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THE HEIGHT OF TWO-DIMENSIONAL COHOMOLOGY CLASSES OF COMPLEX FLAG MANIFOLDS

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ABSTRACT. For a parabolic subgroup H of the general linear group $G = Gl(n, \mathbf{C})$, we characterize the Kähler classes of G/H and give a formula for the height of any two-dimensional cohomology class. As an application, we classify the automorphisms of the cohomology ring of G/H when this ring is generated by two-dimensional classes.

1. **The flag manifolds.** For any sequence $n_1, n_2, \ldots, n_l$ of positive integers with $n_1 + n_2 + \cdots + n_l = n$, let $F(n_1, n_2, \ldots, n_l)$ be the space of flags $0 = p_0 \subset p_1 \subset \cdots \subset p_l = \mathbf{C}^n$ in $\mathbf{C}^n$ with $\dim p_j - \dim p_{j-1} = n_j$. Then $F(n_1, n_2, \ldots, n_l)$ can be considered as the quotient of $Gl(n, \mathbf{C})$ by a parabolic subgroup, and thereby has the structure of a complex manifold of complex dimension $\sum_{p<q} n_p n_q$. In this paper we determine the heights of all elements of $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$

Let $s_j = n_1 + \cdots + n_j$. Then for $1 \le j \le l$, we have canonical s_j-plane bundles ξ_j over $F(n_1, n_2, \ldots, n_l)$. We put $x_1 = c_1(\xi_1)$, $x_j = c_1(\xi_j) - c_1(\xi_{j-1})$ for $2 \le j \le l$. Then $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$ is generated by $x_1, x_2, \ldots, x_l$, with the single relation $x_1 + x_2 + \cdots + x_l = 0$. (For a complete description of $H^*(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$, see [1].)

If $\iota_1, \iota_2, \ldots, \iota_k$ is a subsequence of $1, 2, \ldots, l$ with $\iota_k = l$, then there is a map

$$\iota : F(n_1, n_2, \ldots, n_l) \to F(m_1, m_2, \ldots, m_k),\ m_t = s_{\iota_t} - s_{\iota_{t-1}}.$$

sending $p_1 \subset p_2 \subset \cdots p_l$ to $p_{\iota_1} \subset p_{\iota_2} \subset \cdots \subset p_{\iota_k}$. Let $\xi'_1, \ldots, \xi'_k$ be the canonical bundles over $F(m_1, m_2, \ldots, m_k)$, $x'_1, \ldots, x'_k$ the corresponding generators of $H^2(F(m_1, m_2, \ldots, m_k); \mathbf{Z})$.

PROPOSITION 1.1. *The map ι is holomorphic. Further, ι^* is injective and sends x'_j to $x_{\iota_{j-1}+1} + \cdots + x_{\iota_j}$.*

Proof. Since both $F(n_1, n_2, \ldots, n_l)$ and $F(m_1, m_2, \ldots, m_k)$ are quotients of $Gl(n, \mathbf{C})$ and the map $\iota : F(n_1, n_2, \ldots, n_l) \to F(m_1, m_2, \ldots, m_k)$ is induced by

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the identity on $Gl(n, \mathbf{C})$, we see ι is holomorphic. In fact ι is a fibration, and since its Serre spectral sequence collapses (for degree reasons), ι^* is an injection. Finally, we note that the bundle ξ'_j pulls back by ι to the bundle ξ_{ι_j}, and hence

$$\begin{aligned}\iota^*(x'_j) &= \iota^* c_1(\xi'_j) - \iota^* c_1(\xi'_{j-1}) \\ &= c_1(\xi_{\iota_j}) - c_1(\xi_{\iota_{j-1}}) \\ &= x_{\iota_{j-1}+1} + x_{\iota_{j-1}+2} + \cdots + x_{\iota_j}\end{aligned}$$

by naturality of Chern classes.

2. **Kähler classes.** Let M be a complex manifold. We call a cohomology class $u \in H^2(M; \mathbf{Z})$ Kähler if it projects to a Kähler class in $H^2(M{:}\,\mathbf{C})$. We summarize some facts about Kähler classes in the next result.

PROPOSITION 2.1. *Let M be a Kähler manifold of complex dimension d, and suppose $u \in H^2(M; \mathbf{Z})$ is Kähler.*

1. *The cohomology class u has height d in $H^*(M; \mathbf{Z})$.*
2. *If $f: N \to M$ is a holomorphic embedding, then f^*u is Kähler.*
3. *If $v \in H^2(M; \mathbf{Z})$ is also Kähler, then $u + v$ is Kähler.*

Proof. For (1) and (2), see [4] or [7]. For (3), note that if M has Kähler classes u and v, then $M \times M$ has Kähler class $u \otimes 1 + 1 \otimes v$; this goes to $u + v$ under the map induced by the diagonal embedding $M \to M \times M$, so $u + v$ is Kähler by (2).

