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A Gersten–Witt complex for hermitian Witt groups of coherent algebras over schemes, I: Involution of the first kind

Stefan Gille

Abstract

Let X be a noetherian scheme with dualizing complex and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra with involution of the first kind τ. We develop in this work a coherent hermitian Witt theory for $(\mathcal{A}, \tau)$. As an application we construct a Gersten–Witt spectral sequence which converges to the coherent hermitian Witt theory of $(\mathcal{A}, \tau)$. We show then that the associated Gersten–Witt complex is exact if X is the spectrum of a smooth semilocal ring and $\mathcal{A}$ is locally free as $\mathcal{O}_X$-module.

Introduction

Let X be a regular noetherian scheme of finite Krull dimension n with $1/2$ in its global sections, and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra with involution of the first kind τ. We denote by $k(x)$ the residue field of $x \in X$, and set $\mathcal{A}(x) := \mathcal{A} \otimes k(x)$ and $\mathcal{A}(x)_0 := \mathcal{A}(x)/\operatorname{rad}\mathcal{A}(x)$. The latter is then a semisimple $k(x)$-algebra with involution $\tau(x)_0$ induced by τ.

We define in this work coherent hermitian Witt groups $\tilde{W}^i(\mathcal{A}, \tau)$ of the $\mathcal{O}_X$-algebra with involution $(\mathcal{A}, \tau)$, and construct a spectral sequence $E_1^{p,q}(\mathcal{A}, \tau)$, the hermitian Gersten–Witt spectral sequence of $(\mathcal{A}, \tau)$, which converges to these groups. Our main result about this spectral sequence is the following (see § 6.1).

Theorem. *We have $E_1^{p,q}(\mathcal{A}, \tau) = 0$ for q odd and the even line $q = 2l$ is isomorphic to the following complex which we call the ϵ-hermitian Gersten–Witt complex of $(\mathcal{A}, \tau)$:*

$$\bigoplus_{x \in X^{(0)}} W^\epsilon(\mathcal{A}(x)_0, \tau(x)_0) \xrightarrow{d_\epsilon^0} \bigoplus_{x \in X^{(1)}} W^\epsilon(\mathcal{A}(x)_0, \tau(x)_0) \xrightarrow{d_\epsilon^1} \cdots$$

$$\xrightarrow{d_\epsilon^{n-1}} \bigoplus_{x \in X^{(n)}} W^\epsilon(\mathcal{A}(x)_0, \tau(x)_0) \longrightarrow 0,$$

where $X^{(i)} \subseteq X$ denotes the set of points of codimension i, $\epsilon = (-1)^l$, and $W^\epsilon(\mathcal{A}(x)_0, \tau(x)_0)$ denotes the ϵ-hermitian Witt group of $(\mathcal{A}(x)_0, \tau(x)_0)$.

It turns out that if $\mathcal{A}$ is an Azumaya algebra over X then the differentials of the hermitian Gersten–Witt complex of $(\mathcal{A}, \tau)$ are closely related to the classical second residue maps in quadratic form theory; see § 6.4.

The prototypes of such complexes are Quillen's [Qui73] Gersten complexes for the K-theory of a scheme. They are the lines of the Brown–Gersten–Quillen spectral sequence. This spectral

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sequence converges to the (coherent) K-theory of the scheme in question. Later Colliot-Thélène and Ojanguren [CTO92] introduced such a complex (and such a spectral sequence) for the K-theory of an Azumaya algebra over a noetherian scheme and applied it to prove the Grothendieck conjecture for some special linear groups.

At the end of the 1990s Balmer [Bal00] introduced Witt groups of triangulated categories, and proved a quite general localization sequence for this so-called triangular Witt theory. By this result it has become possible to adapt Quillen's construction to Witt groups and construct a Gersten–Witt spectral sequence for Witt groups of symmetric forms. This has been carried out for regular schemes by Balmer and Walter [BW02] using derived Witt groups of symmetric forms. Later the author [Gil02] and [Gil07a] generalized this construction to a scheme with dualizing complex using coherent Witt groups. We follow here the latter approach. In particular, we construct such a hermitian Gersten–Witt complex for any coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ with involution of the first kind τ over a noetherian scheme X with dualizing complex and $1/2$ in its global sections.

Our main result about the (skew-)hermitian Gersten–Witt complex is the verification of the Gersten conjecture (Theorem 7.2).

Theorem. *If $X = \operatorname{Spec} R$ is the spectrum of a semilocal ring of a smooth variety and $\mathcal{A}$ is locally free as $\mathcal{O}_X$-module then the (skew-)hermitian Gersten–Witt complex of $(\mathcal{A},\tau)$ is exact except in degree 0. If moreover $\mathcal{A}$ is an Azumaya algebra then we have a natural isomorphism*

$$W^{\epsilon}(\mathcal{A},\tau) \xrightarrow{\simeq} \operatorname{Ker} d^0_{\epsilon}.$$

In particular if X is a smooth variety and $(\mathcal{A},\tau)$ is an Azumaya algebra with involution of the first kind over X then the ϵ-hermitian Gersten–Witt complex is a flasque resolution of the Zariski sheaf associated to the presheaf $U \mapsto W^{\epsilon}(\mathcal{A}|_U, \tau|_U)$ on X.

We give a brief overview of the content of this work. After fixing some notation and conventions in § 1, in § 2 we define the coherent hermitian Witt groups $\tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet)$ of a coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ with involution of the first kind τ, where X is a scheme with dualizing complex $\mathcal{I}_\bullet$. These are the Witt groups of the derived category $D^b_c(\mathcal{M}(\mathcal{A}))$ of bounded complexes of $\mathcal{A}$-modules with coherent homology sheaves with respect to the duality $\mathcal{F}_\bullet \mapsto \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet,\mathcal{I}_\bullet)}$. Here for a right $\mathcal{A}$-module $\mathcal{F}$ we denote by $\overline{\mathcal{F}}$ the $\mathcal{O}_X$-module $\mathcal{F}$ with left $\mathcal{A}$-structure $a\cdot m := m\tau(a)$. Note that even if X is a regular scheme of finite Krull dimension and $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module there is in general no obvious relation between the coherent Witt groups of $(\mathcal{A},\tau)$ and the (classical) (skew-)hermitian Witt groups $W^{\pm}(\mathcal{A},\tau)$. However, we show in this section (see § 2.10) that if $\mathcal{A}$ is an Azumaya algebra then there are isomorphisms $W^{+}(\mathcal{A},\tau) \simeq \tilde{W}^0(\mathcal{A},\tau,\mathcal{I}_\bullet)$ and $W^{-}(\mathcal{A},\tau) \simeq \tilde{W}^2(\mathcal{A},\tau,\mathcal{I}_\bullet)$, where $\mathcal{I}_\bullet$ is a finite injective resolution of $\mathcal{O}_X$.

In § 3 we construct a transfer morphism

$$\tilde{W}^i(\mathcal{B},\nu,\mathcal{J}_\bullet) \longrightarrow \tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet)$$

for $f : Y \longrightarrow X$ a finite morphism, $\mathcal{J}_\bullet$ an appropriate dualizing complex of Y, and $(\mathcal{B},\nu)$ a coherent $\mathcal{O}_Y$-algebra with involution of the first kind, such that there is a morphism of $\mathcal{O}_X$-algebras $\alpha : \mathcal{A} \longrightarrow f_*\mathcal{B}$ compatible with the involutions. In § 4 we introduce the coherent hermitian Gersten–Witt spectral sequence and show that the E_1-terms of this spectral sequence have the expected form. We use this spectral sequence to prove a general dévissage theorem (§ 5) which in particular generalizes the known dévissage theorems for symmetric coherent Witt groups. In the last section (§ 7) we prove the Gersten conjecture for our Gersten–Witt complex for regular semilocal rings of geometric type and $\mathcal{A}$ locally free. This result for the Gersten–Witt complex of symmetric forms has been proven by Balmer [Bal01b]. His proof uses the general strategy of Colliot-Thélène, Hoobler, and Kahn [CTHK97]. We would like to point out that we cannot do the same here since we do not know homotopy invariance for the coherent hermitian Witt theory of such algebras.

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1. Notation, conventions and preliminaries

1.1 Let $\mathcal{E}$ be an exact category. We denote by $D^b(\mathcal{E})$ the bounded derived category of $\mathcal{E}$, and by T the translation functor of this triangulated category. As usual in derived and coherent Witt theory we use homological complexes, and therefore $(TM_\bullet)_i = M_{i-1}$ for all $M_\bullet \in D^b(\mathcal{E})$.

If $\mathcal{E}$ has an inner Hom-functor or a tensor product $\otimes$ we use the following sign conventions for $\mathrm{Hom}(M_\bullet, N_\bullet)$ and $M_\bullet \otimes N_\bullet$: the differential in degree l is given by $f \mapsto fd^M + (-1)^{l+1} d^N f$, respectively $m \otimes n \mapsto d^M(m) \otimes n + (-1)^{\deg m} m \otimes d^N(n)$.

1.2 We assume throughout this work that all schemes are noetherian and have $1/2$ in their global sections.

The latter assumption is needed to use Balmer's [Bal00] triangular Witt theory whose main results are only valid for triangulated categories with uniquely 2-divisible morphism groups.

1.3 Let X be a (noetherian) scheme with structure sheaf $\mathcal{O}_X$ and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra. We denote $\mathcal{A}^{\mathrm{op}}$ the opposite algebra of $\mathcal{A}$, i.e. $\mathcal{A} = \mathcal{A}^{\mathrm{op}}$ as $\mathcal{O}_X$-modules, but with reverse multiplication. If not otherwise said $\mathcal{A}$-*module* means *left* $\mathcal{A}$-*module* in this work. We identify (as usual) right $\mathcal{A}$-modules and left $\mathcal{A}^{\mathrm{op}}$-modules.

The following categories related to X and $\mathcal{A}$ appear in this work:

(i) $\mathcal{M}(\mathcal{A})$ is the category of all $\mathcal{A}$-modules;

(ii) $\mathcal{M}_{qc}(\mathcal{A})$ is the category of quasi-coherent $\mathcal{A}$-modules;

(iii) $\mathcal{M}_c(\mathcal{A})$ is the category of coherent $\mathcal{A}$-modules.

If $\mathcal{A} = \mathcal{O}_X$ we use the symbols $\mathcal{M}(X)$, $\mathcal{M}_{qc}(X)$, and $\mathcal{M}_c(X)$ instead.

We denote by $D^b_c(\mathcal{M}(\mathcal{A}))$ (respectively by $D^b_c(\mathcal{M}_{qc}(\mathcal{A}))$) the subcategory of the bounded derived category $D^b(\mathcal{M}(\mathcal{A}))$ (respectively $D^b(\mathcal{M}_{qc}(\mathcal{A}))$) consisting of complexes with coherent homology sheaves. If $X = \operatorname{Spec} R$ the symbol $\mathcal{M}_{qc}(R)$ (respectively $\mathcal{M}_c(R)$) denotes the category of all R-modules (respectively of all finitely generated R-modules); this category is equivalent to $\mathcal{M}_{qc}(X)$ (respectively $\mathcal{M}_c(X)$). We need the following fact for which for lack of reference we give a proof.

Lemma 1.4. *Let X be a scheme and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra. Then the natural functor*

$$D^b_c(\mathcal{M}_{qc}(\mathcal{A})) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A}))$$

is an equivalence.

Proof. Let $D^b_{qc}(\mathcal{M}(\mathcal{A}))$ be the full subcategory of $D^b(\mathcal{M}(\mathcal{A}))$ consisting of complexes with quasi-coherent homology sheaves. It is enough to show that the functor $D^b(\mathcal{M}_{qc}(\mathcal{A})) \longrightarrow D^b_{qc}(\mathcal{M}(\mathcal{A}))$ is an equivalence. By [Har66, ch. I, Proposition 4.8] this functor is an equivalence if any $\mathcal{F} \in \mathcal{M}_{qc}(\mathcal{A})$ can be embedded in a quasi-coherent $\mathcal{A}$-module $\mathcal{J}$ which is injective in $\mathcal{M}(\mathcal{A})$.

