsilon k}.$$

The last inequality implies the assertion of the Lemma 2. ■

Let $F\colon \Omega \to D$ (D is the unit disc) be a conformal mapping and $\varphi = F^{-1}(\colon D \to \Omega)$. Since $\partial\Omega$ is an analytic curve, the mappings F and φ have analytic continuation to some neighborhood of $\bar{\Omega}$ and $\bar{D}$ respectively.

Denote by $T\colon L^2(D) \to L^2(D)$ the linear operator defined by

$$T f(z) = \frac{1}{\pi} \int_D \frac{a\big(\varphi(z)\big) - a\big(\varphi(\xi)\big)}{(1 - z\bar{\xi})^2} f(\xi)\, dA(\xi)$$

(a is an analytic function on some neighborhood of $\bar{\Omega}$).

Remark 1 Theorem 1 can be first proved for the unit disc (then $\varphi(z) \equiv z$), but this proof is not simpler than the proof of the general case (both are based on Lemmata 4 and 6).

Thus, we have formulated and proved the statement for the arbitrary bounded simply connected domain with an analytic boundary.

Lemma 4 *The operators $A = aP - Pa\colon L^2(\Omega) \to L^2(\Omega)$ and $T\colon L^2(D) \to L^2(D)$ have the same singular values.* ■

Since the Bergman reproducing kernel for Ω is given by (see [12])

$$K_0(t,\xi) = \frac{1}{\pi} \frac{F'(t)\overline{F'(\xi)}}{(1 - F(t)\overline{F(\xi)})^2}$$

the operator A is given by

$$A f(z) = \int_\Omega K_0(z,\xi)\big(a(z) - a(\xi)\big) f(\xi)\, dA(\xi).$$

Denote by $V\colon L^2(\Omega) \to L^2(D)$ the operator defined by $V f(z) = f\big(\varphi(z)\big) \cdot \varphi'(z)$. It is verified directly that V is an isometry and that $VA = TV$, which implies that $s_n(A) = s_n(T)$. ■

Since $z \mapsto a\big(\varphi(z)\big)$ is a function analytic on a neighborhood of $\bar{\Omega}$, we have

$$a\big(\varphi(z)\big) - a\big(\varphi(\xi)\big) = a'\big(\varphi(\xi)\big) \cdot \varphi'(\xi)(z - \xi) + (z-\xi)^2 G(z,\xi)$$

where $G(\cdot,\cdot)$ is a function analytic on $D_1 \times D_1$ and D_1 neighborhood of $\bar{D}$.

Lemma 5 *For the operator* $S\colon L^2(D) \to L^2(D)$ *defined by*

$$Sf(z) = \int_D \left(\frac{z-\xi}{1-z\bar{\xi}}\right)^2 G(z,\xi) f(\xi)\, dA(\xi)$$

there holds

$$\lim_{n\to\infty} n s_n(S) = 0. \tag{1}$$

Proof Note first that the singular values of the operator

$$S_0 = \int_D \left(\frac{z-\xi}{1-z\bar{\xi}}\right)^2 \cdot dA(\xi)$$

can be calculated directly, and then we get

$$s_n(S_0) = O(n^{-2}). \tag{2}$$

Since $G(\cdot,\cdot)$ is analytic on $D_1 \times D_1$, we have

$$G(z,\xi) = \sum_{k=0}^{\infty} \frac{G^{(k)}(z,0)}{k!}\xi^k,$$

where the series converges uniformly on $\bar{D} \times \bar{D}$. We obtain

$$\frac{G^{(k)}(z,0)}{k!} = \frac{1}{2\pi i}\int_{\partial D_R} \frac{G(z,\xi)}{\xi^{k+1}}\, d\xi,$$

where $D_R = \{z : |z| < R\} \subset D_1$ and R is number > 1. If

$$M = \max_{(z,\xi)\in\bar{D}\times\overline{D_R}} |G(z,\xi)|$$

then

$$\frac{|G^{(k)}(z,0)|}{k!} \le \frac{M}{R^k} \tag{3}$$

so for $R_n(z,\xi) \stackrel{\text{def}}{=\!=} \sum_{k=n+1}^{\infty} \frac{G^{(k)}(z,0)}{k!}\xi^k$ there holds the estimate

$$|R_n(z,\xi)| \le \frac{M}{R^n(R-1)}, \quad (z,\xi \in \bar{D}). \tag{4}$$

Let A_k and B_k be the operators of multiplication by the function $\frac{G^{(k)}(z,0)}{k!}$ and ξ^k ($k = 0, 1, \ldots, n$) respectively, and

$$C_n = \int_D \left(\frac{z-\xi}{1-z\bar{\xi}}\right)^2 R_n(z,\xi) \cdot dA(\xi).$$

