ON CARTAN PSEUDO GROUPS

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Let M be a domain in an Euclidean space and let Γ be a pseudo group of transformations* of M. We say that Γ is a Cartan pseudo group [1, 2] if the following conditions are satisfied:

1) There exists a domain M' and a projection $\rho : M \to M'$, such that the orbits of Γ are the fibers of the projection ρ. We assume moreover that there is a system of coordinates (x) of M' and a system of coordinates (x, y) of M such that the fibers of ρ are defined by $(x) = \text{constants}$,

2) There are forms ω^i, $\tilde{\omega}^\lambda$, $i = 1 \cdots m$, $\lambda = 1 \cdots n$, defined in D such that

a) $(\omega^i, \tilde{\omega}^\lambda)$ is a basis of the space of linear forms at every point of M,

$$(1) \qquad \text{b)}\ d\omega^i = \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k + a^i_{j\lambda} \omega^j \wedge \tilde{\omega}^\lambda$$

where c^i_{jk}, $a^i_{j\lambda}$ are functions on M which depend on (x) only,

c) $\omega^r = dx^r$ for $1 \le r \le \text{dimension } M'$,

d) The matrices $a_\lambda = \|a^i_{j\lambda}\|$ are linearly independent at every point of M,

e) Let π_1 and π_2 be respectively the projections of $M \times M$ into the first and second factors. The closed differential system Σ on $M \times M$, with independent variables $x \circ \pi_1$, $y \circ \pi_1$ generated by

$$x^r \circ \pi_1 - x^r \circ \pi_2 = 0,\ 1 \le r \le \text{dimension } M',$$
$$\pi_1^* \omega^i - \pi_2^* \omega^i = 0 \qquad 1 \le i \le m$$

is in involution at every integral point,

3) A local transformation f of M is in Γ if and only if f preserves the forms ω^i, i.e. $f^* \omega^i = \omega^i$, $i = 1, \ldots, m$.

In this note we prove that every differential form on M which is invariant under all transformations of a Cartan pseudo group Γ is a linear combination of the forms ω^i the coefficient being functions of x only.

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Put $\sigma^i = \pi_1^*\omega^i - \pi_2^*\omega^i$, $\tau^\lambda = \pi_1^*\tilde{\omega}^\lambda - \pi_2^*\tilde{\omega}^\lambda$. The set $(\pi_1^*\omega^i, \pi_1^*\tilde{\omega}^\lambda, \sigma^i, \tau^\lambda)$ is a basis of differential forms on every point of $M\times M$. Let E be a contact element of $M\times M$ at the integral point $(u_1, u_2)\in M\times M$ such that the forms $\pi_1^*\omega^i$ and $\pi_1^*\tilde{\omega}^\lambda$ are linearly independent on E and denote by σ^i/E the restriction of σ^i to E. Put $x=\rho(u_1)=\rho(u_2)$ and write

$$\sigma^i/E = p_j^i(\pi_1^*\omega^j)/E + p_\lambda^i(\pi_1^*\tilde{\omega}^\lambda)/E$$
$$\tau^\lambda/E = q_j^\lambda(\pi_1^*\omega^j)/E + q_\mu^\lambda(\pi_1^*\tilde{\omega}^\mu)/E.$$

E is an integral element of Σ if and only if

$$p_j^i = p_\lambda^i = q_\mu^\lambda = 0 \tag{2}$$
$$a_{j\lambda}^i(x)q_k^\lambda - a_{k\lambda}^i(x)q_j^\lambda = 0$$

for every choice of the indices i, j, k.

Let V and V' be vector spaces of dimensions v and v' over the real field and let L be a vector space of linear maps of V into V' of dimension n. Take basis of V and V' and let $a_\lambda = \|a_{j\lambda}^i\|$, $i=1, \ldots, v'$, $j=1, \ldots, v$, $\lambda=1, \ldots, n$ be a basis of L. The space $\mathscr{D}(L)$ of all linear maps $b: V\to L$, $b=\|b_j^\lambda\|$ such that

$$a_{j\lambda}^i b_k^\lambda - a_{k\lambda}^i b_j^\lambda = 0$$

for every choice of i, j, k, is called the derived space of L.

Let s_1 be the maximum rank of the matrix $A_1 = \|a_{j\lambda}^i t_1^j\|$ when the vector $(t_1^1, t_1^2, \ldots, t_1^v)$ varies in R^v. Put $A_2 = \|a_{j\lambda}^i t_2^j\|$ and let s_2 be the maximum rank of the matrix $\left\|\begin{matrix}A_1\\A_2\end{matrix}\right\|$ when the vectors $(t_1^1, \ldots, t_1^v)$, $(t_2^1\cdots t_2^v)$, vary independently in R^v. Define an integer s_i in a similar way for each i, $1\leq i\leq v-1$. The integers s_i are called the characters of L. If δ is the dimension of $\mathscr{D}(L)$ it can be proved [3, page 4] that

$$\delta \leq n\cdot v - (s_1 + \cdots + s_{v-1}). \tag{3}$$

The space L is called involutive when the equality holds in (3).