To show this let $X = \bigcup_{i=1}^n U_i$ be a finite affine covering of X, say $U_i = \operatorname{Spec} R_i$ for some noetherian ring R_i, and $M_i = \Gamma(U_i, \mathcal{F})$. For any $1 \leqslant i \leqslant n$ there exists an injective R_i-module I_i and an injection $\alpha_i : M_i \hookrightarrow I_i$. We get a sequence of monomorphisms of left $A_i = \Gamma(U_i, \mathcal{A})$-modules:

$$M_i \xrightarrow{\simeq} \mathrm{Hom}_{A_i}(A_i, M_i) \xrightarrow{\subseteq} \mathrm{Hom}_{R_i}(A_i, M_i) \xrightarrow{\alpha_{i*}} \mathrm{Hom}_{R_i}(A_i, I_i).$$

The A_i-module $J_i := \mathrm{Hom}_{R_i}(A_i, I_i)$ is an injective A_i-module and therefore since R_i is a noetherian ring $\mathcal{J}_i := \tilde{J}_i$ is injective in the category of all $\mathcal{A}|_{U_i}$-modules see e.g. [Gil06, Theorem 3.9]. Let $\iota_{U_i} : U_i \hookrightarrow X$. Since ι_{U_i*} and $\iota_{U_i}^*$ are adjoint functors the $\mathcal{A}$-module $\iota_{U_i*}(\mathcal{J}_i)$ is injective and we have a morphism of $\mathcal{A}$-modules $\mathcal{F} \longrightarrow \iota_{U_i*}(\mathcal{J}_i)$ for all $1 \leqslant i \leqslant n$. The sum of these maps is an embedding of $\mathcal{F}$ into the injective $\mathcal{A}$-module $\mathcal{J} := \bigoplus_{i=1}^n \iota_{U_i*}(\mathcal{J}_i)$ which is quasi-coherent. □

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2. Coherent hermitian Witt groups

2.1 We define in this section coherent hermitian Witt groups of coherent algebras with involution of the first kind, and state some properties of them. To this end we recall first the definition of symmetric coherent Witt groups, which have been introduced in [Gil03].

Let X be a scheme with dualizing complex $\mathcal{I}_\bullet$. Recall that this means that $\mathcal{I}_\bullet \in D^b_c(\mathcal{M}_{qc}(X))$ is a complex of injective $\mathcal{O}_X$-modules and the natural morphism

$$\varpi^{\mathcal{I}}_{\mathcal{F}} : \mathcal{F}_\bullet \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{I}_\bullet), \mathcal{I}_\bullet)$$

is a quasi-isomorphism for all $\mathcal{F}_\bullet \in D^b_c(\mathcal{M}(X))$. It follows that $\mathfrak{D}_{\mathcal{I}} : \mathcal{F}_\bullet \longmapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{I}_\bullet)$ is a (1-exact) duality on $D^b_c(\mathcal{M}(X))$ making this a triangulated category with duality in the sense of Balmer [Bal00], i.e. the isomorphism of functors $\varpi^{\mathcal{I}} : \mathrm{id}_{D^b_c(\mathcal{M}(X))} \xrightarrow{\simeq} \mathfrak{D}_{\mathcal{I}}\mathfrak{D}_{\mathcal{I}}$ satisfies the equations (i) $\varpi^{\mathcal{I}}_{T\mathcal{F}} = T(\varpi^{\mathcal{I}}_{\mathcal{F}})$ and (ii) $\mathfrak{D}_{\mathcal{I}}(\varpi^{\mathcal{I}}_{\mathcal{F}}) \cdot \varpi^{\mathcal{I}}_{\mathfrak{D}_{\mathcal{I}}\mathcal{F}} = \mathrm{id}_{\mathfrak{D}_{\mathcal{I}}\mathcal{F}}$. We denote the associated triangular Witt groups by $\tilde{W}^i(X, \mathcal{I}_\bullet)$.

2.2 Let now $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra which we assume to be equipped with an involution of the first kind τ, i.e. τ is a morphism of $\mathcal{O}_X$-algebras $\mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{op}}$. We use this involution to turn an $\mathcal{A}^{\mathrm{op}}$-module $\mathcal{F}$ into an $\mathcal{A}$-module by defining $am := m \cdot \tau_U(a)$ for $a \in \mathcal{A}(U)$ and $m \in \mathcal{F}(U)$, and $U \subseteq X$ an open subscheme. We denote this $\mathcal{A}$-module $\overline{\mathcal{F}}$.

If $\mathcal{F} \in \mathcal{M}(\mathcal{A})$ then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an $\mathcal{A}^{\mathrm{op}}$-module and therefore $\overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})}$ is an $\mathcal{A}$-module for all $\mathcal{G} \in \mathcal{M}(X)$. Note that the identity induces a natural isomorphism of $\mathcal{A}$-modules $\overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})} \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_X}(\overline{\mathcal{F}}, \mathcal{G})$ for all $\mathcal{O}_X$-modules $\mathcal{G}$.

2.3 *A duality on* $D^b_c(\mathcal{M}(\mathcal{A}))$. Let $\mathcal{F}_\bullet \in D^b_c(\mathcal{M}(\mathcal{A}))$. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{I}_\bullet)$ is a complex of $\mathcal{A}^{\mathrm{op}}$-modules, and therefore we have a contravariant 1-exact functor

$$\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}} : D^b_c(\mathcal{M}(\mathcal{A})) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A})), \quad \mathcal{F}_\bullet \longmapsto \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{I}_\bullet)}.$$

We claim that this is a duality on $D^b_c(\mathcal{M}(\mathcal{A}))$, such that this is a triangulated category with 1-exact duality in the sense of Balmer [Bal00]. This follows from the fact that

$$\varpi^{\mathcal{I}}_{\mathcal{F}} : \mathcal{F}_\bullet \longrightarrow \mathfrak{D}_{\mathcal{I}}(\mathfrak{D}_{\mathcal{I}}(\mathcal{F}_\bullet)) \xrightarrow{=} \overline{\mathcal{H}om_{\mathcal{O}_X}(\overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{I}_\bullet)}, \mathcal{I}_\bullet)} = \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}(\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}(\mathcal{F}_\bullet))$$

is a morphism of complexes of (left-)$\mathcal{A}$-modules. If $Z \subseteq X$ is a closed subset then $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}$ is also a duality on the full subcategory $D^b_{c,Z}(\mathcal{M}(\mathcal{A}))$ of $D^b_c(\mathcal{M}(\mathcal{A}))$ which consists of complexes with support in Z. This is then a triangulated category with 1-exact duality, too.

Definition 2.4. The ith *coherent hermitian Witt group* of $(\mathcal{A}, \tau)$ with respect to $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}$ is the ith triangular Witt group of the triangulated category with duality $(D^b_c(\mathcal{M}(\mathcal{A})), \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}, 1, \varpi^{\mathcal{I}})$. We denote this group $\tilde{W}^i(\mathcal{A}, \tau, \mathcal{I}_\bullet)$. The ith triangular Witt group of $(D^b_{c,Z}(\mathcal{M}(\mathcal{A})), \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}, 1, \varpi^{\mathcal{I}})$, which we denote $\tilde{W}^i_Z(\mathcal{A}, \tau, \mathcal{I}_\bullet)$, is called the ith coherent hermitian Witt group of $(\mathcal{A}, \tau)$ *with support* in Z.

2.5 *Remarks.* (1) By Lemma 1.4 the functor $D^b_c(\mathcal{M}_{qc}(\mathcal{A})) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A}))$ is an equivalence. It follows then from [BW02, Lemma 4.3] that $D^b_c(\mathcal{M}_{qc}(\mathcal{A}))$ is a triangulated category with duality, too, and that the above equivalence induces an isomorphism between the triangular Witt groups of these categories.

(2) If $X = \operatorname{Spec} R$ is an affine scheme, $A = \Gamma(X, \mathcal{A})$, and $I_\bullet = \Gamma(X, \mathcal{I}_\bullet)$, then the bounded derived category $D^b_c(\mathcal{M}_{qc}(A))$ of A-modules with finitely generated homology modules is equivalent to $D^b_c(\mathcal{M}_{qc}(\mathcal{A}))$. The functor $\mathfrak{D}^{(A,\tau)}_{I} : M_\bullet \mapsto \overline{\mathrm{Hom}_R(M_\bullet, I_\bullet)}$ is a duality on $D^b_c(\mathcal{M}_{qc}(A))$, making this a

triangulated category with duality. With these duality structures the functor $D^b_c(\mathcal{M}_{qc}(\Gamma(X,\mathcal{A}))) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A}))$ is an equivalence of triangulated categories with duality and therefore the Witt theories of these two categories coincide. We will use this fact in the following without further comment.

2.6 *The module structure over the total graded Witt ring of X.* Let $\mathcal{P}(X)$ denote the category of locally free $\mathcal{O}_X$-modules of finite rank. The (derived) contravariant functor

$$\mathfrak{D}_{\mathcal{O}_X} := \mathcal{H}om_{\mathcal{O}_X}(-,\mathcal{O}_X) : D^b(\mathcal{P}(X)) \longrightarrow D^b(\mathcal{P}(X))$$

is a duality on the bounded derived category $D^b(\mathcal{P}(X))$, making this a triangulated category with duality; see [Bal01a]. The triangulated Witt groups $W^i(X)$ of this category are called *derived Witt groups*.

The tensor product $\otimes_{\mathcal{O}_X} : D^b(\mathcal{P}(X)) \times D^b(\mathcal{P}(X)) \longrightarrow D^b(\mathcal{P}(X))$ induces a product, say the left product, on the *total graded Witt ring* $W^{\mathrm{tot}}(X) := \bigoplus_{i\in\mathbb{Z}} W^i(X)$, making this a graded skew-commutative ring with $1 = [\mathcal{O}_X, \mathrm{id}]$; see [GN03].

The tensor product induces also a pairing

$$D^b(\mathcal{P}(X)) \times D^b_c(\mathcal{M}(\mathcal{A})) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A})), \quad (\mathcal{P}_\bullet,\mathcal{F}_\bullet) \longmapsto \mathcal{P}_\bullet \otimes_{\mathcal{O}_X} \mathcal{F}_\bullet,$$

which we claim to be a dualizing pairing in the sense of [GN03, Definition 1.11]. For this we have to show that the natural isomorphism

$$\eta_{\mathcal{P},\mathcal{F}} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_\bullet,\mathcal{O}_X) \otimes_{\mathcal{O}_X} \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet,\mathcal{I}_\bullet)} \xrightarrow{\simeq} \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_\bullet \otimes_{\mathcal{O}_X} \mathcal{F}_\bullet,\mathcal{I}_\bullet)}$$

satisfies some compatibility axioms, and is $\mathcal{A}$-linear. Since the definition of η is as in the symmetric case we have only to check the latter, which is straightforward and left to the reader. Hence by the main result of [GN03] the *total graded coherent hermitian Witt group*

$$\tilde{W}^{\mathrm{tot}}(\mathcal{A},\tau,\mathcal{I}_\bullet) := \bigoplus_{i\in\mathbb{Z}} \tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet)$$

is a graded $W^{\mathrm{tot}}(X)$-module. The same applies to the total graded Witt group with support in a closed subset $Z \subseteq X$, i.e. to $\tilde{W}^{\mathrm{tot}}_Z(\mathcal{A},\tau,\mathcal{I}_\bullet) := \bigoplus_{i\in\mathbb{Z}} \tilde{W}^i_Z(\mathcal{A},\tau,\mathcal{I}_\bullet)$.

2.7 *The localization sequence.* Let κ be an open subscheme $U \subseteq X$ or an embedding $U = \operatorname{Spec}\mathcal{O}_{X,x} \hookrightarrow X$ for some $x \in X$. The natural isomorphism

$$\kappa^*(\overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet,\mathcal{I}_\bullet)}) \xrightarrow{\simeq} \overline{\mathcal{H}om_{\mathcal{O}_U}(\kappa^*\mathcal{F}_\bullet,\kappa^*\mathcal{I}_\bullet)}$$

is a morphism of complexes of $\kappa^*\mathcal{A}$-modules making the pull-back κ^* duality preserving. Hence there is a well-defined homomorphism of coherent hermitian Witt groups $\kappa^* : \tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet) \longrightarrow \tilde{W}^i(\kappa^*\mathcal{A},\kappa^*\tau,\kappa^*\mathcal{I}_\bullet)$.