From (3) and (4) it follows that

$$\|A_k\| \le \frac{M}{R^k}, \quad \|B_k\| \le 1, \quad \|C_k\| \le \frac{M\pi}{R^k(R-1)}. \tag{5}$$

Since $G(z,\xi) = \sum_{k=0}^{n} \frac{G^{(k)}(z,0)}{k!}\xi^k + R_n(z,\xi)$, we obtain

$$S = \sum_{k=0}^{n} A_k S_0 B_k + C_n,$$

so by using (5) and the properties of the singular values of the sum of operators, we obtain

$$\begin{aligned} s_{(n+2)m}(S) &\le \sum_{k=0}^{n} s_m(A_k S_0 B_k) + s_m(C_n) \\ &\le \sum_{k=0}^{n} \frac{M}{R^k} s_m(S_0) + \|C_n\| \\ &\le \frac{MR}{R-1} s_m(S_0) + \frac{M\pi}{R^n(R-1)}. \end{aligned} \tag{6}$$

From (2) it follows that

$$s_m(S_0) \le \frac{d_0}{m^2} \quad (d_0 \text{ does not depend on } m) \tag{7}$$

so from (6) it follows

$$s_{(n+2)m}(S) \le \frac{MRd_0}{R-1}\frac{1}{m^2} + \frac{M\pi}{(R-1)}\frac{1}{R^n}.$$

Let $0 < \alpha < 1$ be a fixed number. Putting $n = [m^\alpha] - 2$ (where $[x]$ is the integer part of x) we obtain $[m^\alpha]m \sim m^{\alpha+1}$ $(m \to \infty)$, so from (7) we get (because $R > 1$)

$$s_n(S) = O(n^{-\frac{2}{\alpha+1}}), \quad n \to \infty.$$

Since $0 < \alpha < 1$, then $1 < \frac{2}{\alpha+1} < 2$, so from the previous asymptotic formula we obtain (1). ■

Let $K_a = \{z : -a < \operatorname{Re} z < 0, 0 < \operatorname{Im} z < a\}$ for $a > 0$.

https://doi.org/10.4153/CJM-2004-013-8 Published online by Cambridge University Press

Lemma 6 *There hold the following asymptotic formulae:*

(a)
$$s_n\left(\int_{K_{2\pi}} \frac{(e^z - e^\xi)e^{z+\bar\xi}}{(1-e^{z+\bar\xi})^2} \cdot dA(\xi)\right) \sim \frac{\pi}{n}$$

(b)
$$s_n\left(\int_{K_{2\pi}} \frac{e^z - e^\xi}{1-e^{z+\bar\xi}} \cdot dA(\xi)\right) = O(n^{-\frac{3}{2}})$$

(c)
$$s_n\left(\int_{K_{2\pi}} \frac{e^z - e^\xi}{(z+\bar\xi \pm 2\pi i)^2} \cdot dA(\xi)\right) = o(n^{-1})$$

(d)
$$s_n\left(\int_{K_{2\pi}} \frac{e^z - e^\xi}{(z+\bar\xi)^2} \cdot dA(\xi)\right) \sim \frac{\pi}{n}$$

(e)
$$s_n\left(\int_{K_a} \frac{z - \xi}{(z+\bar\xi)^2} \cdot dA(\xi)\right) \sim \frac{a}{2n}$$ ■

