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ON THE WEAK GLOBAL DIMENSION OF PSEUDO-VALUATION DOMAINS

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1. Introduction. In [7], Hedstrom and Houston introduce a type of quasilocal integral domain, therein dubbed a pseudo-valuation domain (for short, a PVD), which possesses many of the ideal-theoretic properties of valuation domains. For the reader's convenience and reference purposes, Proposition 2.1 lists some of the ideal-theoretic characterizations of PVD's given in [7]. As the terminology suggests, any valuation domain is a PVD. Since valuation domains may be characterized as the quasilocal domains of weak global dimension at most 1, a homological study of PVD's seems appropriate. This note initiates such a study by establishing (see Theorem 2.3) that the only possible weak global dimensions of a PVD are 0, 1, 2 and ∞. One upshot (Corollary 3.4) is that a coherent PVD cannot have weak global dimension 2: hence, none of the domains of weak global dimension 2 which appear in [10, Section 5.5] can be a PVD.

As detailed in [5, Proposition 4.9(i)], an ample supply of PVD's is provided by the "$D+M$ construction". (Not all PVD's are so constructed, even in the coherent case: see Remark 2.2.) Happily, the conclusions of [1] and [3], which were designed for the $D+M$ construction, extend naturally to the PVD context, albeit with more complicated proofs. The reworking of [1] also yields facts about coherent PVD's which generalize results established, by very different means, in [5] under the assumption of finite Krull dimension.

We caution that familiarity with [1] and [3] will be assumed. As usual, weak dimension and weak global dimension will be denoted by w.d. and w.gl.dim, respectively. Throughout, R will denote a quasilocal integral domain, with maximal ideal M.

2. Background and statement of main result. We begin by recalling some results of Hedstrom and Houston [7, Theorems 1.4, 2.7 and 2.10] concerning our principal object of study.

PROPOSITION 2.1. *Let K be the quotient field of the quasilocal domain R. Then the following two conditions are equivalent:*

(a) *There is an overring of R (i.e., a subring of K which contains R) which is a valuation domain whose maximal ideal coincides with the maximal ideal, M, of R;*

(b) *For any ideals I and J of R, either $I \subset J$ or $MJ \subset MI$.*

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If R satisfies (a) *and* (b), *then R is said to be a pseudo-valuation domain* (PVD). *If R is not a valuation domain, then* (a) *and* (b) *are equivalent to*
(c) $M^{-1}=\{x \text{ in } K:xM\subset R\}$ *is a valuation overring of R with maximal ideal M. Moreover, if R is not a valuation domain and if* (c) *holds, then* $M^{-1}=\{x \text{ in } K:xM\subset M\}$ *and the prime ideals of* M^{-1} *coincide with those of R.*

REMARK 2.2. Since the major themes of this note concerning coherence and PVD's have already been pursued in [1] and [3] in the context of the $D+M$ construction, it seems just to pause and exhibit a coherent PVD which is not of the form $D+M$. Let S be the Noetherian (hence, coherent) PVD given in [7, Example 3.6]. Explicitly, let m be a positive square-free integer with $m\equiv 5 \pmod 8$, let $T=\mathbb{Z}[\sqrt{m}]$, let $N=(2, 1+\sqrt{m})$ and set $S=T_N$. If S assumes the form $D+M$, then S has a valuation overring $L+M$ with maximal ideal $M(\neq 0)$, such that L is a field containing the ring D. As S has Krull dimension 1, the integral closure of S is of the form $F+M$, where F is a field contained in L. Then $\mathbb{Q}(\subset F)$ is integral over S. By multiplying an integrality equation for $\frac{1}{2}$ over S by a sufficiently large power of 2, we infer $\frac{1}{2}\in S$, the desired absurdity.

We may now state the main result of this note.

THEOREM 2.3. *Suppose that R is a* PVD, *but not a valuation domain. Then:*
(a) *The following four conditions are equivalent*:
(1) $M=M^2$;
(2) *M is R-flat;*
(3) *Each prime ideal of R is R-flat;*
(4) $\text{w.gl.dim}(R)=2$.
(b) *The following three conditions are equivalent:*
(i) $M\neq M^2$;
(ii) $\text{w.d.}_R(M)=\infty$;
(iii) $\text{w.gl.dim}(R)=\infty$.