If $\kappa : U \subseteq X$ is the open immersion this pull-back homomorphism fits into a localization sequence. Denote by $Z = X \setminus U$ the closed complement. Since $D^b_{c,Z}(\mathcal{M}(\mathcal{A}))$ is a thick saturated subcategory of $D^b_c(\mathcal{M}(\mathcal{A}))$ we have an exact sequence of triangulated categories

$$D^b_{c,Z}(\mathcal{M}(\mathcal{A})) \rightarrowtail D^b_c(\mathcal{M}(\mathcal{A})) \twoheadrightarrow D^b_c(\mathcal{M}(\mathcal{A}))/D^b_{c,Z}(\mathcal{M}(\mathcal{A})),$$

and the restriction functor induces an exact functor $D^b_c(\mathcal{M}(\mathcal{A}))/D^b_{c,Z}(\mathcal{M}(\mathcal{A})) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A}|_U))$ which we claim to be an equivalence of triangulated categories. By Lemma 1.4 this is equivalent to the statement that $D^b_c(\mathcal{M}_{qc}(\mathcal{A}))/D^b_{c,Z}(\mathcal{M}_{qc}(\mathcal{A}))$ is equivalent to $D^b_c(\mathcal{M}_{qc}(\mathcal{A}|_U))$. If $\mathcal{A} = \mathcal{O}_X$ this has been proven e.g. in [Gil02], and essentially the same arguments prove the general case. Therefore by Balmer's [Bal00] localization theorem there is an exact sequence of coherent hermitian Witt groups

$$\cdots \longrightarrow \tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet) \longrightarrow \tilde{W}^i(\mathcal{A}|_U,\tau|_U,\mathcal{I}_\bullet|_U) \xrightarrow{\partial} \tilde{W}^{i+1}_Z(\mathcal{A},\tau,\mathcal{I}_\bullet) \longrightarrow \cdots$$

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(here ∂ denotes the connecting homomorphism). By well-known arguments, see e.g. [Bal01b], we get from this also a Mayer–Vietoris exact sequence.

2.8 Contrary to K-theory of coherent sheaves a flat morphism of schemes $f : Y \longrightarrow X$ does not induce a pull-back morphism of coherent Witt groups. However if both X and Y are Gorenstein schemes of finite Krull dimension we have such a pull-back. In this case a locally free $\mathcal{O}_X$-module $\mathcal{L}$ of rank 1 has a finite resolution $0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_{-1} \longrightarrow \cdots \longrightarrow \mathcal{I}_{-\dim X} \longrightarrow 0$ by quasi-coherent injective $\mathcal{O}_X$-modules. The complex $\mathcal{I}_\bullet$ considered as an object of $D^b_c(\mathcal{M}_{qc}(X))$ which lives in the indicated degrees is then a dualizing complex of X and (since f is flat) the pull-back $f^*(\mathcal{I}_\bullet)$ is a resolution of $f^*\mathcal{L}$. Therefore there is a canonical quasi-isomorphism $f^*(\mathcal{I}_\bullet) \xrightarrow{\simeq} \mathcal{J}_\bullet$, where $\mathcal{J}_\bullet$ is a finite resolution of $f^*\mathcal{L}$ by injective quasi-coherent $\mathcal{O}_Y$-modules. This quasi-isomorphism induces a natural isomorphism

$$f^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{I}_\bullet) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{F}_\bullet, \mathcal{J}_\bullet) \tag{1}$$

for all $\mathcal{F}_\bullet \in D^b_c(\mathcal{M}(X))$.

Let $(\mathcal{A}, \tau)$ be as above. We set $\tilde{W}^i(\mathcal{A}, \tau, \mathcal{L}) := \tilde{W}^i(\mathcal{A}, \tau, \mathcal{I}_\bullet)$. Let further $(\mathcal{B}, \nu)$ be a coherent $\mathcal{O}_Y$-algebra with involution of the first kind, and $\alpha : \mathcal{B} \longrightarrow f^*\mathcal{A}$ a morphism of $\mathcal{O}_Y$-algebras, such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}_Y & \longrightarrow & \mathcal{B} & \xrightarrow{\nu} & \mathcal{B}^{\mathrm{op}} \\ \big\downarrow{\scriptstyle =} & & {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^{\mathrm{op}}} \\ f^*\mathcal{O}_X & \longrightarrow & f^*\mathcal{A} & \xrightarrow{f^*\tau} & f^*\mathcal{A}^{\mathrm{op}} \end{array} \tag{2}$$

Then if $\mathcal{F}_\bullet \in D^b_c(\mathcal{M}(\mathcal{A}))$ the morphism (1) is an isomorphism in the derived category $D^b_c(\mathcal{M}(\mathcal{B}^{\mathrm{op}}))$ and induces a duality transformation for the pull-back

$$\alpha^* : D^b_c(\mathcal{M}(\mathcal{A})) \xrightarrow{f^*} D^b_c(\mathcal{M}(f^*\mathcal{A})) \xrightarrow{\subseteq} D^b_c(\mathcal{M}(\mathcal{B}))$$

with respect to the dualities $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}$ and $\mathfrak{D}^{(\mathcal{B},\nu)}_{\mathcal{J}}$, respectively. We get a homomorphism

$$\alpha^* : \tilde{W}^i(\mathcal{A}, \tau, \mathcal{L}) \longrightarrow \tilde{W}^i(\mathcal{B}, \nu, f^*\mathcal{L})$$

for all $i \in \mathbb{Z}$.

Example 2.9. If $(\mathcal{A}, \tau)$ is a coherent $\mathcal{O}_X$-algebra with involution of the first kind τ then $(f^*\mathcal{A}, f^*\tau)$ is a coherent $\mathcal{O}_Y$-algebra with involution of the first kind, too. Taking $\alpha = \mathrm{id}_{f^*\mathcal{A}}$ diagram (2) commutes and we have a pull-back map $\alpha^* : \tilde{W}^i(\mathcal{A}, \tau, \mathcal{L}) \longrightarrow \tilde{W}^i(f^*\mathcal{A}, f^*\tau, f^*\mathcal{L})$.

2.10 *Comparison with derived Witt groups.* Let X, $(\mathcal{A}, \tau)$ and $\mathcal{L}$ be as above. Let further $\mathcal{P}_b(\mathcal{A}) := \overline{\mathcal{P}(X) \cap \mathcal{M}_c(\mathcal{A})}$. This is an exact category. Then, see § 2.3, the contravariant functor $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{L}} := \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{L})$ is a duality on $D^b(\mathcal{P}_b(\mathcal{A}))$, making this a triangulated category with duality. We denote the triangular Witt groups of this category with respect to the duality $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{L}}$ by $W^i(\mathcal{A}, \tau, \mathcal{L})$. Since X is Gorenstein, $\mathcal{L}$ has a finite injective resolution $\mathcal{I}_\bullet$ and a quasi-isomorphism $\gamma : \mathcal{L} \xrightarrow{\simeq} \mathcal{I}_\bullet$ induces a duality transformation for the natural functor $F_{\mathcal{A}} : D^b(\mathcal{P}_b(\mathcal{A})) \longrightarrow D^b_c(\mathcal{M}(\mathcal{A}))$. Hence we get a homomorphism $F_{\mathcal{A},\gamma} : W^i(\mathcal{A}, \tau, \mathcal{L}) \longrightarrow \tilde{W}^i(\mathcal{A}, \tau, \mathcal{L})$ for all $i \in \mathbb{Z}$ which depends on γ. This is an isomorphism if X is moreover regular and $\mathcal{A}$ locally free as $\mathcal{O}_X$-module.

If $\mathcal{A}$ is an Azumaya algebra the category $\mathcal{P}_b(\mathcal{A})$ is equal to the category $\mathcal{P}(\mathcal{A})$ of all coherent $\mathcal{A}$-modules $\mathcal{P}$ with the property that $\mathcal{P}_x$ is a projective $\mathcal{A}_x$-module for all $x \in X$. Furthermore the reduced trace, see [KO74, ch. IV], induces an isomorphism of $\mathcal{A}$-bimodules $\theta : \mathcal{A} \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X)$.

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We get

$$\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{O}_X}(\mathcal{F}_\bullet) = \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_\bullet, \mathcal{O}_X)} \simeq \overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{F}_\bullet, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X))} \simeq \overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{F}_\bullet, \mathcal{A})}$$

for all $\mathcal{F}_\bullet \in D^b_c(\mathcal{M}(\mathcal{A}))$. Hence the Witt group $W^i(\mathcal{A}, \tau, \mathcal{O}_X)$ is isomorphic to the usual ith derived Witt group of $(\mathcal{A}, \tau)$, i.e. to the ith derived Witt group of $\mathcal{P}_b(\mathcal{A})$ with respect to the duality $\mathcal{F} \mapsto \overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})}$.

However for a coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ which is not an Azumaya algebra there seems to be no obvious relation between the groups $W^i(\mathcal{A}, \tau, \mathcal{L})$ and the usual derived Witt groups of $\mathcal{A}$.

2.11 The following is used to simplify the proof of the zero theorem, Theorem 5.4. Assume that $X = \operatorname{Spec} R$ is an affine Gorenstein scheme and (A, τ) an R-algebra with involution of the first kind which is finitely generated as R-module. Let $\mathcal{E}(R)$ be the category of R-modules M, such that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \geqslant 1$. This is an exact category containing $\mathcal{P}(R)$ the category of projective R-modules of finite rank. By [Gil02, Theorem 2.15] all modules in $\mathcal{E}(R)$ are reflexive and $\mathfrak{D}_R := \operatorname{Hom}_R(-, R)$ is a duality on $\mathcal{E}(R)$. Let $\mathcal{E}_b(A) := \mathcal{E}(R) \cap \mathcal{M}_c(A)$. Then $\mathfrak{D}^{(A,\tau)}_R := \overline{\operatorname{Hom}_R(-, R)}$ is a duality on this category. We denote the derived Witt groups of this exact category with duality by $W^i(\mathcal{E}_b(A), \tau, R)$.

Assume now moreover that A is projective as R-module. Let M be a finitely generated A-module and $\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$ be a resolution of M by finitely generated projective A-modules. Then this resolution is also a resolution by finitely generated projective R-modules, and hence $\ker d_n \in \mathcal{E}_b(A)$ for n big enough by a result of Bass [Bas63, Theorem 8.2]. It follows that the natural functor $F_A : D^b(\mathcal{E}_b(A)) \longrightarrow D^b_c(\mathcal{M}_{qc}(A))$ is an equivalence. Choosing a quasi-isomorphism $\gamma : R \xrightarrow{\simeq} I_\bullet$, where $I_\bullet$ is a finite injective resolution of the R-module R, we get as above an isomorphism $F_{A,\gamma} : W^i(\mathcal{E}_b(A), \tau, R) \xrightarrow{\simeq} \tilde{W}^i(A, \tau, I_\bullet)$ for all $i \in \mathbb{Z}$ which depends on γ.

If $f : R \longrightarrow R'$ is a flat morphism and $A' = R' \otimes_R A$ then $R' \otimes_R M \in \mathcal{E}_b(A')$ for all $M \in \mathcal{E}_b(A)$ and we have as in Example 2.9 a pull-back map $f^* : W^i(\mathcal{E}_b(A), \tau, R) \longrightarrow W^i(\mathcal{E}_b(A'), \tau', R')$, where $\tau' = \tau \otimes \mathrm{id}_{R'}$.