Proof (a) Since the operator $H_0 = \frac{1}{\pi}\int_D \frac{z-\xi}{(1-z\bar\xi)^2} \cdot dA(\xi)$ can be expressed in the form

$$H_0 = -\frac{1}{\pi}(\cdot,\xi)_{L^2(D)} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)}}(\cdot,g_n)_{L^2(D)} f_{n+1}$$

where $f_{n+1}(z) = \sqrt{\frac{n+1}{\pi}}z^n$, $g_{n(z)} = \sqrt{\frac{n+3}{\pi}}z^n\big(n+1-(n+2)|z|^2\big)$ $(n = 1,2,\dots)$ are orthonormal system of functions in $L^2(D)$ we have

$$s_n(H_0) \sim \frac{1}{\pi} \quad (n \to \infty).$$

Let $D_0 = \{z : |z| < e^{-2\pi}\}$, $D_1 = \{z : e^{-2\pi} < |z| < 1\}$ and $P_i\colon L^2(D) \to L^2(D)$, $i = 1, 2$, be the operators defined by $P_i f(z) = \mathcal{X}_{D_i}(z) f(z)$ (where $\mathcal{X}_S(\cdot)$ is the characteristic function of S). Then $P_0 + P_1 = I$, hence

$$H_0 = P_0H_0P_0 + P_1H_0P_0 + P_0H_0P_1 + P_1H_0P_1.$$

By the Birman-Solomjak theorem [3] the singular values of $P_0H_0P_0$, $P_1H_0P_0$, $P_0H_0P_1$ have exponential decrease, so by the Ky-Fan theorem [6]

$$s_n(P_1H_0P_1) \sim \frac{1}{n}, (n \to \infty),$$

i.e.,

$$s_n\left(\int_D \frac{z-\xi}{(1-z\bar\xi)^2} \cdot dA(\xi)\right) \sim \frac{\pi}{n}.$$

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Since the operator $\int_{D_1} \frac{z-\xi}{(1-z\bar{\xi})^2} \cdot dA(\xi)$ is unitarily equivalent with the operator $\int_{K_{2\pi}} \frac{e^z-e^{\xi}}{(1-e^{z+\bar{\xi}})^2} e^{z+\bar{\xi}} \cdot dA(\xi)$, we get the assertion (a).

(b) It suffices to prove that

$$s_n\Big(\int_{K_{2\pi}} \frac{e^z-e^{\xi}}{1-e^{z+\bar{\xi}}} e^{z+\bar{\xi}} \cdot dA(\xi)\Big) = O(n^{-\frac{3}{2}}),$$

because the operators of multiplication by e^{-z} and $e^{-\bar{\xi}}$ are bounded on $L^2(K_{2\pi})$. Since

$$s_n\Big(\int_{K_{2\pi}} \frac{e^z-e^{\xi}}{1-e^{z+\bar{\xi}}} e^{z+\bar{\xi}} \cdot dA(\xi)\Big) \sim s_n\Big(\int_D \frac{z-\xi}{1-z\bar{\xi}} \cdot dA(\xi)\Big)$$

(the proof is similar to (a)), and since $s_n\big(\int_D \frac{z-\xi}{1-z\bar{\xi}} \cdot dA(\xi)\big) = O(n^{-\frac{3}{2}})$ we obtain (b).

(c) It suffices to prove that

$$s_n\Big(\int_{K_{2\pi}} \frac{e^z-e^{\xi}}{(z+\bar{\xi}+2\pi i)^2} \cdot dA(\xi)\Big) = o(n^{-1}).$$

Since the function $z \mapsto e^z$ is bounded on $K_{2\pi}$ it suffices to prove that

$$s_n\Big(\int_{K_{2\pi}} \frac{e^{z-\bar{\xi}+2\pi i}-1}{(z+\bar{\xi}+2\pi i)^2} \cdot dA(\xi)\Big) = o(n^{-1}),$$

i.e.,

$$s_n\Big(\int_{K_{2\pi}} \frac{z-\bar{\xi}+2\pi i}{(z+\bar{\xi}+2\pi i)^2} \cdot dA(\xi)\Big) = o(n^{-1}). \tag{8}$$

(We are taking only the first term in the expansion $e^{z-\bar{\xi}+2\pi i}-1 = z-\bar{\xi}+2\pi i+\cdots$.)