We shall obtain a formula for the height of elements of $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$ by showing certain elements of $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$ are Kähler.

The flag manifold $F(n_1, n_2)$ is the Grassmannian of n_1-planes in $\mathbf{C}^{n_1+n_2}$. The next result describes the Kähler classes in $H^2(F(n_1, n_2); \mathbf{Z})$.

PROPOSITION 2.2. *If $a > 0$, then $ax_2 = -ax_1 \in H^2(F(n_1, n_2); \mathbf{Z})$ is Kähler.*

Proof. The flag manifold $F(1, N)$ is just the N-dimensional complex projective space. Let ξ'_1 be the canonical line bundle over $F(1, N)$, and x'_1 the corresponding generator of $H^2(F(1, N); \mathbf{Z})$. Then $F(1, N)$ is known to have Kähler class $-x'_1$ (in fact $-x'_1$ projects to the class in $H^2(F(1, N); \mathbf{C})$ induced by the Fubini-Study metric on $F(1, N)$: see [7, p. 218]). Now the Plücker embedding

$$F(n_1, n_2) \to F(1, N), \qquad N = \binom{n_1 + n_2}{n_1} - 1$$

pulls back the line bundle ξ'_1 to the line bundle $\Lambda^{n_1}\xi_1$ over $F(n_1, n_2)$: thus, x'_1 pulls back to $x_1 \in H^2(F(n_1, n_2); \mathbf{Z})$ by naturality of Chern classes. By (2) of 2.1, $-x_1$ is Kähler; so $-ax_1$ is Kähler for any $a > 0$ by (3) of 2.1.

Now we can show certain classes in $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$ are Kähler: this will in fact turn out to be a complete description of Kähler classes of $F(n_1, n_2, \ldots, n_l)$.

THEOREM 2.3. *A cohomology class* $a_1x_1 + \cdots + a_lx_l \in H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$ *is Kähler if* $a_1 < a_2 < \cdots < a_l$.

Proof. For each $1 \leq j \leq l-1$, there is a map

$$F(n_1, n_2, \ldots, n_l) \to F(s_j, n-s_j)$$

given by picking out the jth subspace in the flag. We denote the generators of $H^2(F(s_j, n-s_j); \mathbf{Z})$ by $x_1^{(j)}, x_2^{(j)}$. (Of course $x_1^{(j)} = -x_2^{(j)}$.) Then the product of these maps,

$$F(n_1, n_2, \ldots, n_l) \to \prod_{j=1}^{l-1} F(s_j, n-s_j),$$

is a holomorphic embedding. By 1.1, $x_2^{(j)}$ pulls back to $x_{j+1} + x_{j+2} + \cdots + x_l$. It follows that $a_1x_1 + \cdots + a_lx_l$ is Kähler, since it is the pullback of

$$\sum_{j=1}^{l-1} (a_{j+1} - a_j)x_2^{(j)} \in H^2\left(\prod_{j=1}^{l-1} F(s_j, n-s_j); \mathbf{Z}\right).$$

(We use the relation $x_1 + x_2 + \cdots + x_l = 0$.)

3. **Height formula and applications.**

We use Theorem 2.3 to prove the following formula for the height of elements of $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$.

THEOREM 3.1. *For*

$$a_1x_1 + a_2x_2 + \cdots + a_lx_l \in H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z}),$$

let $\{b_1 < b_2 < \cdots < b_k\}$ *be the set of distinct values taken on the by the* a_t, *and let*

$$m_j = \sum_{a_t = b_j} n_t, \qquad 1 \leq j \leq k.$$

Then the height of $a_1x_1 + \cdots + a_lx_l$ *is*

$$\sum_{p<q} m_p m_q.$$

Proof. For any permutation σ of $1, 2, \ldots, l$, there is a homeomorphism

$$F(n_{\sigma(1)}, n_{\sigma(2)}, \ldots, n_{\sigma(l)}) \to F(n_1, n_2, \ldots, n_l)$$

which permutes the x_j in cohomology. Thus, we can assume $a_1 \leq a_2 \leq \cdots \leq a_l$. If we put $\iota_r =$ order of $\{t \mid a_t \leq b_r\}$, then the map

$$\iota : F(n_1, n_2, \ldots, n_l) \to F(m_1, m_2, \ldots, m_k)$$

of 1.1 has the property

$$\iota^*(b_1x_1' + \cdots + b_kx_k') = a_1x_1 + \cdots + a_lx_l.$$

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Now $b_1x'_1+\cdots+b_kx'_k$ is Kähler by 2.3, hence of height

$$\dim_{\mathbf{C}} F(m_1, m_2, \ldots, m_k)=\sum_{p<q} m_pm_q$$

in $H^*(F(m_1, m_2, \ldots, m_k); \mathbf{Z})$: since ι^* is an injection, $a_1x_1+\cdots+a_lx_l$ must have the same height in $H^*(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$.