3. A transfer morphism

3.1 We consider in this section the following situation. Let $f : Y \longrightarrow X$ be a finite morphism of schemes, and assume that X has a dualizing complex $\mathcal{I}_\bullet \in D^b_c(\mathcal{M}_{qc}(X))$. Let further $(\mathcal{A}, \tau)$ be a coherent $\mathcal{O}_X$-algebra with involution of the first kind, and $(\mathcal{B}, \nu)$ be a coherent $\mathcal{O}_Y$-algebra with involution of the first kind. We assume that there is a morphism of coherent $\mathcal{O}_X$-algebras $\alpha : \mathcal{A} \longrightarrow f_*(\mathcal{B})$, such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}_X & \longrightarrow & \mathcal{A} & \xrightarrow{\tau} & \mathcal{A}^{\mathrm{op}} \\ \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha^{\mathrm{op}}} \\ f_*(\mathcal{O}_Y) & \longrightarrow & f_*(\mathcal{B}) & \xrightarrow{f_*\nu} & f_*(\mathcal{B}^{\mathrm{op}}) \end{array} \qquad (3)$$

The aim of this section is to construct under these assumptions a transfer morphism

$$\operatorname{Tr}_{(\mathcal{B},\nu)/(\mathcal{A},\tau)} : \tilde{W}^i(\mathcal{B}, \nu, \mathcal{J}_\bullet) \longrightarrow \tilde{W}^i(\mathcal{A}, \tau, \mathcal{I}_\bullet)$$

for a suitable dualizing complex $\mathcal{J}_\bullet$ of Y.

3.2 Let $\bar{f} : (Y, \mathcal{O}_Y) \longrightarrow (X, f_*\mathcal{O}_Y)$ be the induced morphism of ringed spaces. The pull-back $\bar{f}^*$ is an exact functor and

$$f^\natural(\mathcal{I}_\bullet) := \bar{f}^* \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{I}_\bullet)$$

is a dualizing complex of Y; cf. [Har66, ch. V, § 2]. Moreover there is a morphism of complexes $\mathrm{ev}: f_* f^\natural(\mathcal{I}_\bullet) \longrightarrow \mathcal{I}_\bullet$, such that the composition $\rho_{\mathcal{F}}$:

$$\begin{array}{ccc} f_*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}_\bullet, f^\natural(\mathcal{I}_\bullet)) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{F}_\bullet, f_*f^\natural(\mathcal{I}_\bullet)) \\ & \searrow^{\rho_{\mathcal{F}}} & \downarrow \mathrm{ev} \\ & & \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{F}_\bullet, \mathcal{I}_\bullet) \end{array}$$

is a quasi-isomorphism for all $\mathcal{F}_\bullet \in D^b_c(\mathcal{M}(Y))$. This is a duality transformation for the push-forward $f_*: D^b_c(\mathcal{M}(Y)) \longrightarrow D^b_c(\mathcal{M}(X))$ with respect to the dualities $\mathfrak{D}_{f^\natural(\mathcal{I})}$ and $\mathfrak{D}_{\mathcal{I}}$, respectively see [Gil03, § 4] or [Gil07b, § 2]. The induced morphism of Witt groups $\mathrm{Tr}_{Y/X}: \tilde{W}^i(Y, f^\natural(\mathcal{I}_\bullet)) \longrightarrow \tilde{W}^i(X, \mathcal{I}_\bullet)$ is called *transfer*.

We claim that $\rho_{\mathcal{F}}$ is a morphism of complexes of right $\mathcal{A}$-modules for all complexes of left $\mathcal{B}$-modules $\mathcal{F}$. This follows easily from the commutative diagram (3) and the local description of ev (cf. [Gil03] or [Gil07b]): let $U = \operatorname{Spec} R \subseteq X$ be an affine open subscheme and $\operatorname{Spec} R' = f^{-1}(U)$. Then

$$\Gamma(U, \mathrm{ev}): \mathrm{Hom}_R(R', I_\bullet) = \Gamma(U, f^\natural(\mathcal{I}_\bullet)) \longrightarrow \Gamma(U, \mathcal{I}_\bullet) = I_\bullet$$

is given in degree l by $h \longmapsto (-1)^l h(1)$.

Let α_* be the restriction of f_* to $D^b_c(\mathcal{M}(\mathcal{B}))$, i.e.

$$\alpha_*: D^b_c(\mathcal{M}(\mathcal{B})) \xrightarrow{f_*} D^b_c(\mathcal{M}(f_*\mathcal{B})) \xrightarrow{\subseteq} D^b_c(\mathcal{M}(\mathcal{A})).$$

Then it follows from the considerations above that ρ induces an isomorphism of functors $\bar{\rho}: \alpha_* \mathfrak{D}^{(\mathcal{B},\nu)}_{f^\natural(\mathcal{I})} \xrightarrow{\simeq} \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}} \alpha_*$ which makes α_* duality preserving, i.e.

$$(\alpha_*, \bar{\rho}): (D^b_c(\mathcal{M}(\mathcal{B})), \mathfrak{D}^{(\mathcal{B},\nu)}_{f^\natural(\mathcal{I})}, 1, \varpi^{f^\natural(\mathcal{I})}) \longrightarrow (D^b_c(\mathcal{M}(\mathcal{A})), \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}, 1, \varpi^{\mathcal{I}})$$

is a duality preserving functor. Hence we have a homomorphism

$$\mathrm{Tr}_{(\mathcal{B},\nu)/(\mathcal{A},\tau)} := (\alpha_*, \bar{\rho})_*: \tilde{W}^i(\mathcal{B}, \nu, f^\natural(\mathcal{I}_\bullet)) \longrightarrow \tilde{W}^i(\mathcal{A}, \tau, \mathcal{I}_\bullet)$$

for all $i \in \mathbb{Z}$, which we call the *transfer* of coherent hermitian Witt groups for the finite extension $\alpha: (\mathcal{A}, \tau) \longrightarrow (\mathcal{B}, \nu)$ of algebras with involution of the first kind.

Example 3.3.

(i) If $X = Y$, $f = \mathrm{id}_X$, and $(\mathcal{A}, \tau) = (\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$, then the natural morphism $\alpha: \mathcal{O}_X \longrightarrow \mathcal{B}$ makes the diagram (3) above commute and we have a transfer map

$$\mathrm{Tr}_{(\mathcal{B},\nu)/(\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})}: \tilde{W}^i(\mathcal{B}, \nu, \mathcal{I}_\bullet) \longrightarrow \tilde{W}^i(X, \mathcal{I}_\bullet).$$

(ii) If $\mathcal{B} = f^*\mathcal{A}$ and $\nu = f^*\tau$ then the natural morphism $\mathcal{A} \longrightarrow f_*\mathcal{B} = f_*f^*\mathcal{A}$ makes (3) commute and we have a transfer homomorphism

$$\mathrm{Tr}_{f^*(\mathcal{A},\tau)/(\mathcal{A},\tau)}: \tilde{W}^i(f^*\mathcal{A}, f^*\tau, f^\natural(\mathcal{I}_\bullet)) \longrightarrow \tilde{W}^i(\mathcal{A}, \tau, \mathcal{I}_\bullet).$$

4. The hermitian Gersten–Witt spectral sequence

4.1 One of the most useful tools to calculate Witt groups of symmetric bilinear forms over schemes is the Gersten–Witt spectral sequence. We construct in this section the analog of this spectral sequence in coherent hermitian Witt theory, i.e. a spectral sequence which converges to the coherent hermitian Witt theory.

4.2 *A filtration on $D_c^b(\mathcal{M}(\mathcal{A}))$.* Throughout this section X is a scheme with dualizing complex $\mathcal{I}_\bullet$ and $(\mathcal{A},\tau)$ is a coherent $\mathcal{O}_X$-algebra with involution of the first kind. Let $\mu_{\mathcal{I}} : X \longrightarrow \mathbb{Z}$ be the codimension function of the dualizing complex $\mathcal{I}_\bullet$. Recall that $\mu_{\mathcal{I}}(x)$ is the unique integer $i \in \mathbb{Z}$, such that $\operatorname{Ext}^{-i}_{\mathcal{O}_{X,x}}(k(x), \mathcal{I}_{\bullet\, x}) \neq 0$, where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ denotes the residue field of $x \in X$ (for the '$-$' sign: recall that we use homological complexes).

Associated with the codimension function $\mu_{\mathcal{I}}$ we have a finite filtration on $D_c^b(\mathcal{M}(\mathcal{A}))$. Let $D^p_{\mathcal{A},\mathcal{I}} = D_c^b(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}$ be the full subcategory of $D_c^b(\mathcal{M}(\mathcal{A}))$ whose objects are complexes $\mathcal{F}_\bullet$, such that $\mu_{\mathcal{I}}(x) \geqslant p$ for all x in the (homological) support of $\mathcal{F}_\bullet$. Then we have a finite filtration

$$D_c^b(\mathcal{M}(\mathcal{A})) = D^m_{\mathcal{A},\mathcal{I}} \supseteq D^{m+1}_{\mathcal{A},\mathcal{I}} \supseteq \cdots \supseteq D^{n-1}_{\mathcal{A},\mathcal{I}} \supseteq D^n_{\mathcal{A},\mathcal{I}} \supseteq (0),$$

where $m = \min \mu_{\mathcal{I}}$ and $n = \max \mu_{\mathcal{I}}$. Similarly we have a finite filtration on $D^b_{c,Z}(\mathcal{M}(\mathcal{A}))$ for any closed subset $Z \subseteq X$ by setting $D^{Z,p}_{\mathcal{A},\mathcal{I}} = D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}} := D^b_{c,Z}(\mathcal{M}(\mathcal{A})) \cap D^b_c(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}$.

Since $D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}$ is a thick saturated subcategory of $D^b_{c,Z}(\mathcal{M}(\mathcal{A}))$ the quotient $D^{Z,p}_{\mathcal{A},\mathcal{I}}/D^{Z,p+1}_{\mathcal{A},\mathcal{I}}$ exists for all $p \in \mathbb{Z}$. The restriction of the duality $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}$ to these subcategories is a duality on them and therefore we have an exact sequence of triangulated categories with duality

$$D^{Z,p+1}_{\mathcal{A},\mathcal{I}} \rightarrowtail D^{Z,p}_{\mathcal{A},\mathcal{I}} \twoheadrightarrow D^{Z,p}_{\mathcal{A},\mathcal{I}}/D^{Z,p+1}_{\mathcal{A},\mathcal{I}}$$

for all $p \in \mathbb{Z}$. By Balmer's [Bal00] localization theorem we get then long exact sequences of triangulated Witt groups

$$\cdots \longrightarrow W^i(D^{Z,p}_{\mathcal{A},\mathcal{I}}) \longrightarrow W^i(D^{Z,p}_{\mathcal{A},\mathcal{I}}/D^{Z,p+1}_{\mathcal{A},\mathcal{I}}) \xrightarrow{\partial} W^{i+1}(D^{Z,p+1}_{\mathcal{A},\mathcal{I}}) \longrightarrow \cdots$$

which constitute an exact couple. This gives rise to a spectral sequence.

Definition 4.3. The associated spectral sequence

$$E_1^{p,q}(\mathcal{A},\tau,\mathcal{I}_\bullet,Z) := W^{p+q}(D^{Z,p}_{\mathcal{A},\mathcal{I}}/D^{Z,p+1}_{\mathcal{A},\mathcal{I}}) \quad \Longrightarrow \quad \tilde{W}^{p+q}_Z(\mathcal{A},\tau,\mathcal{I}_\bullet)$$

is called the *coherent hermitian Gersten–Witt spectral sequence* of $(\mathcal{A},\tau)$ with *support* in Z. If $Z = X$ we use the symbol $E_1^{p,q}(\mathcal{A},\tau,\mathcal{I}_\bullet)$ instead and call this the *coherent hermitian Gersten–Witt spectral sequence* of $(\mathcal{A},\tau)$.

Example 4.4. Let X be a regular scheme of finite Krull dimension and $\mathcal{A}$ an Azumaya algebra over X with involution of the first kind τ. Then $\mathcal{A} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A},\mathcal{O}_X)$ as $\mathcal{A}$-bimodules. Let $\mathcal{I}_\bullet$ be a finite injective resolution of the $\mathcal{O}_X$-module $\mathcal{O}_X$. We have then $\tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet) \simeq W^i(\mathcal{A},\tau,\mathcal{O}_X)$, and the latter group is isomorphic to the usual derived Witt group of $(\mathcal{A},\tau)$, see § 2.10, i.e. $E_1^{p,q}(\mathcal{A},\tau,\mathcal{I}_\bullet)$ converges to the derived Witt theory of $(\mathcal{A},\tau)$.