Let $\varepsilon > 0$, $\varepsilon < 2\pi$, $\Omega_1 = K_\varepsilon$, $\Omega_2 = K_\varepsilon + (2\pi-\varepsilon)i$, $\Omega_3 = K_{2\pi} \setminus (\Omega_1 \cup \Omega_2)$, $P_i' f(z) = \mathfrak{X}_{\Omega_i}(z) f(z) \colon L^2(K_{2\pi}) \to L^2(K_{2\pi})$ $(n = 1,2,3)$ and

$$C = \int_{K_{2\pi}} \frac{z-\bar{\xi}+2\pi i}{(z+\bar{\xi}+2\pi i)^2} \cdot dA(\xi).$$

Then $C = (\sum_1^3 P_i')C(\sum_1^3 P_i')$. Since $s_n(P_i'CP_j') = O\big(e^{-d(\varepsilon)\sqrt{n}}\big)$ (Birman-Solomjak theorem [3]) for all i, j except for $i=1$, $j=2$ and $i=2$, $j=1$, we obtain, using the properties of the singular values of the sum of operators,

$$s_n(C_0) = O\big(e^{-d'(\varepsilon)\sqrt{n}}\big) \quad \big(d_0'(\varepsilon) > 0\big), \tag{9}$$

where

$$C_0 = P_1'CP_1' + P_3'CP_1' + P_2'CP_2' + P_3'CP_2' + P_1'CP_3' + P_2'CP_3' + P_3'CP_3'.$$

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The singular values of $P_1' C P_2'$ are equal to the singular values of the operator

$$\int_{K_\varepsilon} \frac{z-\bar{\xi}+\varepsilon i}{(z+\bar{\xi}+\varepsilon i)^2} \cdot dA(\xi)$$

(because the mapping $f \mapsto f\big(\xi+(2\pi-\varepsilon)i\big)$ is an isometry $L^2(\Omega_1)$ to $L^2(\Omega_2)$. Taking into account that

$$s_n\left(\int_{K_\varepsilon} \frac{z-\bar{\xi}+\varepsilon i}{(z+\bar{\xi}+\varepsilon i)^2} \cdot dA(\xi)\right) = \frac{\varepsilon}{2\pi} s_n(C),$$

we get

$$s_n(P_1' C P_2') = \frac{\varepsilon}{2\pi} s_n(C),$$

and similarly

$$s_n(P_2' C P_1') = \frac{\varepsilon}{2\pi} s_n(C).$$

Since $C = C_0 + P_1' C P_2' + P_2' C P_1'$, from the last relation and (9) it follows that

$$s_{3n}(C) \le D_0 e^{-d'(\varepsilon)\sqrt{n}} + \frac{\varepsilon}{\pi} s_n(C)$$

(D_0 is independent of n), so by Lemma 3, choosing $\varepsilon > 0$ so that $\frac{\varepsilon}{\pi} < \frac{1}{3}$, we find that C is a nuclear operator, which proves (8). In a similar way one proves that

$$s_n\left(\int_{K_{2\pi}} \frac{e^z - e^{\xi}}{(z+\bar{\xi}-2\pi i)^2} \cdot dA(\xi)\right) = o(n^{-1}).$$

(d) Since the function $z \mapsto \frac{1}{(1-e^z)^2} - \frac{1}{z^2} - \frac{1}{(z-2\pi i)^2} - \frac{1}{(z+2\pi i)^2}$ is analytic in the disc $\{z : |z| < 4\pi\}$, then according to the Birman-Solomjak theorem of the singular values of integral operator with analytic kernels [3], the Ky-Fan theorem, and the assertions (a), (b), (c) we obtain the assertion (d).

(e) It is enough to prove the assertion in the case $a = 2\pi$, *i.e.*,

$$s_n\left(\int_{K_{2\pi}} \frac{z-\xi}{(z+\bar{\xi})^2} \cdot dA(\xi)\right) \sim \frac{\pi}{n}.$$