The proof of Theorem 2.3 will be obtained in the next section after some preliminaries.

3. **Proofs.** The statements of the next two lemmas are suggested by the results of [1] and [3]. Indeed, Lemma 3.1 explores a technique studied in [2, Proposition 4.5], [1, Theorem 3] and [3, Proposition 3.1]. The ancestry of Lemma 3.2 includes [1, Theorem 7 and Remark 9] and [3, Propositions 2.3 and 3.1].

As usual, $E^{(n)}$ will denote the direct sum of n copies of an R-module E.

LEMMA 3.1. *Let R be a* PVD, *let A be a nonzero ideal of R, and let* $n=m+1$ *be the cardinality of a minimal R-generating set of A. Then there is a short exact sequence of R-modules*

$$0\to M^{(m)}\to R^{(n)}\to A\to 0.$$

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Proof. Let $S=\{x_i\}$ be a minimal R-generating set of A. Well-order S; let x_1 be its first element, x_2 its second element, etc. Consider the R-module epimorphism $f:R^{(n)}\to A$ which sends the i-th basis element, e_i, of $R^{(n)}$ to x_i. By minimality of S, we have $\ker(f)\subset MR^{(n)}$. It suffices to prove that the R-module homomorphism $g:\ker(f)\to M^{(m)}$, given by $g(\sum m_ie_i)=(m_2, m_3, \ldots)$, is an isomorphism. Now, g is a monomorphism: if $\sum m_ix_i=0$ with $m_i=0$ for each $i>1$, then $m_1x_1=0$, whence cancellation of $x_1(\neq 0)$ gives $m_1=0$. The proof that g is surjective depends upon condition (b) in Proposition 2.1. Indeed, given a finite set $\{m_2, \cdots, m_k\}\subset M$ with $k\leq n$, produce m_1 in M such that $m_1x_1+\cdots+m_kx_k=0$ as follows. Consider the ideals $I=Rx_1$ and $J=Rx_2+\cdots+Rx_k$. By minimality of S, note $I\not\subset J$. If R is a valuation domain then $J\subset I$; if R isn't valuation, (b) yields that $MJ\subset MI$. In either case, $m_2x_2+\cdots+m_kx_k\in Mx_1$, to complete the proof.

Lemma 3.1 will be used to treat finitely generated ideals A, as any such has a minimal generating set. For the ideal M, which need not be finitely generated, it will be convenient to record the following companion result.

LEMMA 3.1. (*bis*). *Let R be a* PVD *which is not a valuation domain.. Let V be its valuation overring described in Proposition* 2.1(c). *Let $n=m+1$ be the dimension of V/M as an R/M-vector space. If $M\neq M^2$, then there is a short exact sequence of R-modules*

$$0\to M^{(m)}\to R^{(n)}\to M\to 0.$$

Proof (sketch). Note that $M=Vu$, for any u in $M\backslash M^2$. Consider $\{v_i+M\}$, a well-ordered R/M-basis of V/M, with $v_1=1$. The R-module homomorphism $f:R^{(n)}\to M$, given by $f(e_i)=v_iu$, is surjective since its image contains $\sum Rv_iu+Mv_1u=(\sum Rv_i+M)u=Vu$. As before, one constructs an isomorphism $g:\ker(f)\to M^{(m)}$. (Show that $(m_2,\ldots, m_k, 0, 0, \ldots)$ is in the image of g with the aid of $I=Rv_1u=Ru$ and $J=Rv_2u+\cdots+Rv_ku$.) Details may safely be omitted.

LEMMA 3.2. *Let R be a* PVD *which is not a valuation domain. Then:*

(a) *Let P be an ideal of R. If $P=MP$, then P is R-flat. If P is prime and R-flat, then $P=MP$.*

(b) *If $M\neq M^2$, then* $\text{w.d.}_R(M)=\infty$.