4.5 *Functorial properties of the spectral sequence.* We start with localization. Let for this $\kappa : U \hookrightarrow X$ be either an open immersion $U \subseteq X$ or an embedding $\operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow X$ for some $x \in X$. The pullback $\kappa^* : D_c^b(\mathcal{M}(\mathcal{A})) \longrightarrow D_c^b(\mathcal{M}(\kappa^*\mathcal{A}))$ respects the filtration induced by the codimension functions $\mu_{\mathcal{I}}$ and $\mu_{\kappa^*\mathcal{I}} = \mu_{\mathcal{I}}|_U$, respectively. Hence by the functorial properties of the localization sequence, see [Gil02, Theorem 2.9], the duality preserving functor described in § 2.7 induces a morphism of localization sequences associated with these filtrations. Hence we get a homomorphism of spectral sequences: $\kappa^* : E_1^{p,q}(\mathcal{A},\tau,\mathcal{I}_\bullet) \longrightarrow E_1^{p,q}(\kappa^*\mathcal{A},\kappa^*\tau,\kappa^*\mathcal{I}_\bullet)$.

We consider now the situation of § 3.1, i.e. $f : Y \longrightarrow X$ is a finite morphism, $(\mathcal{B},\nu)$ is a coherent $\mathcal{O}_Y$-algebra with involution of the first kind, such that there is a morphism of $\mathcal{O}_X$-algebras $\alpha : \mathcal{A} \longrightarrow f_*(\mathcal{B})$ making diagram (3) commute. In this situation we defined in § 3.2 a duality preserving functor $(\alpha_*, \bar{\rho})$ between $D_c^b(\mathcal{M}(\mathcal{B}))$ and $D_c^b(\mathcal{M}(\mathcal{A}))$, where the dualities are induced by $\mathcal{I}_\bullet$ and $f^\natural(\mathcal{I}_\bullet)$, respectively. Since the push-forward f_* respects the filtrations coming from the codimension functions of these dualizing complexes we get as above a morphism of spectral sequences

$$\mathrm{Tr}_{(\mathcal{B},\nu)/(\mathcal{A},\tau)} : E_1^{p,q}(\mathcal{B},\nu,f^\natural(\mathcal{I}_\bullet)) \longrightarrow E_1^{p,q}(\mathcal{A},\tau,\mathcal{I}_\bullet).$$

4.6 We aim now to give a local description of the terms of the coherent hermitian Gersten–Witt spectral sequence. Let for this $X_{\mathcal{I}}^{(p)} := \{x \in X \mid \mu_{\mathcal{I}}(x) = p\}$ be the set of points of $\mu_{\mathcal{I}}$-codimension p. Localization induces a functor

$$D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}/D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p+1)}_{\mathcal{I}} \longrightarrow \coprod_{x\in X_{\mathcal{I}}^{(p)}\cap Z} D^b_{c,\mathfrak{m}_x}(\mathcal{M}_{qc}(\mathcal{A}_x)), \tag{4}$$

where $D^b_{c,\mathfrak{m}_x}(\mathcal{M}_{qc}(\mathcal{A}_x))$ denotes the bounded derived category of $\mathcal{A}_x$-modules whose homology modules are annihilated by some power of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$. This is an equivalence for all $p \in \mathbb{Z}$. To show this we can (using Lemma 1.4) replace the categories $D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}$ by $D^b_{c,Z}(\mathcal{M}_{qc}(\mathcal{A}))^{(p)}_{\mathcal{I}} := D^b_{c,Z}(\mathcal{M}_{qc}(\mathcal{A})) \cap D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}$. For these categories and $\mathcal{A} = \mathcal{O}_X$ a proof can be found e.g. in [Gil07a, § 5]. Essentially the same arguments work for any coherent $\mathcal{O}_X$-algebra.

We get from this equivalence an isomorphism

$$E_1^{p,q}(\mathcal{A},\tau,\mathcal{I}_\bullet,Z) \xrightarrow{\simeq} \bigoplus_{x\in X_{\mathcal{I}}^{(p)}\cap Z} \tilde{W}^{p+q}_{\mathfrak{m}_x}(\mathcal{A}_x,\tau_x,\mathcal{I}_{\bullet,x}), \tag{5}$$

where $\tilde{W}^i_{\mathfrak{m}_x}(\mathcal{A}_x,\tau_x,\mathcal{I}_{\bullet,x})$ denotes the coherent Witt group of $(\mathcal{A}_x,\tau_x)$ with support in the closed point $\mathfrak{m}_x$ of $\operatorname{Spec}\mathcal{O}_{X,x}$ and with respect to the duality $\mathfrak{D}^{(\mathcal{A}_x,\tau_x)}_{\mathcal{I}_x}$, cf. § 2.5. We show that these groups are either zero or isomorphic to (skew-)hermitian Witt groups of $(\mathcal{A}\otimes k(x))/\mathrm{rad}(\mathcal{A}\otimes k(x))$.

4.7 Let for this $(R,\mathfrak{m},k)$ be a local (noetherian) ring with dualizing complex $I_\bullet$ and A an R-algebra which is finitely generated as R-module and possesses an involution τ of the first kind. Since R is noetherian we can assume, see e.g. [Gil07a, § 1], that $I_\bullet$ is a residual dualizing complex, and therefore $E := I_{-n} = \mathrm{E}_R(k)$ is the R-injective hull of the residue field k, where $n = \max\mu_{\mathcal{I}} = \mu_{\mathcal{I}}(\mathfrak{m})$. Let further $\mathcal{M}_{\mathrm{fl}}(A)$ be the abelian category of all A-modules which have finite length as R-modules.

It follows from the assumption that R is a noetherian ring that the natural functor

$$D^b(\mathcal{M}_{\mathrm{fl}}(A)) \longrightarrow D^b_{c,\mathfrak{m}}(\mathcal{M}_{qc}(A))$$

is an equivalence. We equip the left-hand side of this equivalence with a duality as follows. As well known, see e.g. [Gil07a, § 3], $\mathfrak{D}_E := \mathrm{Hom}_R(-,E)$ is a duality on the bounded derived category $D^b(\mathcal{M}_{\mathrm{fl}}(R))$ of finite length R-modules, and therefore $\mathfrak{D}^{(A,\tau)}_E := \overline{\mathrm{Hom}_R(-,E)}$ is a duality on $D^b(\mathcal{M}_{\mathrm{fl}}(A))$, making this a triangulated category with duality; see § 2.3. We denote the associated triangular Witt groups $W^i(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$.

Considering E as a complex concentrated in degree 0 we have an isomorphism of complexes of (left-)A-modules $\overline{\mathrm{Hom}_R(M_\bullet,E)} \xrightarrow{\simeq} T^n\overline{\mathrm{Hom}_R(M_\bullet,I_\bullet)}$ which is given by $(-1)^{in}$ id in degree i for all $M_\bullet \in D^b(\mathcal{M}_{\mathrm{fl}}(A))$. This makes the equivalence $D^b(\mathcal{M}_{\mathrm{fl}}(A)) \longrightarrow D^b_{c,\mathfrak{m}}(\mathcal{M}_{qc}(A))$ duality preserving and so we get an isomorphism

$$W^i(\mathcal{M}_{\mathrm{fl}}(A),\tau,E) \xrightarrow{\simeq} \tilde{W}^{i+n}_{\mathfrak{m}}(A,\tau,I_\bullet)$$

for all $i \in \mathbb{Z}$. Since $\mathcal{M}_{\mathrm{fl}}(A)$ is an abelian category we have $W^{2l+1}(\mathcal{M}_{\mathrm{fl}}(A),\tau,E) = 0$ for all $l \in \mathbb{Z}$ by [BW02, Proposition 5.2]. This proves one half of the following lemma.

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Lemma 4.8. *Let* $(R,\mathfrak{m},k)$, $I_\bullet$, *and* (A,τ) *be as above and* $A_k := A\otimes_R k$ *and* $\tau_k := \tau\otimes \mathrm{id}_k$. *Then we have*

$$\tilde{W}^i_{\mathfrak{m}}(A,\tau,I_\bullet) \simeq \begin{cases} W^+(A_k,\tau_k,k) & i\equiv n \bmod 4,\\ W^-(A_k,\tau_k,k) & i\equiv n+2 \bmod 4,\\ 0 & \text{otherwise,}\end{cases}$$

where $W^\pm(A_k,\tau_k,k)$ *denotes the (skew-)hermitian Witt group of the exact category* $\mathcal{P}_b(A_k) = \mathcal{M}_c(A_k)$ *with respect to the duality* $\mathfrak{D}_k^{(A_k,\tau_k)} := \overline{\mathrm{Hom}_k(-,k)}$.

Proof. Let $F_A : \mathcal{M}_{\mathrm{fl}}(A) \longrightarrow D^b(\mathcal{M}_{\mathrm{fl}}(A))$ be the natural functor. By the main result of [Bal01a] we know that F_A and TF_A induce isomorphisms

$$W^+(\mathcal{M}_{\mathrm{fl}}(A),\tau,E) \xrightarrow{\simeq} W^0(\mathcal{M}_{\mathrm{fl}}(A),\tau,E) \simeq W^{4i}(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$$

and

$$W^-(\mathcal{M}_{\mathrm{fl}}(A),\tau,E) \xrightarrow{\simeq} W^2(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)) \simeq W^{4i+2}(\mathcal{M}_{\mathrm{fl}}(A),\tau,E),$$

respectively, where $W^\pm(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$ denotes the (skew-)hermitian Witt group of the abelian category $\mathcal{M}_{\mathrm{fl}}(A)$ with respect to the duality $\mathfrak{D}_E^{(A,\tau)} = \overline{\mathrm{Hom}_R(-,E)}$, and the latter isomorphisms are due to 4-periodicity of triangular Witt groups.

We fix now an embedding $\iota : k \hookrightarrow E = \mathrm{E}_R(k)$. Then $h\mapsto \iota h$ is an isomorphism of A-modules $\overline{\mathrm{Hom}_k(M,k)} \xrightarrow{\simeq} \overline{\mathrm{Hom}_R(M,E)}$ for all $M\in\mathcal{P}_b(A_k)$ which makes the natural functor $G_A : \mathcal{P}_b(A_k)\longrightarrow \mathcal{M}_{\mathrm{fl}}(A)$ duality preserving. We claim that the induced homomorphism

$$G_{A,\iota} : W^\epsilon(A_k,\tau_k,k) \longrightarrow W^\epsilon(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$$

is an isomorphism for $\epsilon=+$ and $\epsilon=-$ (note that $G_{A,\iota}$ depends on the embedding ι).

Since $\mathcal{M}_{\mathrm{fl}}(A)$ is an abelian category we know by [QSS79, Corollary 6.4] that an element of the group $W^\epsilon(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$ is zero if and only if it is metabolic. It follows that $G_{A,\iota}$ is injective. To show that it is surjective, too, we adapt an argument of [Sch85].

Let the ϵ-hermitian space (M,φ) represent an element of $W^\epsilon(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$ and

$$h_\varphi : M\times M \longrightarrow E, \quad (m,m')\longmapsto \varphi(m)(m'),$$

be the associated ϵ-hermitian sesquilinear form. If M has length 1 as R-module then $\mathfrak{m}M=0$ and so $M\in\mathcal{P}_b(A_k)$. Since $\mathrm{Im}\,\iota = \{x\in E \mid \mathfrak{m}\cdot x = 0\}$ it follows that the class $[M,\varphi]$ of the space (M,φ) is in the image of $G_{A,\iota}$.