First prove that for the operator $W\colon L^2(K_{2\pi}) \to L^2(K_{2\pi})$ defined by

$$W f(z) = \int_{K_{2\pi}} \left(\frac{z-\xi}{z+\bar{\xi}}\right)^2 f(\xi)\, dA(\xi)$$

there holds

$$\lim_{n\to\infty} n s_n(W) = 0. \tag{10}$$

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Let r be a fixed positive integer and

$$\Omega_r^1 = \left\{ z : -2\pi < \operatorname{Re} z < \frac{-2\pi}{r}, 0 < \operatorname{Im} z < 2\pi \right\}$$
$$\Omega_r^2 = \left\{ z : \frac{-2\pi}{r} < \operatorname{Re} z < 0, 0 < \operatorname{Im} z < 2\pi \right\}$$
$$Q_i' : L^2(K_{2\pi}) \to L^2(K_{2\pi}),\ Q_i' f(z) = \mathcal{X}_{\Omega_r^i}(z) f(z) \quad i = 1, 2.$$

Then
$$W = Q_1' W Q_1' + Q_2' W Q_1' + Q_1' W Q_2' + Q_2' W Q_2'.$$
From the Birman-Solomjak theorem [3] it follows that the operator

$$W_1^r = Q_1' W Q_1' + Q_2' W Q_1' + Q_1' W Q_2'$$

is nuclear. Let

$$K_r^j = \left\{ z : \frac{-2\pi}{r} < \operatorname{Re} z < 0, (j-1)\frac{2\pi}{r} < \operatorname{Im} z < \frac{2\pi}{r} j \right\}$$

and $Q_j^r f(z) = \mathcal{X}_{K_r^j}(z) f(z)$ $j = 1, 2, \ldots, r$. Then

$$Q_2' W Q_2' = \sum_{i,j=1}^{r} Q_j^r Q_2' W Q_2' Q_i^r = \sum_{i,j=1}^{r} Q_j^r W Q_i^r.$$

By the Birman-Solomjak theorem [3], for $|i - j| \geq 2$, all the operators $Q_j^r W Q_i^r$ are nuclear. In the case $|i - j| = 1$ all the operators $Q_j^r W Q_i^r$ are also nuclear (which is proved as in Lemma 5(c).

Thus the operator W can be written as $W = W^r + F^r$ where $F^r = \sum_{i=1}^{r} Q_i^r W Q_i^r$ and W^r is a nuclear operator for every positive integer r. Since $s_n(Q_i^r W Q_i^r) = \frac{1}{r^2} s_n(W)$ and $Q_i^r W Q_i^r \cdot Q_j^r W Q_j^r = 0$ for $i \neq j$, we have

$$s_{nr}(F^r) = \frac{1}{r^2} s_n(W)$$

for every $n = 1, 2, \ldots,$ and therefore

$$s_{2nr}(W) \leq s_{nr}(W^r) + s_{nr}(F^r) \leq s_n(W^r) + \frac{1}{r^2} s_n(W).$$

Since the operator W^r is nuclear, we conclude by choosing r so that $\frac{1}{r^2} < \frac{1}{2r}$ and, using Lemma 3, that W is nuclear which proves (10).

Since the function $s \mapsto h_0(s) = \frac{e^s - 1 - s}{s}$ is entire, it follows from (10) that

$$\lim_{n\to\infty} n s_n \left(\int_{K_{2\pi}} \frac{z - \xi}{(z + \bar{\xi})^2} h_0(z - \xi) \cdot dA(\xi) \right) = 0. \tag{11}$$

Having in mind that

$$\frac{e^z - e^\xi}{(z+\bar{\xi})^2} = e^z \frac{z-\xi}{(z+\bar{\xi})^2} + e^\xi \frac{z-\xi}{(z+\bar{\xi})^2} h_0(z-\xi),$$

from (11), Lemma 6(d) and Ky-Fan's theorem it follows that

$$s_n\Big(\int_{K_{2\pi}} e^\xi \frac{z-\xi}{(z+\bar{\xi})^2} \cdot dA(\xi)\Big) \sim \frac{\pi}{n} \quad n \to \infty. \tag{12}$$

Since

$$\lim_{n\to\infty} ns_n\Big(\int_{K_{2\pi}} (e^z - 1)\frac{z-\xi}{(z+\bar{\xi})^2} \cdot dA(\xi)\Big) = 0,$$

(the proof is the copy of the procedure of proving that $s_n(W) = o(n^{-1})$), we get from (12) by Ky-Fan's theorem the formula

$$s_n\Big(\int_{K_{2\pi}} \frac{z-\xi}{(z+\bar{\xi})^2} \cdot dA(\xi)\Big) \sim \frac{\pi}{n}.$$