Let $L(x)$ be the space of endomorphisms of R^m generated by the matrices $a_\lambda(x) = \|a_{j\lambda}^i(x)\|$ and denote by $s_1(x), \ldots, s_{m-1}(x)$, $\delta(x)$ the characters and the dimension of the derived space of $L(x)$; Σ is in involutions at every integral point if and only if $\delta(x)$ is constant and $L(x)$ is involutive for every x. When Σ is involutive the characters $s_i(x)$ are independent of x.

Let now f be a transformation of Γ. Applying f^* to equation (1) we have

$$d\omega^i = \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k + a^i_{jk}\omega^j \wedge f^*\tilde{\omega}^\lambda$$

hence,

$$a^i_{j\lambda}(f^*\tilde{\omega}^\lambda - \tilde{\omega}^\lambda) \wedge \omega^j = 0$$

and the linear form $a^i_{j\lambda}(f^*\tilde{\omega}^\lambda - \tilde{\omega}^\lambda)$ is a linear combination of the form ω^j. Since the matrices a_λ are linearly independent we have

$$f^*\tilde{\omega}^\lambda = \tilde{\omega}^\lambda + h^\lambda_j\omega^j. \tag{4}$$

Substituting (4) in (1) we have

$$a^i_{j\lambda}h^\lambda_k - a^i_{j\lambda}h^\lambda_j = 0.$$

Conversely let u_1, u_2 be two points of M such that $\rho(u_1) = \rho(u_2) = x$ and let h^λ_j be an element of the derived space of $L(x)$. Let E be the integral contact element of Σ at the point $(u_1, u_2) \in M \times M$ whose coordinates are $p^i_j = p^i_\lambda = q^\lambda_\mu = 0$, $q^\lambda_j = h^\lambda_j$. Let f be the transformation of M defined by an integral manifold of Σ whose tangent space at the point (u_1, u_2) is E. Then $f(u_1) = u_2$ and, at the point u_1

$$f^*\tilde{\omega}^\lambda = \tilde{\omega}^\lambda + h^\lambda_i\omega^i$$

for $1 \leq \lambda \leq n$.

Assume now that the differential form ω is invariant under Γ and write

$$\omega = \alpha_i\omega^i + \beta_\lambda\tilde{\omega}^\lambda.$$

Given $u_1, u_2 \in M$ with $\rho(u_1) = \rho(u_2) = x$ there exists $f \in \Gamma$ such that $f(u_1) = u_2$ and, at the point u_1, $f^*\tilde{\omega}^\lambda = \tilde{\omega}^\lambda$. It follows that α_i, β_λ depend only on x. Assume that there exists x such that not all coefficients $\beta_\lambda(x)$ are zero. Then we can take ω to be one of the forms $\tilde{\omega}^\lambda$. Hence, there exists a system of forms $(\omega^i, \tilde{\omega}^\lambda)$ which satisfy conditions 2) and 3) and such that $\tilde{\omega}^n$ is an invariant form of Γ. Then, if h^λ_i is an element of $\mathscr{D}(L(x))$ we have necessarily $h^n_i = 0$ for every i. Let L_0 be the subspace $L(x)$ generated by $a_1(x), \ldots, a_{n-1}(x)$. Any element of $\mathscr{D}(L(x))$ has values in L_0 hence, the dimension of $\mathscr{D}(L_0)$ is $\delta(x)$. Let $s'_1 \cdots s'_{m-1}$ be the characters of (L_0). By (3)

$$\delta(x) \leq m(n-1) - [s'_1 + \cdots + s'_{m-1}].$$

By the definition of the characters $s_i \leq s'_i + 1$. Therefore

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$$\delta(x) < m \cdot n - [s_1 + \cdots + s_{m-1}]$$

and $L(x)$ is not involutive. Hence all coefficients β are zero and ω is a linear combination of the ω^i with coefficients depending on x only.

The following example shows that the result is not true if we drop the condition that Σ is in involution even if the coefficients $a^i_{j\lambda}$, c^i_{jk} are constant and Γ is transitive. Let Γ be the pseudo group operating on R^n obtained by localization of the group of rigid motions. Γ is a Lie pseudo group of order 1. Let $\mathscr{F}$ be the space of orthonormal frames of R^n and denote by $\tilde{\Gamma}$ the prolongation of Γ to $\mathscr{F}$. In $\mathscr{F}$ we have differential forms ω^i, $\omega^i_j(\omega^i_j + \omega^j_i = 0)$, canonically defined which satisfy equations (1) with constant coefficients. A transformation f of $\mathscr{F}$ is in $\tilde{\Gamma}$ if and only if f preserves the forms ω^i. On the other hand all the forms ω^i, ω^i_j are invariant by the elements of $\tilde{\Gamma}$.

References

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