Otherwise there exists $d\geqslant 2$, such that $\mathfrak{m}^{d-1}M\neq 0$ but $\mathfrak{m}^d M = 0$. Then the subspace $\mathfrak{m}^{d-1}M\subseteq M$ is totally isotropic and therefore φ induces an ϵ-hermitian non-degenerate form φ' on $M' := N/\mathfrak{m}^{d-1}M$, where we have set

$$N := (\mathfrak{m}^{d-1}M)^\perp := \{m\in M \mid h_\varphi(m,x)=0 \text{ for all } x\in\mathfrak{m}^{d-1}M\}.$$

It is easy to see that the A-submodule $\{(x, x+\mathfrak{m}^{d-1}M) \mid x\in N\}$ of $M\oplus M'$ is a Lagrangian of $(M,\varphi)\perp(M',-\varphi')$ and so we have $[M,\varphi]=[M',\varphi']$ in the Witt group $W^\epsilon(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)$. Since $\mathrm{length}\,M' < \mathrm{length}\,M$ it follows by induction that $[M',\varphi']$ is in the image of $G_{A,\iota}$, and we are done. ∎

4.9 Using the notation of the last section we denote the composition of isomorphisms

$$W^+(A_k,\tau_k,k)\longrightarrow W^0(\mathcal{M}_{\mathrm{fl}}(A),\tau,E)\longrightarrow \tilde{W}^i_{\mathfrak{m}}(A,\tau,I_\bullet)$$

by $H_{A,\iota}$. Let $\mathfrak{a}\subseteq\mathfrak{m}$ be an ideal of R and $R' := R/\mathfrak{a}$, $A' := A\otimes_R R'$, and $\tau' := \tau\otimes\mathrm{id}_{R'}$. Then $\mathfrak{m}' := \mathfrak{m}/\mathfrak{a}$ is the maximal ideal of R' and the residue field of R' is k, too. We have $A'_k = A'\otimes_{R'} k = A_k$

and $\tau'_k = \tau' \otimes \mathrm{id}_k = \tau_k$, and a diagram

$$\begin{array}{ccc} W^+(A'_k, \tau'_k, k) & \xrightarrow{H_{A',\iota'}} & \tilde{W}^n_{\mathfrak{m}'}(A', \tau', \pi^\natural(I_\bullet)) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{(A',\tau')/(A,\tau)}} \\ W^+(A_k, \tau_k, k) & \xrightarrow{H_{A,\iota}} & \tilde{W}^n_{\mathfrak{m}}(A, \tau, I_\bullet) \end{array} \qquad (6)$$

where $\pi : R \longrightarrow R'$ is the natural quotient morphism. We claim that for suitable choices of injective hull injections $\iota : k \hookrightarrow \mathrm{E}_R(k)$ and $\iota' : k \hookrightarrow \mathrm{E}_{R'}(k) = \mathrm{Hom}_R(R', E)$ this diagram commutes. Recall for this, see § 3.2, that the duality transformation

$$\overline{\mathrm{Hom}_{R'}(-, \pi^\natural(I_\bullet))} = \overline{\mathrm{Hom}_{R'}(-, \mathrm{Hom}_R(R', I_\bullet))} \xrightarrow{\simeq} \overline{\mathrm{Hom}_R(-, I_\bullet)}$$

for the push-forward $D^b_{c,\mathfrak{m}'}(\mathcal{M}_{qc}(A')) \longrightarrow D^b_{c,\mathfrak{m}}(\mathcal{M}_{qc}(A))$ is induced by the evaluation morphism $\mathrm{ev} = \mathrm{ev}_\bullet : \pi^\natural(I_\bullet) = \mathrm{Hom}_R(R', I_\bullet) \longrightarrow I_\bullet$ which is given in degree l by $h \mapsto (-1)^l h(1)$. Hence if we choose $\iota' : k \hookrightarrow \mathrm{Hom}_R(R', E)$ arbitrary we can take $\iota = \mathrm{ev}_n \iota'$ to make (6) commute.

The same arguments apply to the skew-hermitian Witt groups and so we obtain the following result.

Lemma 4.10. *The transfer homomorphism*

$$\mathrm{Tr}_{(A',\tau')/(A,\tau)} : \tilde{W}^i_{\mathfrak{m}'}(A', \tau', \pi^\natural(I_\bullet)) \longrightarrow \tilde{W}^i_{\mathfrak{m}}(A, \tau, I_\bullet)$$

is an isomorphism for all $i \in \mathbb{Z}$.

4.11 We denote by $\mathrm{rad}\, A_k$ the Jacobson radical of A_k, and $(A_k)_0 := A_k/\mathrm{rad}\, A_k$. Since $\dim_k A < \infty$ the latter is a semisimple k-algebra and so by a theorem of Eilenberg and Nakayama [EN55, Proposition 5] a symmetric k-algebra, i.e. there is an isomorphism of $(A_k)_0$-bimodules $\theta : (A_k)_0 \simeq \mathrm{Hom}_k((A_k)_0, k)$. Recall that if $(A_k)_0$ is moreover a separable k-algebra then there is a canonical isomorphism $(A_k)_0 \xrightarrow{\simeq} \mathrm{Hom}_k((A_k)_0, k)$ induced by the reduced trace; see e.g. [Rei75, Theorem 9.26].

The involution τ_k of A_k sends maximal left ideals to maximal right ideals and therefore we have $\tau_k(\mathrm{rad}\, A_k) = \mathrm{rad}\, A_k$. It follows that τ_k induces an involution of the first kind on $(A_k)_0$ which we denote $(\tau_k)_0$. There are two dualities on the exact category $\mathcal{P}((A_k)_0) = \mathcal{M}_c((A_k)_0)$: the usual one $\overline{\mathrm{Hom}_{(A_k)_0}(-, (A_k)_0)}$ and $\mathfrak{D}_k^{((A_k)_0,(\tau_k)_0)} = \overline{\mathrm{Hom}_k(-, k)}$. We denote the (skew-)hermitian Witt group of this category with respect to the usual duality by $W^\pm((A_k)_0, (\tau_k)_0)$ and with respect to the other duality by $W^\pm((A_k)_0, (\tau_k)_0, k)$. The isomorphism $\theta : (A_k)_0 \xrightarrow{\simeq} \mathrm{Hom}_k((A_k)_0, k)$ induces an isomorphism between these two duality structures and hence we have an isomorphism (which depends on θ)

$$W^\epsilon((A_k)_0, (\tau_k)_0) \xrightarrow{\simeq} W^\epsilon((A_k)_0, (\tau_k)_0, k)$$

for $\epsilon = +$ and $\epsilon = -$. Therefore the forgetful functor $s_0 : \mathcal{M}_c((A_k)_0) \longrightarrow \mathcal{M}_c(A_k)$ induces a homomorphism $s_{0*} : W^\epsilon((A_k)_0, (\tau_k)_0) \longrightarrow W^\epsilon(A_k, \tau_k, k)$ for $\epsilon = +$ and $\epsilon = -$. The categories $\mathcal{M}_c((A_k)_0)$ and $\mathcal{M}_c(A_k)$ are both abelian, and by Nakayama we have $(\mathrm{rad}\, A_k)^l M = 0$ for all $M \in \mathcal{M}_c(A_k)$ for some $l \geqslant 0$ (which depends on M). Hence as in the proof of Lemma 4.8 we see that s_{0*} is bijective. We get the following corollary.

Corollary 4.12. *We have*

$$W^\epsilon((A_k)_0, (\tau_k)_0) \simeq W^\epsilon(A_k, \tau_k, k)$$

for $\epsilon = +$ and $\epsilon = -$.

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5. Dévissage and zero theorem

5.1 We start with dévissage. Let $\pi : Z \hookrightarrow X$ be a closed subscheme, $\mathcal{I}_\bullet$ a dualizing complex of X, and $(\mathcal{A},\tau)$ a coherent $\mathcal{O}_X$-algebra with involution of the first kind. The canonical morphism of $\mathcal{O}_X$-algebras $\alpha : \mathcal{A} \longrightarrow \pi_*\pi^*\mathcal{A}$ makes diagram (3) in § 3.1 with $(\mathcal{B},\nu) = \pi^*(\mathcal{A},\tau)$ and $Y = Z$ commute, and so we have a transfer map

$$\mathrm{Tr}_{\pi^*(\mathcal{A},\tau)/(\mathcal{A},\tau)} : \tilde{W}^i(\pi^*\mathcal{A},\pi^*\tau,\pi^\natural(\mathcal{I}_\bullet)) \longrightarrow \tilde{W}^i_Z(\mathcal{A},\tau,\mathcal{I}_\bullet),$$

cf. Example 3.3(ii).

Theorem 5.2. *The above transfer morphism is an isomorphism.*

Proof. Let $D^p := D^b_{c,Z}(\mathcal{M}(\mathcal{A}))^{(p)}_{\mathcal{I}}$ and $D^p_Z := D^b_c(\mathcal{M}(\pi^*\mathcal{A}))^{(p)}_{\pi^\natural(\mathcal{I})}$. By the functorial properties of the localization sequence, cf. § 4.5, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & W^i(D^p_Z) & \longrightarrow & W^i(D^p_Z/D^{p+1}_Z) & \xrightarrow{\partial} & W^{i+1}(D^{p+1}_Z) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} & \\
\cdots \longrightarrow & W^i(D^p) & \longrightarrow & W^i(D^p/D^{p+1}) & \xrightarrow{\partial} & W^{i+1}(D^{p+1}) & \longrightarrow \cdots
\end{array}$$

(with the obvious dualities) for all $i,p \in \mathbb{Z}$ and therefore (since the filtration is finite) it is enough to show that the induced morphism $\mathrm{Tr} : W^i(D^p_Z/D^{p+1}_Z) \longrightarrow W^i(D^p/D^{p+1})$ is an isomorphism for all $p \in \mathbb{Z}$. The transfer commutes with localization and therefore we have a commutative diagram

$$\begin{array}{ccc}
W^i(D^p_Z/D^{p+1}_Z) & \xrightarrow{\simeq} & \bigoplus\limits_{\mu_{\pi^\natural(\mathcal{I})}(x)=p} \tilde{W}^i_{\mathfrak{m}'_x}(\pi^*(\mathcal{A})_x,\pi^*(\tau)_x,\pi^\natural(\mathcal{I}_\bullet)_x) \\
\downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle (\mathrm{Tr}_{(\pi^*(\mathcal{A})_x,\pi^*(\tau)_x)/(\mathcal{A}_x,\tau_x)})_{\mu_{\pi^\natural(\mathcal{I})}(x)=p}} \\
W^i(D^p/D^{p+1}) & \xrightarrow{\simeq} & \bigoplus\limits_{x \in X^{(p)}_{\mathcal{I}} \cap Z} \tilde{W}^i_{\mathfrak{m}_x}(\mathcal{A}_x,\tau_x,\mathcal{I}_{\bullet\, x})
\end{array}$$

for all $i,p \in \mathbb{Z}$, where $\mathfrak{m}'_x$ denotes the maximal ideal of the local ring of x in the subscheme Z. The theorem follows from this commutative diagram and Lemma 4.10. □

5.3 We consider now the following situation. Let R be a Gorenstein ring of finite Krull dimension, $I_\bullet \in D^b_c(\mathcal{M}_{qc}(R))$ a minimal injective resolution of the R-module R, and A an R-algebra which is finitely generated and projective as R-module and has an involution of the first kind τ. Let further $t \in R$ be a non-zero divisor. Then $R' = R/Rt$ is a Gorenstein ring, too. We denote by π the quotient map $R \longrightarrow R'$, and set $(A',\tau') = (A \otimes_R R', \tau \otimes \mathrm{id}_{R'})$.

For each $p \geqslant 0$ there is a commutative diagram of duality preserving functors

$$\begin{array}{ccc}
(D^b_c(\mathcal{M}_{qc}(A'))^{(p+1)}_{\pi^\natural(I)},\mathfrak{D}^{(A',\tau')}_{\pi^\natural(I)},1,\varpi^{\pi^\natural(I)}) & \xrightarrow{(\pi_*,\bar{\rho})} & (D^b_c(\mathcal{M}_{qc}(A))^{(p+1)}_I,\mathfrak{D}^{(A,\tau)}_I,1,\varpi^I) \\
& \searrow{\scriptstyle (\pi_*,\xi)} & \downarrow{\scriptstyle \subseteq} \\
& & (D^b_c(\mathcal{M}_{qc}(A))^{(p)}_I,\mathfrak{D}^{(A,\tau)}_I,1,\varpi^I)
\end{array}$$

see Example 3.3, where (π_*,ξ) is the composition of $(\pi_*,\bar{\rho})$ and the column arrow on the right-hand side, and $\bar{\rho}$ is the isomorphism described in § 3.2. We have then the following result which is called the zero theorem for the transfer.