■

Let N be a positive integer,

$$D_0^N = \{z : |z| \le e^{-\frac{2\pi}{N}}\},$$

$$D_i^N = \{z : e^{-\frac{2\pi}{N}} < |z| < 1, (i-1)\frac{2\pi}{N} < \arg z < \frac{2\pi}{N} i\}$$

and $R_i^N : L^2(D_i^N) \to L^2(D_i^N)$, $i = 1, 2, \dots, N$, the linear operators defined by

$$R_i^N f(z) = \frac{1}{\pi} \int_{D_i^N} \frac{z-\xi}{(1-z\bar{\xi})^2} f(\xi)\, dA(\xi).$$

Now we prove the main lemma.

Lemma 7

(a) *For every positive integer N and $i = 1, 2, \dots, N$ there holds*

$$s_n(R_i^N) \sim \frac{1}{n \cdot N}, \quad n \to \infty$$

(b) *If $m \in C(\bar{D})$, then*

$$s_n(H) \sim \frac{1}{2\pi n} \int_0^{2\pi} |m(e^{i\theta})|\, d\theta, \quad n \to \infty$$

where

$$H = \frac{1}{\pi} \int_D \frac{z-\xi}{(1-z\bar{\xi})^2} m(\xi) \cdot dA(\xi).$$

■

Proof (a) Since

$$s_n(R_i^N) = s_n\Big(\frac{1}{\pi}\int_{K_{\frac{2\pi}{N}}} \frac{e^z - e^\xi}{1-e^{z+\bar\xi}} e^{z+\bar\xi}\cdot dA(\xi)\Big)$$

$$\sim \text{(the same reasoning as in Lemma 6(a))}$$

$$\sim s_n\Big(\int_{K_{\frac{2\pi}{N}}} \frac{z-\xi}{(z+\bar\xi)^2}\cdot dA(\xi)\Big) \sim \frac{1}{2n}\cdot\frac{2\pi}{N}\cdot\frac{1}{N}$$

$$= \frac{1}{n\cdot N} \quad \text{(using the methods from the proof of Lemma 6(d), (e)).}$$

(b) Let $\xi_\nu = e^{i\theta_\nu}$, $(\nu-1)\frac{2\pi}{N} \le \theta_\nu \le \frac{2\pi}{N}\nu$, $\nu = 1,2,\dots,N$. By the assertion (a), for the operators

$$\mathcal{R}_i^N = \frac{1}{\pi}\int_{D_i^N} \frac{z-\xi}{(1-z\bar\xi)^2} m(\xi_i)\cdot dA(\xi)$$

we have

$$s_n(\mathcal{R}_i^N) \sim \frac{|m(\xi_i)|}{n\cdot N}, \quad i = 1,2,\dots,N,\ n\to\infty.$$

Whence

$$\mathcal{N}_t \sim \frac{|m(\xi_i)|}{Nt}, \quad t\to 0^+,\ i = 1,2,\dots,N. \tag{13}$$

Let $T_i\colon L^2(D)\to L^2(D)$ $(i = 0,1,2,\dots,N)$, be the linear operators defined by

$$T_i f(z) = \mathcal{X}_{D_i^N}(z) f(z).$$

Then

$$H = T_0HT_0 + \sum_{i\neq j}^{N} T_iHT_j + \sum_{i=1}^{N} T_iHT_i.$$

Since $s_n(T_0HT_0) = O(e^{-d_0\sqrt{n}})$ $(d_0 > 0)$ by the Birman-Solomjak theorem [3] and since $s_n(T_iHT_j) = o(n^{-1})$ $(i\neq j)$ which is consequence of lemma, we have

$$\lim_{n\to\infty} ns_n(E_N) = 0, \tag{14}$$

where $E_N = T_0HT_0 + \sum_{i\neq j}^N T_iHT_j$, $F_N = \sum_{i=1}^N T_iHT_i$ and $H = E_N + F_N$. Let G_i^N, $i = 1,2,\dots,N$, be the operators on $L^2(D)$ defined by

$$G_i^N f(z) = \mathcal{X}_{D_i^N}(z)\frac{1}{\pi}\int_{D_i^N} \frac{z-\xi}{(1-z\bar\xi)^2}\big(m(\xi)-m(\xi_i)\big)\mathcal{X}_{D_i^N}(\xi) f(\xi)\, dA(\xi),$$

and let

$$F_N'' = \sum_{i=1}^N \mathcal{X}_{D_i^N}(z)\frac{1}{\pi}\int_{D_i^N} \frac{z-\xi}{(1-z\bar\xi)^2} m(\xi_i)\mathcal{X}_{D_i^N}(\xi)\cdot dA(\xi).$$