Proof. (a) Suppose that $P=MP$. To establish R-flatness of P, we show that any relation $\sum_i r_ip_i=0$ (with r_i in R, p_i in P) arises from elements v_j in P, r_{ij} in R and equations $p_i=\sum_j r_{ij}v_j$ (for each i) and $\sum_i r_ir_{ij}=0$ (for each j).

Let $V=M^{-1}$ be the valuation overring of R described in Proposition 2.1(c). As PV is V-flat (*any* ideal of a valuation domain is flat), the given relation yields elements n_j in PV and v_{ij} in V such that $p_i=\sum_j v_{ij}n_j$ and $\sum_i r_iv_{ij}=0$. Since V is a valuation domain, we may write $n_j=xw_j$, with x in P and w_j in V. Use the fact that V is valuation, this time in tandem with the hypothesis that $P = MP$, to write $x = yz$, with y in M and z in P. A computation verifies that setting $v_j = z$ and $r_{ij} = v_{ij}w_jy$ produces the required equations. (To show $p_i = \sum_j r_{ij}v_j$, first note that $v_{ij}n_j = r_{ij}z$.) Thus, P is R-flat, as required.

Conversely, we show that if a prime ideal P is R-flat, then $P = MP$. Without loss of generality, $P \neq 0$. By [8, Lemma 2.1], it suffices to eliminate the possibility that P is a principal ideal. Since R is not a valuation domain, this is accomplished by an appeal to [7, Corollary 2.9].

(b) Assume that $M \neq M^2$. By part (a), M is not R-flat. Then, if $m+1$ is the cardinality of an R/M-basis of V/M, Lemma 3.1 (bis) implies that $\text{w.d.}_R(M) = 1 + \text{w.d.}_R(M^{(m)}) = 1 + \text{w.d.}_R(M)$, whence $\text{w.d.}_R(M) = \infty$, as required.

COROLLARY 3.3. *If R is a* PVD *and P is a nonmaximal prime ideal of R, then P is R-flat.*

Proof. Of course, R may be assumed nonvaluation. As noted in [5], it follows readily that any PVD is divided, in the sense of [4]; i.e., any prime ideal of R is comparable to any principal ideal of R under inclusion. In particular, if b is in $M \backslash P$, then $P \subset Rb$, whence $P = Pb \subset PM$, and so R-flatness of P follows from Lemma 3.2(a).

Proof of Theorem 2.3. Note that $\text{w.gl.dim}(R) \geq 2$ since R is not a valuation domain. Moreover, for each finitely generated ideal A of R, the exact sequence guaranteed by Lemma 3.1 yields, as in the proof of Lemma 3.2(b), that $\text{w.d.}_R(A) \leq 1 + \text{w.d.}_R(M)$. Thus, $\text{w.gl.dim}(R) \leq 1 + \sup_A \text{w.d.}_R(A) \leq 2 + \text{w.d.}_R(M)$. In particular, (2)$\Rightarrow$(4). By Lemma 3.2, we have (1)$\Leftrightarrow$(2) and (i)$\Rightarrow$(ii); Corollary 3.3 yields (2)$\Rightarrow$(3); and the implications (3)$\Rightarrow$(2) and (ii)$\Rightarrow$(iii) are trivial. Since exactly one of the conditions "$M = M^2$" and "$M \neq M^2$" holds, the implications (1)$\Rightarrow$(4) and (i)$\Rightarrow$(iii), which have already been established, now yield (4)$\Rightarrow$(1) and (iii)$\Rightarrow$(i), to complete the proof.

It was noted in the proof of Corollary 3.3 that, if R is a PVD, then R is divided and, hence by [4, Proposition 2.1], R is a going-down ring. If, in addition, R is coherent but not valuation, [2, Proposition 2.5] shows that R has infinite global dimension. The next result strengthens the conclusion in this case to $\text{w.gl.dim}(R) = \infty$.

COROLLARY 3.4. *If R is a coherent* PVD, *then the only possible values of* $\text{w.gl.dim}(R)$ *are* 0, 1 *and* ∞.

Proof. According to Theorem 2.3, it suffices to rule out the possibility that $\text{w.gl.dim}(R) = 2$. However, in that case, M is R-flat (thanks to Theorem 2.3), and coherence then entails that R is a valuation domain [9, Lemma 3.9], so that $\text{w.gl.dim}(R) \leq 1$, the desired contradiction.