Theorem 5.4. *If $\pi : R \longrightarrow R'$ has a flat splitting morphism $q : R' \longrightarrow R$, i.e. q is a flat morphism, such that $\pi q = \mathrm{id}_{R'}$, then for any i-symmetric space $(M_\bullet, \phi)$ in $D^b_c(\mathcal{M}_{qc}(A'))^{(p+1)}_{\pi^\natural(I)}$ with respect to the duality $\mathfrak{D}^{(A',\tau')}_{\pi^\natural(I)}$ the push-forward $(\pi_*, \xi)_*(M, \phi)$ is a neutral i-symmetric space in $D^b_c(\mathcal{M}_{qc}(A))^{(p)}_I$ with respect to the duality $\mathfrak{D}^{(A,\tau)}_I$.*

Proof. Let $(\mathcal{A}, \tau)$ be a coherent algebra with involution of the first kind over a scheme X with dualizing complex $\mathcal{I}_\bullet$. We will use in this proof the duality preserving functor $\mathfrak{sh} := (\mathrm{id}_{D^b_c(\mathcal{M}(\mathcal{A}))}, \sigma)$:

$$(D^b_c(\mathcal{M}(\mathcal{A})), \mathfrak{D}^{(\mathcal{A},\tau)}_{T\mathcal{I}}, 1, \varpi^{T\mathcal{I}}) \longrightarrow (D^b_c(\mathcal{M}(\mathcal{A})), T\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{I}}, -1, -\varpi^{\mathcal{I}}),$$

where σ is $(-1)^l\,\mathrm{id}$ in degree l. The induced homomorphism $\mathfrak{sh}_* : \tilde{W}^i(\mathcal{A}, \tau, T\mathcal{I}_\bullet) \longrightarrow \tilde{W}^{i+1}(\mathcal{A}, \tau, \mathcal{I})$ is an isomorphism for all $i \in \mathbb{Z}$. Let further

$$\begin{array}{lcccccccccc}
K_\bullet(t) & : & \cdots \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot t} & R & \longrightarrow & 0 & \longrightarrow \cdots \\
\Big\downarrow \zeta & & & \Big\downarrow & & \Big\downarrow {-\mathrm{id}} & & \Big\downarrow \mathrm{id} & & \Big\downarrow & \\
T(\mathfrak{D}_R(K_\bullet(t))) & : & \cdots \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot(-t)} & R & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

be the standard 1-symmetric space on the Koszul complex of length 1 (living in degrees 0 and 1). The idea of the proof is to show that in this situation the transfer is essentially multiplication by the 1-symmetric space $(K_\bullet(t), \zeta)$ which is zero in $W^1(R)$ (but in general not in $W^1_{Rt}(R)$).

The complex $\pi^\natural(I_\bullet)$ is a finite injective resolution of the R'-module R' (starting in degree -1). Let $\gamma' : R' \simeq T(\pi^\natural(I_\bullet))$ be a quasi-isomorphism (we shift here the complex $\pi^\natural(I_\bullet)$ by 1 so that it starts in degree 0). Then by § 2.11 we have $(M_\bullet, \phi) \simeq \mathfrak{sh}_* F_{A',\gamma'}(N_\bullet, \psi)$ for some $(i-1)$-symmetric space $(N_\bullet, \psi)$ in $D^b(\mathcal{E}_b(A'))$ with respect to the duality $\mathfrak{D}^{(A',\tau')}_{R'}$. We can assume that ψ is not a fraction, i.e. a morphism in the category of complexes given by A'-linear maps $\psi_l : N_l \longrightarrow \overline{\mathrm{Hom}_{R'}(N_{-l+(i-1)}, R')}$.

Denote by $\star$ the product introduced in § 2.6. We claim now that there exists quasi-isomorphisms $\gamma' : R' \simeq T(\pi^\natural(I_\bullet))$ and $\gamma : R \simeq I_\bullet$, such that

$$F_{A,\gamma}((K_\bullet(t), \zeta) \star q^*(N_\bullet, \psi)) \simeq (\pi_*, \xi)_*(\mathfrak{sh}_* F_{A',\gamma'}(N_\bullet, \psi)) \tag{7}$$

for all $(i-1)$-symmetric spaces $(N_\bullet, \psi)$ in $D^b(\mathcal{E}_b(A'))$. This implies the theorem since $q^*(N_\bullet)$ is a Lagrangian of the space on the left-hand side and $q^*(N_\bullet) \simeq q^*(M_\bullet) \in D^b_c(\mathcal{M}_{qc}(A))^{(p)}_I$ since q is flat.

We choose γ and γ' as follows. Note first that $I_0 \xrightarrow{\cdot t} I_0$ is an isomorphism since I_0 is the R-injective hull of R and t is not a zero divisor in R. We denote by t^{-1} the inverse. Let $\gamma : R \simeq I_\bullet$ be arbitrary. Then $\pi(r) \longmapsto d^I_0(t^{-1}\gamma(r))$ defines a monomorphism $\gamma'' : R' \longrightarrow I_{-1}$ with image equal to the image of $\mathrm{ev}_{-1} : \mathrm{Hom}_R(R', I_{-1}) \longrightarrow I_{-1}$. We set $\gamma' := (\mathrm{ev}_{-1})^{-1} \cdot \gamma''$ (note that ev_{-1} is injective in this case).

To prove the claim we note first that the underlying complex of the space on the left-hand side of (7) is $K_\bullet(t) \otimes_{R'} N_\bullet$, and the one of the space on the right-hand side is $N_\bullet$ considered as a complex of A-modules. There is a quasi-isomorphism of complexes of A-modules $g_N : K_\bullet(t) \otimes_{R'} N_\bullet \longrightarrow N_\bullet$ which is given in degree l by

$$(R \otimes_{R'} N_l) \oplus (R \otimes_{R'} N_{l-1}) \longrightarrow N_l, \quad (r \otimes n, r' \otimes n'), \longmapsto \pi(r) \cdot n.$$

Denote by ϕ_1 (respectively ϕ_2) the form of the space on the left- (respectively right-) hand side of (7). We claim that the morphisms of complexes ϕ_1 and $T^i\mathfrak{D}^{(A,\tau)}_I(g_N) \cdot \phi_2 \cdot g_N$:

$$K_\bullet(t) \otimes_{R'} N_\bullet \longrightarrow T^i\mathfrak{D}^{(A,\tau)}_I(K_\bullet(t) \otimes_{R'} N_\bullet)$$

are homotopic. Set $N^R := R \otimes_{R'} N$ and $\mathfrak{D}_0(N^R) := \overline{\mathrm{Hom}_R(N^R, I_0)}$ for $N \in \mathcal{M}_c(A')$. We leave it to the reader to check that

$$s_l := \begin{pmatrix} 0 & 0 \\ t^{-1}\gamma(\mathrm{id}_R \otimes \psi_l) & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \quad N_l^R \oplus N_{l-1}^R \longrightarrow \mathfrak{D}_0(N^R_{-(l+1)+(i-1)}) \oplus \mathfrak{D}_0(N^R_{-l+(i-1)}) \oplus \cdots$$

defines an A-linear homotopy between them; cf. [GH05, § 2]. We are done. □

6. The hermitian Gersten–Witt complex

6.1 *Definition of the complex.* Let X be a scheme with dualizing complex $\mathcal{I}_\bullet$ and $(\mathcal{A}, \tau)$ a coherent $\mathcal{O}_X$-algebra with involution of the first kind. By Lemma 4.8 and Corollary 4.12 we know that the odd lines of the hermitian Gersten–Witt spectral sequence are zero, and the lines $2l$, $l \in \mathbb{Z}$, are isomorphic to the following complexes:

$$\bigoplus_{\mu_{\mathcal{I}}(x)=m} W^+(\mathcal{A}(x)_0, \tau(x)_0) \longrightarrow \bigoplus_{\mu_{\mathcal{I}}(x)=m+1} W^+(\mathcal{A}(x)_0, \tau(x)_0) \longrightarrow \cdots \tag{8}$$

if l is even, and

$$\bigoplus_{\mu_{\mathcal{I}}(x)=m} W^-(\mathcal{A}(x)_0, \tau(x)_0) \longrightarrow \bigoplus_{\mu_{\mathcal{I}}(x)=m+1} W^-(\mathcal{A}(x)_0, \tau(x)_0) \longrightarrow \cdots \tag{9}$$

if l is odd. Here we have set $m = \min \mu_{\mathcal{I}}$, $\mathcal{A}(x) := \mathcal{A} \otimes k(x)$, $\tau(x) := \tau \otimes \mathrm{id}_{k(x)}$, and $\mathcal{A}(x)_0 = \mathcal{A}(x)/\mathrm{rad}\,\mathcal{A}(x)$, and $\tau(x)_0$ denotes the involution induced by $\tau(x)$ on $\mathcal{A}(x)_0$. These complexes depend on the choice of embeddings $\iota_x : k(x) \hookrightarrow \mathrm{E}_{\mathcal{O}_{X,x}}(k(x))$ and isomorphisms $\theta_x : \mathcal{A}(x)_0 \simeq \mathrm{Hom}_{k(x)}(\mathcal{A}(x)_0, k(x))$. We denote the complex (8) by $\mathrm{GWh}^+(\mathcal{A}, \tau, \mathcal{I}_\bullet)$ and call it the *hermitian Gersten–Witt complex*, and the complex (9) by $\mathrm{GWh}^-(\mathcal{A}, \tau, \mathcal{I}_\bullet)$ and call it the *skew-hermitian Gersten–Witt complex* of $\mathcal{A}$ with respect to the duality $\mathfrak{D}_{\mathcal{I}}^{(\mathcal{A},\tau)}$. We denote their differentials by d_+^* and d_-^*, respectively. If we want to emphasize the dependences of (8) and (9) on the families $\underline{\iota} = \{\iota_x\}_{x\in X}$ and $\underline{\theta} = \{\theta_x\}_{x\in X}$ we use the notation $\mathrm{GWh}^\pm(\mathcal{A}, \tau, \mathcal{I}_\bullet, \underline{\iota}, \underline{\theta})$ and $d_\pm^*(\underline{\iota}, \underline{\theta})$ instead.

6.2 *The differential of the (skew-)hermitian Gersten–Witt complex.* We consider now the differentials of the (skew-)hermitian Gersten–Witt complex more closely. We denote $d_\epsilon^p(\underline{\iota}, \underline{\theta})_{xy}$ the xy-component of the differential

$$d_\epsilon^p(\underline{\iota}, \underline{\theta}) : \bigoplus_{\mu_{\mathcal{I}}(x)=p} W^\epsilon(\mathcal{A}(x)_0, \tau(x)_0) \longrightarrow \bigoplus_{\mu_{\mathcal{I}}(y)=p+1} W^\epsilon(\mathcal{A}(y)_0, \tau(y)_0)$$

of the complex $\mathrm{GWh}^\epsilon(\mathcal{A}, \tau, \mathcal{I}_\bullet, \underline{\iota}, \underline{\theta})$, where $\epsilon \in \{\pm\}$.

