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On the Hereditary Paracompactness of Locally Compact, Hereditarily Normal Spaces

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Abstract. We establish that if it is consistent that there is a supercompact cardinal, then it is consistent that every locally compact, hereditarily normal space that does not include a perfect pre-image of ω_1 is hereditarily paracompact.

This is the sixth in a series of papers ([12], [19], [7], [11], [16] being the logically previous ones) that establish powerful topological consequences in models of set theory obtained by starting with a particular kind of Souslin tree S, iterating partial orders that do not destroy S, and then forcing with S. The particular case of the theorem stated in the abstract when X is perfectly normal (and hence has no perfect pre-image of ω_1) was proved in [11], using essentially that locally compact perfectly normal spaces are locally hereditarily Lindelöf and first countable. Here we avoid these last two properties by combining the methods of [2] and [16]. To apply [2], we establish the new set-theoretic result that $\mathrm{PFA}^{++}(S)[S]$ implies Fleissner's "Axiom R". This notation is explained below; the model is a strengthening of those used in the previous five papers.

The results established here were actually proved around 2004, modulo results of Todorcevic announced in 2002 (which now appear in [7] and [19]) and of the second author [16]. We delayed submission until a correct version of [16] existed.

Definition A continuous map is *perfect* if images of closed sets are closed, and pre-images of points are compact.

It is easy to find locally compact hereditarily normal spaces that are not paracompact; ω_1 is one such. Nontrivial perfect pre-images of ω_1 may also be hereditarily normal, but are not paracompact. Our result says that consistently, any example must in fact include such a canonical example.

Theorem 1 *If it is consistent that there is a supercompact cardinal, it is consistent that every locally compact hereditarily normal space that does not include a perfect pre-image of ω_1 is (hereditarily) paracompact.*

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This is not a ZFC result, since there are many consistent examples of locally compact perfectly normal spaces that are not paracompact. For example, the Cantor tree over a Q-set, which is the standard example of a locally compact, normal, non-metrizable Moore space; see *e.g,* [15], which has essentially the same example. Other examples include the Ostaszewski and Kunen lines, as in [5].

Let us state some axioms we will be using.

PFA^{++}: Suppose P is a proper partial order, $\{D_\alpha\}_{\alpha<\omega_1}$ is a collection of dense subsets of P, and $\{\dot{S}_\alpha : \alpha < \omega_1\}$ is a sequence of terms such that $(\forall\alpha < \omega_1) \Vdash_P \dot{S}_\alpha$ is stationary in ω_1. Then there is a filter $G \subseteq P$ such that

(i) $(\forall\alpha < \omega_1)\, G \cap D_\alpha \neq 0$,

(ii) $(\forall\alpha < \omega_1)\, S_\alpha(G) = \{\xi < \omega_1 : (\exists p \in G) p \Vdash \xi \in \dot{S}_\alpha\}$ is stationary in ω_1.

Baumgartner [3] introduced this axiom and called it "PFA$^+$". Since then, others have called this "PFA^{++}", using "PFA$^+$" for the weaker one-term version. As Baumgartner observed, the usual consistency proof for PFA, which uses a supercompact cardinal, yields a model for what we are calling PFA^{++}.

Definition $\Gamma \subseteq [X]^{<\kappa}$ is *tight* if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from Γ and $\omega < cf\delta < \kappa$, $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$. *Axiom R:* if $\Sigma \subseteq [X]^{<\omega_1}$ is stationary and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and cofinal, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma$ is stationary in $[Y]^{<\omega_1}$. *Axiom R^{++}:* if $\Sigma_\alpha(\alpha < \omega_1)$ are stationary subsets of $[X]^{<\omega_1}$ and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and cofinal, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma_\alpha$ is stationary in $[Y]^{<\omega_1}$ for each $\alpha < \omega_1$.

Fleissner introduced Axiom R in [6] and showed that it held in the usual model for PFA.

$\sum$: Let X be a compact countably tight space. Let $Y \subseteq X$, $|Y| = \aleph_1$. Suppose $\{W_\alpha\}_{\alpha\in\omega_1}$, $\{V_\alpha\}_{\alpha\in\omega_1}$ are open subsets of X such that

(i) $\overline{W}_\alpha \subseteq V_\alpha$,

(ii) $|V_\alpha \cap Y| \leq \aleph_0$,

(iii) $Y \subseteq \bigcup\{W_\alpha : \alpha < \omega_1\}$.