COROLLARY 3.2. *Any Kähler class in* $H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$ *must be of the form given in Theorem* 2.3.

Proof. Any Kähler class

$$u=a_1x_1+\cdots+a_lx_l\in H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$$

must have maximal height in $H^*(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$, so by the previous result the coefficients a_j must all be distinct. Suppose now that the coefficients are not in ascending order: let $a_j>a_{j+1}$. Choose a strictly increasing sequence $b_1<b_2<\cdots<b_l$ with $b_j=a_{j+1}$ and $b_{j+1}=a_j$. Then

$$v=b_1x_1+\cdots+b_lx_l\in H^2(F(n_1, n_2, \ldots, n_l); \mathbf{Z})$$

is Kähler by 2.3. If u is Kähler, then $u+v$ is Kähler by 2.1: but this is impossible, since the coefficients of x_j and x_{j+1} in $u+v$ are the same.

Let $F(1^{n-m}, m)$ denote $F(n_1, n_2, \ldots, n_l)$ with $n_1=n_2=\cdots=n_{l-1}=1$ and $n_l=m$. Then 3.1 gives us the following result.

COROLLARY 3.3. *Let*

$$a_1x_{\sigma(1)}+\cdots+a_tx_{\sigma(t)}\in H^2(F(1^{n-m}, m); \mathbf{Z}),\ t\le n-m.$$

where σ *is a permutation of* $1, 2, \ldots, n$ *(if* $m=1$*) or of* $1, 2, \ldots, n-m$ *(if* $m\ge 2$*) and* $a_1, \ldots, a_t\ne 0$. *Then*

$$t(n-t)\le \textit{height}\left(\sum_{j=1}^{t} a_jx_{\sigma(j)}\right)\le t(n-t)+\binom{t}{2}.$$

with the lower bound attained if and only if all the a_j *are equal, and the upper bound attained if and only if all the* a_j *are distinct.*

Proof. Let $b_1, b_2, \ldots, b_k$ be as in 3.1: one of the b_j is zero (since $t\le n-m$), say b_r. Then

$$m_1+m_2+\cdots+m_{r-1}+m_{r+1}+\cdots+m_k=t$$

and $m_r=n-t$. Applying 3.1,

$$\text{height}\left(\sum_{j=1}^{t} a_jx_{\sigma(j)}\right)=\sum_{p<q} m_pm_q=m_r\sum_{p\ne r} m_p+\sum_{\substack{p<q\\ p,q\ne r}} m_pm_q$$

$$=(n-t)t+\sum_{\substack{p<q\\ p,q\ne r}} m_pm_q.$$

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Since

$$0 \leq \sum_{\substack{p<q \\ p,q \neq r}} m_p m_q \leq \binom{t}{2}$$

with the minimum attained precisely when there are no terms in the sum (i.e., all the a_j are equal) and the maximum attained precisely when $m_p = 1$ for $p \neq r$, the conclusion follows.

From the preceding result it follows that any $u \in H^2(F(1^{n-m}, m); \mathbf{Z})$ with $u^n = 0$ is of the form ax_j. This fact is proved in [5] for the case $m \geq n-m$, and in [6] and [2] for the case $m = 1$. It is used in [5] to classify automorphisms of $H^*(F(1^{n-m}, m); \mathbf{Z})$ for $m \geq n-m$. Since 3.3 removes the restriction $m \geq n-m$, the argument of [5] gives immediately the following.

COROLLARY 3.4. *Any automorphism of* $H^*(F(1^{n-m}, m): Z)$ *has the form*

$$x_j \to \varepsilon x_{\sigma(j)}, \quad 1 \leq j \leq n-m.$$

where $\varepsilon = \pm 1$ *and* σ *is a permutation of* $1, 2, \ldots, n$ (*if* $m = 1$) *or of* $1, 2, \ldots, n-m$ (*if* $m \geq 2$).

REMARK. The same result holds for rational coefficients, provided "$\varepsilon = \pm 1$" is replaced by "$\varepsilon \neq 0$". It then follows that the manifolds $F(1^{m-n}, m)$ are all generically rigid, by the main result of [3].

REFERENCES

1. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

2. J. Ewing and A. Liulevicius, *Homotopy rigidity of linear actions on homogeneous spaces*, J. of Pure and Applied Alg. **18** (1980), 259–267.

3. H. Glover and G. Mislin, *On the genus of generalized flag manifolds*, Enseign. Math. XXVII (1981), 211–219.

4. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

5. A. Liulevicius, *Homotopy rigidity of linear actions: characters tell all*, Bull. AMS **84** (1978), 213–221.

6. J. D. Monk, *Geometry of flag manifolds*, Proc. London Math. Soc. (3) **9** (1959) 253–286.

7. R. O. Wells, *Differential Analysis on Complex Manifolds*. Springer-Verlag, New York, 1980.

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