Then it follows as in the coherent Witt theory of symmetric forms, see [Gil07a, Lemma 7.2], that $d_\epsilon^p(\underline{\iota}, \underline{\theta})_{xy} = 0$ if $y \notin \overline{\{x\}}$ and $d_\epsilon^p(\underline{\iota}, \underline{\theta})_{xy}$ coincides with a so called *generalized second residue map*. This is the differential of a (skew-)hermitian Gersten–Witt complex of a finite algebra with involution (A, τ) over a one dimensional local Cohen–Macaulay domain $(R, \mathfrak{m}, k)$ with canonical module Ω, i.e. the differential of $\mathrm{GWh}^\epsilon(A, \tau, I_\bullet, \underline{\iota}, \underline{\theta})$:

$$W^\epsilon((A_K)_0, (\tau_K)_0) \xrightarrow{\delta_\epsilon^{(A,\tau)}(\underline{\iota},\underline{\theta})} W^\epsilon((A_k)_0, (\tau_k)_0)$$

for suitable embeddings $\underline{\iota} = \{\iota_{(0)}, \iota_{\mathfrak{m}}\}$ and isomorphisms $\underline{\theta} = \{\theta_{(0)}, \theta_{\mathfrak{m}}\}$, where $I_\bullet$ is a finite injective resolution of Ω, $A_K = A \otimes_R K$, $\tau_K = \tau \otimes \mathrm{id}_K$, $A_k = A \otimes_R k$, and $\tau_k = \tau \otimes \mathrm{id}_k$.

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Example 6.3. Let $(R, \mathfrak{m}, k)$ be a discrete valuation ring with quotient field K and $(A, \tau) = (R, \mathrm{id}_R)$. Then we can chose for $\theta_{(0)}$ and $\theta_{\mathfrak{m}}$ the natural identifications $K \simeq \mathrm{Hom}_K(K, K)$ and $k \simeq \mathrm{Hom}_k(k, k)$. The map $\delta_+^{(A,\tau)}(\underline{\iota}, \underline{\theta})$ is then a homomorphism of Witt groups of symmetric spaces $W(K) \longrightarrow W(k)$. It is shown in [Gil07a, Lemma 6.3] that there exists a uniformizer π of R such that this map is equal to the 'classical' second residue map δ^π.

Conversely, if π is a uniformizer of R then there exists $\underline{\iota}$ such that $\delta^\pi = \delta_+^{(A,\tau)}(\underline{\iota}, \underline{\theta})$. This follows from the fact that two uniformizers differ by a unit $u \in R^\times$ and $u \cdot \iota_{\mathfrak{m}}$ is also an R-linear embedding $k \hookrightarrow \mathrm{E}_R(k)$ if $\iota_{\mathfrak{m}}$ is one.

6.4 *Generalized and classical second residue maps.* Let $(R, \mathfrak{m}, k)$ and K be as in Example 6.3 above, and (A, τ) an R-Azumaya algebra with R-linear involution. By Example 3.3(i) and § 4.5 the structure map $R \longrightarrow A$ induces a transfer morphism of hermitian Gersten–Witt spectral sequences

$$\mathrm{Tr}_{(A,\tau)/(R,\mathrm{id}_R)} : E_1^{p,q}(A, \tau, I_\bullet) \longrightarrow E_1^{p,q}(R, \mathrm{id}_R, I_\bullet).$$

We get from this a commutative diagram whose lines are complexes which are short exact sequences by Theorem 7.2 if R is the local ring of a smooth curve

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^+(A,\tau) & \longrightarrow & W^+(A_K,\tau_K) & \xrightarrow{\delta_+^{(A,\tau)}(\iota_{(0)},\iota_{\mathfrak{m}})} & W^+(A_k,\tau_k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t_{(A,\tau)/R}} & & \downarrow{\scriptstyle t_{(A_K,\tau_K)/K}} & & \downarrow{\scriptstyle t_{(A_k,\tau_k)/k}} & & \\
0 & \longrightarrow & W(R) & \longrightarrow & W(K) & \xrightarrow{\delta^\pi} & W(k) & \longrightarrow & 0
\end{array}$$

for some uniformizer π of R, where the column maps $t_{(A,\tau)/R}$, $t_{(A_K,\tau_K)/K}$ and $t_{(A_k,\tau_k)/k}$ are defined as follows: let S be a commutative ring and B an S-Azumaya algebra with S-linear involution ν, and (M, φ) a hermitian space over (B, ν). The isomorphism φ is then an isomorphism $M \xrightarrow{\simeq} \overline{\mathrm{Hom}_B(M, B)}$. Composing it with the natural isomorphisms (the one on the left-hand side is induced by the reduced trace)

$$\overline{\mathrm{Hom}_B(M, B)} \xrightarrow{\simeq} \overline{\mathrm{Hom}_B(M, \mathrm{Hom}_S(B, S))} \xrightarrow{\simeq} \overline{\mathrm{Hom}_S(M, S)}$$

we get a $\mathrm{Hom}_S(-, S)$-symmetric isomorphism $\varphi' : M \xrightarrow{\simeq} \mathrm{Hom}_S(M, S)$. The map $(M, \varphi) \mapsto (M, \varphi')$ defines a homomorphism of Witt groups $W^+(B, \nu) \longrightarrow W(S)$ which we denote $t_{(B,\nu)/S}$.

7. The Gersten conjecture for the hermitian Gersten–Witt complex

7.1 We state and prove in this section the Gersten conjecture for our (skew-)hermitian Gersten–Witt complex. Our proof follows closely Quillen's [Qui73] original proof of the same statement for the Gersten complex in K-theory of schemes except that the concluding argument is replaced by the zero theorem (Theorem 5.4). The analogous statement for the K-theory of an Azumaya algebra has been proven by Colliot-Thélène and Ojanguren [CTO92] if the Azumaya algebra is extended from the base and some years later by Panin and Suslin [PS98] in general.

Theorem 7.2. *Let k be a field, $\tilde{R}$ a smooth k-algebra, and $P_1, \dots, P_m$ be prime ideals of $\tilde{R}$. Set $S := \tilde{R} \setminus \bigcup_{i=1}^m P_i$, and $R := S^{-1}\tilde{R}$. Further let A be an R-algebra which is a free R-module of finite rank, and has an involution of the first kind τ. Then the hermitian and the skew-hermitian Gersten–Witt complexes*

$$\mathrm{GWh}^+(A, \tau, I_\bullet) \quad \textit{and} \quad \mathrm{GWh}^-(A, \tau, I_\bullet),$$

where $I_\bullet$ is a finite injective resolution of R, are exact. Furthermore the kernel of the first differential:

$$W^\epsilon(A(x_0)_0, \tau(x_0)_0) \longrightarrow \bigoplus_{\mathrm{ht}\, x = 1} W^\epsilon(A(x)_0, \tau(x)_0)$$

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(x_0 the generic point of $X = \operatorname{Spec} R$) is naturally isomorphic to $W^{\epsilon}(A,\tau,R)$ for $\epsilon \in \{\pm\}$, where $W^{\epsilon}(A,\tau,R)$ denotes the (skew-)symmetric Witt group of $\mathcal{P}_b(A)$ with respect to the duality $\mathfrak{D}_R^{(A,\tau)} = \operatorname{Hom}_R(-,R)$.

Proof. Replacing $\tilde{R}$ by $\tilde{R}_b$ for some $b \in S$ and we can assume that (A,τ) is extended from $\tilde{R}$, i.e. there is a $\tilde{R}$-algebra $\tilde{A}$ which is a free $\tilde{R}$-module of finite rank and has an involution of the first kind $\tilde{\tau}$, such that $A = \tilde{A} \otimes_{\tilde{R}} R$ and $\tau = \tilde{\tau} \otimes \mathrm{id}_R$.

As is well known, see e.g. [Qui73, § 7, Proposition 5.6], the theorem follows if we show that the morphism $W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p+1)}, \tau, I_\bullet) \longrightarrow W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p)}, \tau, I_\bullet)$ is zero for all $p \geqslant 0$ and all $i \in \mathbb{Z}$, where $W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p)}, \tau, I_\bullet)$ denotes the Witt group of $D^b_c(\mathcal{M}_{qc}(A))_I^{(p)}$ with respect to the duality $\mathfrak{D}_I^{(A,\tau)}$.

Let $x \in W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p+1)}, \tau, I_\bullet)$. Replacing $\tilde{R}$ by $\tilde{R}_b$ for some $b \in S$ if necessary we can also assume that x is in the image of the localization morphism

$$W^i(D^b_c(\mathcal{M}_{qc}(\tilde{A}))_{\tilde{I}}^{(p+1)}, \tilde{\tau}, \tilde{I}_\bullet) \longrightarrow W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p+1)}, \tau, I_\bullet),$$

where $\tilde{I}_\bullet$ is a finite injective resolution of the $\tilde{R}$-module $\tilde{R}$, such that $S^{-1}\tilde{I}_\bullet = I_\bullet$. Therefore the theorem follows from the next lemma. $\square$

Lemma 7.3. *Let R be a smooth k-algebra and (A,τ) a flat coherent R-algebra with involution of the first kind, and $I_\bullet$ a finite injective resolution of the R-module R. Further let $x \in W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p+1)}, \tau, I_\bullet)$ and $S \subset R$ the complement of finitely many prime ideals of R. Then there exists $b \in S$, such that the image of x under the composition*

$$W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p+1)}, \tau, I_\bullet) \longrightarrow W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p)}, \tau, I_\bullet)$$

$$\longrightarrow W^i(D^b_c(\mathcal{M}_{qc}(A_b))_{I_b}^{(p)}, \tau_b, I_{\bullet\, b})$$

is zero.

Proof. Since $p+1 \geqslant 1$ there exists by the dévissage theorem (Theorem 5.2) a non-zero divisor $t \in R$, such that x is in the image of

$$W^i(D^b_c(\mathcal{M}_{qc}(A'))_{\pi^\natural(I)}^{(p+1)}, \tau', \pi^\natural(I_\bullet)) \xrightarrow{\operatorname{Tr}_{(A',\tau')/(A,\tau)}} W^i(D^b_c(\mathcal{M}_{qc}(A))_I^{(p+1)}, \tau, I_\bullet),$$

where $A' = A/tA$, $\tau' = \tau \otimes \mathrm{id}_{R/Rt}$, and $\pi : R \longrightarrow R' = R/Rt$ is the natural quotient map. Note that $\mu_{\pi^\natural(I_\bullet)}(P/Rt) = \mu_I(P)$ for all prime ideals P which contain t.

We use now Quillen's normalization lemma [Qui73, § 7, Lemma 5.12]. By this result there exists $b \in S$, a Gorenstein ring C, and a commutative diagram

$$\begin{array}{ccc} C & \xleftarrow{\;v\;} & R_b \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle \pi_b} & \\ R'_b & & \end{array}$$

such that (i) v is a finite morphism, (ii) u is surjective and $\operatorname{Ker} u = C \cdot g$ for some regular $g \in C$, and (iii) there is a flat splitting $q : R'_b \longrightarrow C$ of the surjective morphism u.

Let $B = A_b \otimes_{R_b} C$, $\nu = \tau_b \otimes \mathrm{id}_C$, and $J_\bullet = v^\natural(I_{\bullet\, b})$. By Theorem 5.4 the transfer map

$$W^i(D^b_c(\mathcal{M}_{qc}(A'_b))_{\pi_b^\natural(I_b)}^{(p+1)}, \tau'_b, \pi_b^\natural(I_{\bullet\, b})) \xrightarrow{\operatorname{Tr}_{(A'_b,\tau'_b)/(A',\tau')}} W^i(D^b_c(\mathcal{M}_{qc}(B))_J^{(p)}, \nu, J_\bullet)$$

is zero, and hence since the transfer is transitive also the homomorphism

$$W^i(D^b_c(\mathcal{M}_{qc}(A'_b))^{(p+1)}_{\pi^\natural_b(I_b)}, \tau'_b, \pi^\natural_b(I_{\bullet\,b})) \xrightarrow{\mathrm{Tr}_{(A'_b,\tau'_b)/(A_b,\tau_b)}} W^i(D^b_c(\mathcal{M}_{qc}(A_b))^{(p)}_{I_b}, \tau_b, I_{\bullet\,b})$$

is zero. The lemma follows since the transfer commutes with localization. □

Remark 7.4. If A is an R-Azumaya algebra then the kernel of the first differential of $\mathrm{GWh}^{\pm}(A,\tau,I_\bullet)$ is naturally isomorphic to the (skew-)hermitian Witt group $W^{\pm}(A,\tau)$. This follows from § 2.10.

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Stefan Gille gille@mathematik.uni-muenchen.de
Mathematisches Institut, Universität München, Theresienstrasse 39, 80333 München, Germany

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