Then Y is σ-closed-discrete in $\bigcup\{W_\alpha : \alpha < \omega_1\}$.

Balogh [1] proved that MA_{ω_1} implies $\sum$.

Definition A space is *(strongly) κ-collectionwise Hausdorff* if for each closed discrete subspace $\{x_d\}_{d\in D}$, $|D| \leq \kappa$, there is a disjoint (discrete) family of open sets $\{U_d\}_{d\in D}$ with $x_d \in U_d$. A space is *(strongly) collectionwise Hausdorff* if it is (strongly) κ-collectionwise Hausdorff for all κ.

It is easy to see that normal (κ-) collectionwise Hausdorff spaces are strongly (κ-) collectionwise Hausdorff.

Balogh [2] proved the following lemma.

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Lemma 2 *MA_{ω_1} + Axiom R implies that locally compact hereditarily strongly $\aleph_1$-collectionwise Hausdorff spaces that do not include a perfect pre-image of ω_1 are paracompact.*

The consequences of MA_{ω_1} Balogh used are $\sum$ and Szentmiklóssy's result [14] that *compact spaces with no uncountable discrete subspaces are hereditarily Lindelöf*. Our plan is to find a model in which these two consequences and Axiom R hold, as well as normality implying (strongly) $\aleph_1$-collectionwise Hausdorffness for the spaces under consideration. The model we will consider is of the same genre as those in [12], [19], [7], [11], and [16]. One starts with a particular kind of Souslin tree S, a *coherent* one, which is obtainable from $\diamondsuit$ or by adding a Cohen real. One then iterates in standard fashion as in establishing MA_{ω_1} or PFA, but omitting partial orders that adjoin uncountable antichains to S. In the PFA case, for example, this will establish PFA(S), which is like PFA except restricted to partial orders that do not kill S. In fact it will also establish $\text{PFA}^{++}(S)$, which is the corresponding modification of PFA^{++}. We then force with S. For more information on such models, see [13] and [10]. We use $\text{PFA}^{++}(S)[S]$ *implies* φ to mean that whenever we force over a model of $\text{PFA}^{++}(S)$ with S, φ holds. Similarly for PFA(S)[S], etc.

In [16] the following lemma is established.

Lemma 3 PFA(S)[S] *implies that locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff.*

By doing some preliminary forcing (as in [11]), one can actually get full collectionwise Hausdorffness, but we won't need that here.

We will assume all spaces are Hausdorff, and use "X^*" to refer to the one-point compactification of a locally compact space X.

There is a bit of a gap in Balogh's proof of Lemma 2. Balogh asserted that:

Lemma 4 *If X is locally compact and does not include a perfect pre-image of ω_1, then X^* is countably tight.*

and referred to [1] for the proof. However in [1], he only proved this for the case in which X is countably tight. It is not obvious that that hypothesis can be omitted, but in fact it can. We need a definition and lemma.

Definition A space Y is ***ω**-bounded* if each separable subspace of Y has compact closure.

Lemma 5 ([4, 8]) *If Y is ω-bounded and does not include a perfect pre-image of ω_1, then Y is compact.*

We then can establish Lemma 4 as follows.

Proof By Lemma 5, every ω-bounded subspace of X is compact. By [1], it suffices to show that X is countably tight. Suppose, on the contrary, that there is a $Y \subseteq X$ that is not closed, but is such that for all countable $Z \subseteq Y$, $\overline{Z} \subseteq Y$. Since X is a k-space,

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there is a compact K such that $K \cap Y$ is not closed. Then $K \cap Y$ is not ω-bounded, so there is a countable $Z \subseteq K \cap Y$ such that $\overline{Z} \cap K \cap Y$ is not compact. But $\overline{Z} \subseteq Y$, so $\overline{Z} \cap K \cap Y = \overline{Z} \cap K$, which is compact, a contradiction.