We close by showing how the methods of [1] serve to generalize some results in [5]. Corollary 3.6 and the equivalence of (a), (b), and (c) in Proposition 3.5 were obtained in [5, Corollary 4.7] and [5, Lemma 4.5(ii) and Remark 4.8(b)], respectively, under the additional hypothesis that R has finite Krull dimension.

PROPOSITION 3.5. *Let R be a* PVD *which is not a valuation domain. Let V be its valuation overring described in Proposition* 2.1(c). *The following conditions are equivalent:*

(a) *R is coherent;*
(b) *R is finite-conductor; i.e., the intersection of any two principal ideals of R is finitely generated as an ideal of R;*
(c) *M is a finitely generated ideal of R;*
(d) *V is a finitely generated R-module and $M \neq M^2$.*

Proof. (a)$\Rightarrow$(b) trivially, while (b)$\Rightarrow$(c) was established in [5, Lemma 4.5(ii)].

To prove that (c)$\Rightarrow$(d), assume (c). As $M \neq 0$, Nakayama's lemma guarantees $M \neq M^2$. To show that V is a finitely-generated R-module, we ape the proof of [1, Lemma 1]. Observe that M/M^2 is both a finite-dimensional R/M-vector space and (since M is a principal ideal of V) also a cyclic V/M-space. Then, $V/M \cong M/M^2$, whence there exists a finite R/M-basis $\{v_1+M, \ldots, v_n+M\}$ of V/M, thus forcing $V = M + \Sigma\, Rv_i$, a sum of two finitely-generated R-submodules.

Finally, to establish (d)$\Rightarrow$(a), assume (d). With the aid of Ferrand's desent result as in the proof of [1, Theorem 3], our task is reduced to showing that V is a finitely presented R-module. To this end, write $M = Vm$ and $V = \Sigma\, Rv_i$ (with m in $M \backslash M^2$; $v_1, \ldots, v_n$ in V), so that $M = \Sigma\, R(v_i m)$ is finitely generated over R. Now, Lemma 3.1 supplies an exact sequence

$$0 \to M^{(t-1)} \to R^{(t)} \to M \to 0$$

where t is the cardinality of a minimal R-generating set of M; thus, M is finitely presented over R. As $V \cong Vm = M$, the proof is complete.

COROLLARY 3.6 *If R is a coherent* PVD, *then each overring of R is coherent.*

Proof. As overrings of valuation domains are valuations domains (and, hence, coherent), we may suppose that R is not a valuation domain. By [5, Proposition 4.2], the integral closure of R *is* a valuation domain which, as explained in [5, Remark 4.8(a)], must coincide with the overring $V = M^{-1}$ described in Proposition 2.1(c). It now follows readily (as, e.g., in [6, Proposition 8]) that each overring of R compares with V under inclusion. It remains only to prove that each integral overring $T (\neq V)$ of R is coherent. Now, any such T is a PVD (by [5, Proposition 4.2]) with maximal ideal M. Coherence of R assures that M is finitely generated over R (by Proposition 3.5) and, *a*

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fortiori, finitely generated over T, whence another application of Proposition 3.5 establishes coherence of T.

REMARK 3.7. (a) By virtue of Lemma 3.2(a), the appeal to [9] in the proof of Corollary 3.4 may be replaced by an appeal to the equivalence (a)$\Leftrightarrow$(d) in Proposition 3.5.

(b) In view of Corollary 3.4, it is of interest to note that a PVD of infinite weak global dimension need not be coherent. For an example, let $L \subset F$ be an infinite-dimensional algebraic extension of fields, let $F+N$ be a valuation domain with maximal ideal $N \neq N^2$, and set $S = L + N$. Then S is a PVD (by [7, Example 2.1] or [5, Proposition 4.9(i)]), has maximal ideal N with infinite weak dimension (by Theorem 2.3), and is not coherent (since S does not satisfy condition (d) of Proposition 3.5). Observe finally that, despite the noncoherence of S, each overring of S is a PVD.

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