Then $F_N = F'_N + F''_N$, where $F'_N = \sum_{i=1}^N G_i^N$. Since $m \in C(\bar{D})$ for every given $\varepsilon > 0$ there exists N large enough so that $|m(\xi) - m(\xi_i)| < \varepsilon$ for $\xi \in D_i^N$ and every $i = 1, 2, \dots, N$, so by Lemma 7(a) we obtain

$$s_n(G_i^N) \le \frac{c'_1 \varepsilon}{nN} \quad (c'_1 \text{ does not depend on } \varepsilon, n, N)$$

i.e.,

$$\mathcal{N}_t(G_i^N) \le \frac{c'_1 \varepsilon}{Nt} \quad , t > 0,\ i = 1, 2, \dots, N. \tag{15}$$

Having in mind that $G_i^N \cdot G_j^N = 0$ (for $i \ne j$) we obtain $\mathcal{N}_t(F'_N) = \sum_{i=1}^N \mathcal{N}_t(G_i^N)$, so from (15) it follows that

$$\mathcal{N}_t(F'_N) \le \frac{c'_1 \varepsilon}{t}, \quad t > 0,$$

i.e.,

$$s_n(F'_N) \le \frac{c'_1 \varepsilon}{n}.$$

The previous inequality and (14) show that if N is large enough then

$$\overline{\lim_{n\to\infty}}\, n s_n(E_N + F'_N) \le c'_2 \cdot \varepsilon \tag{16}$$

where c'_2 is independent of ε.

Since

$$s_n\left(\mathcal{X}_{D_i^N}(z) \frac{1}{\pi} \int_{D_i^N} \frac{z - \xi}{(1 - z\bar{\xi})^2} m(\xi_i) \mathcal{X}_{D_i^N}(\xi) \cdot dA(\xi) \right) = s_n(\mathcal{R}_i^N) \sim \frac{|m(\xi_i)|}{nN},$$

we have

$$\mathcal{N}_t\left(\mathcal{X}_{D_i^N}(z) \frac{1}{\pi} \int_{D_i^N} \frac{z - \xi}{(1 - z\bar{\xi})^2} m(\xi_i) \mathcal{X}_{D_i^N}(\xi) \cdot dA(\xi) \right) \sim \frac{|m(\xi_i)|}{Nt}, \quad t \to 0^+,$$

and having in mind that

$$\mathcal{N}_t(F''_N) = \sum_{i=1}^N \mathcal{N}_t\left(\mathcal{X}_{D_i^N}(z) \frac{1}{\pi} \int_{D_i^N} \frac{z - \xi}{(1 - z\bar{\xi})^2} m(\xi_i) \mathcal{X}_{D_i^N}(\xi) \cdot dA(\xi) \right),$$

we obtain

$$\lim_{t\to 0^+} t \mathcal{N}_t(F''_N) = \frac{1}{N} \sum_{i=1}^N |m(\xi_i)|. \tag{17}$$

From (16) and (17) we get, by Lemma 1,

$$\lim_{t\to 0^+} t \mathcal{N}_t(H) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N |m(\xi_i)| = \frac{1}{2\pi} \int_0^{2\pi} |m(e^{i\theta})|\, d\theta.$$

In particular, for $t = s_n(H)$ we get

$$s_n(H) \sim \frac{1}{2\pi n} \int_0^{2\pi} |m(e^{i\theta})| \, d\theta. \qquad \blacksquare$$

Proof of Theorem 1 The operator $aP - Pa$ is Volterra, because, under the conditions of Theorem 1 we have

$$(aP - Pa)^2 = 0$$

and therefore by the Dunford Theorem on mapping of the spectrum it follows that its spectrum reduces to the point $\lambda = 0$.