Lemma 3 takes care of the hereditary strong $\aleph_1$-collectionwise Hausdorffness we need, since if open subspaces are $\aleph_1$-collectionwise Hausdorff, then all subspaces are, and open subspaces of locally compact spaces are locally compact. The proposition that

$\sum^-$: in a compact countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.

is implied by $\mathrm{PFA}(S)[S]$ was announced by Todorcevic in the Toronto Set Theory Seminar in 2002. ■

From $\sum^-$ it is standard to get the result of Szentmiklóssy quoted earlier. Since the compact space has no uncountable discrete subspace, it has countable tightness. If it were not hereditarily Lindelöf, it would have a right-separated subspace of size $\aleph_1$. But $\sum^-$ implies it has an uncountable discrete subspace, a contradiction.

$\sum$ is established by a minor variation of the forcing for $\sum^-$. There is a proof from $\mathrm{PFA}(S)[S]$ in [7], depending on [19].

Thus all we have to do is prove that $\mathrm{PFA}^{++}(S)[S]$ implies Axiom R.

In order to prove that, we first note that a straightforward argument using the forcing $\mathrm{Coll}(\omega_1, X)$ (whose conditions are countable partial functions from ω_1 to X, ordered by inclusion) shows that $\mathrm{PFA}^{++}(S)$ implies Axiom R^{++}.

It then suffices to prove the following lemma.

Lemma 6 *If Axiom R^{++} holds and S is a Souslin tree, then Axiom R^{++} still holds after forcing with S.*

Proof First note that if X is a set, P is a c.c.c. forcing, and τ is a P-name for a tight cofinal subset of $[X]^{<\omega_2}$, then the set of $a \in [X]^{<\omega_2}$ such that every condition in P forces that a is in the realization of τ is itself tight and cofinal. The tightness of this set is immediate. To see that it is cofinal, let b_0 be any set in $[X]^{<\omega_2}$. Define sets b_α $(\alpha \leq \omega_1)$ and σ_α $(\alpha < \omega_1)$ recursively by letting σ_α be a P-name for a member of the realization of τ containing b_α and letting $b_{\alpha+1}$ be the set of members of X that are forced by some condition in P to be in σ_α. For limit ordinals $\alpha \leq \omega_1$, let b_α be the union of the b_β $(\beta < \alpha)$. Then b_{ω_1} is forced by every condition in P to be in τ.

Since we are assuming that the Axiom of Choice holds, Axiom R^{++} does not change if we require X to be an ordinal. Fix an ordinal γ and let $\rho_\alpha(\alpha < \omega_1)$ be S-names for stationary subsets of $[\gamma]^{<\omega_1}$. Let T be a tight cofinal subset of $[\gamma]^{<\omega_2}$. For each countable ordinal α and each node $s \in S$, let $\tau_{s,\alpha}$ be the set of countable subsets a of γ such that some condition in S extending s forces that a is in the realization of ρ_α. Applying Axiom R^{++}, we have a set $Y \in [\gamma]^{<\omega_2}$ such that each $\mathcal{P}(Y) \cap \tau_{s,\alpha}$ is stationary in $[Y]^{<\omega_1}$.

Since S is c.c.c., every club subset of $[Y]^{<\omega_1}$ that exists after forcing with S includes a club subset of $[Y]^{<\omega_1}$ existing in the ground model. Letting $(\rho_\alpha)_G$ (for each $\alpha < \omega_1$)

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be the realization of ρ_α, we have by genericity then that after forcing with S, each $\mathcal{P}(Y) \cap (\rho_\alpha)_G$ will be stationary in $[Y]^{<\omega_1}$. ■

This completes the proof of Theorem 1. ■

We do not know the answer to the following question; a positive answer would likely enable us to dispense with Axiom R, and possibly with the supercompact cardinal.

Question Does MA_{ω_1} imply every locally compact, hereditarily strongly collectionwise Hausdorff space that does not include a perfect pre-image of ω_1 is paracompact?

We conjecture that in our main result, it is possible to replace "perfect pre-image of ω_1" by "copy of ω_1". The second author has proved this could be done if $\mathrm{PFA}(S)[S]$ implies every first countable perfect preimage of ω_1 includes a copy of ω_1.

Remark That $\mathrm{PFA}(S)[S]$ does not imply Axiom R is proved in [17].

The problem of finding in models of $\mathrm{PFA}(S)[S]$ necessary and sufficient conditions for locally compact normal spaces to be paracompact is studied in [18] by extending the methods of [2] and this note.

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