Since

$$T = \frac{1}{\pi} \int_D \frac{z - \xi}{(1 - z\bar{\xi})^2} a'\big(\varphi(\xi)\big)\, \varphi'(\xi) \cdot dA(\xi) + \frac{1}{\pi} S,$$

by using Lemma 7(b) (taking $m(\xi) = a'(\varphi(\xi))\varphi'(\xi)$), Lemma 5, and the Ky-Fan theorem, we find that

$$s_n(T) \sim \frac{1}{2\pi n} \int_0^{2\pi} \left| a'\big(\varphi(e^{i\theta})\big) \right| |\varphi'(e^{i\theta})| \, d\theta = \frac{1}{2\pi n} \int_{\partial\Omega} |a'(z)| \, |dz|.$$

Since $s_n(T) = s_n(A)$, by Lemma 4 we obtain

$$s_n(aP - Pa) \sim \frac{1}{2\pi n} \int_{\partial\Omega} |a'(z)| \, |dz|. \qquad \blacksquare$$

Remark 2 If the function a is anti-analytic, then

$$aP - Pa = H_a P.$$

Here $H_a \colon L_a^2(\Omega) \to L^2(\Omega)$ is the Hankel operator with an anti-analytic symbol. It follows from Theorem 1 that for singular values of the operator H_a,

$$s_n(H_a) \sim \frac{1}{2\pi n} \int_{\partial\Omega} |a'(z)| \, |dz|, \quad n \to \infty.$$

From this asymptotic relation it follows that

$$\mathrm{Tr}_\omega |H_a| = \frac{1}{2\pi} \int_{\partial\Omega} |a'(z)| \, |dz|,$$

where Tr_ω is Dixmier trace (see [4], p. 303) and $|H_a| = \sqrt{H_a^* H_a}$. Another interesting formula,

$$\|H_a\|_2^2 = \frac{1}{\pi} \int_\Omega |a'(z)|^2 \, dA(z),$$

where $\| \cdot \|_2$ is the Hilbert-Schmidt norm, is proved in [1].

Many papers (see *e.g.* [2], [8], [10], [14]) consider some other properties of Hankel: operator boundedness, compactness, the property of belonging to Schatten ideal, but not the precise spectral asymptotics.

Remark 3 The assumptions about the boundary $\partial\Omega$ and the analytic function a could be weakened. We chose the stronger assumptions in order to simplify the proof of Lemma 4 and to make clear the dependence of the spectral asymptotics on the geometry of domain Ω. In order for Theorem 1 to be true, it is enough that Lemma 4 holds. From the theorem of Paraska (see [11] p. 625) it follows that Lemma 4 will hold if

$$\frac{\partial}{\partial\xi}\left(\left(\frac{z-\xi}{1-\bar{\xi}z}\right)^2 G(z,\xi)\right) \in L^2(D\times D),$$
$$\frac{\partial}{\partial\bar{\xi}}\left(\left(\frac{z-\xi}{1-\bar{\xi}z}\right)^2 G(z,\xi)\right) \in L^2(D\times D).$$

This will be true if, for example, $(a\circ\varphi)''' \in C(\bar{D})$. The last condition is fulfilled if $a''' \in C(\bar{D})$ and if domain Ω is such that the function $s \mapsto z(s)$ (where $z(s)$ is the natural parametrization of $\partial\Omega$) has the third derivative that belongs to Lip α $(0<\alpha<1)$ (see [13]).

Note that the condition $(a\circ\varphi)''' \in C(\bar{D})$ implies that the function $(a\circ\varphi)'$ belongs to the Besov space B^1 (see [14]).

The following related although less precise result is proved in [9]. In the case of the half plane holds the estimate:

$$s_n(b\cdot P_{\Psi^\alpha} - P_{\Psi^\alpha}\cdot b) \le \text{const}\,\frac{\|b\|_{B_1}}{n}.$$

Here Ψ^α is the Bergman wavelet and b belongs to the Besov space B^1 (in the half plane).

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https://doi.org/10.4153/CJM-2004-013-8 Published online by Cambridge University Press