3

Resonances

3.1 How to examine unphysical sheets of the amplitude

In the previous lecture we have discussed in detail that owing to unitarity, analyticity and crossing symmetry – the general properties of the theory – all the physics of hadron interactions is determined, in principle, by the spectrum of real particles.

We saw that the interaction constant – the measure of the interaction strength – entered only as a residue in the pole of the scattering amplitude. Can it be true that the plethora of phenomena in the hadron world is described by a single quantitative characteristic – the residue?

This situation looks strange and profoundly unsatisfactory from a theoretical point of view. It makes one wonder whether there is not something *hidden* beyond the mass spectrum that we have introduced.

Philosophy aside, it is important to know how the amplitude behaves in the vicinity of the cut. We cannot say a priori that it changes smoothly there, since the question of smoothness of a *multi-valued* function is a delicate one.

If we position ourselves near the cut on the physical sheet then, rather close in the energy variable, we have an *unphysical* sheet about which nothing had been said so far. If there is a singularity (for example, a pole) on the unphysical sheet close to the physical one, the amplitude on the physical sheet would be changing fast.

Thus, knowing the analytic structure of the physical sheet alone turns out to be insufficient. We need information about what is happening on the other sheets of the scattering amplitude, what sort of singularities could be there.

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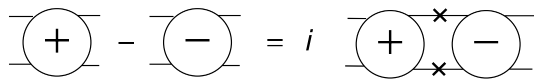
We have to find means to extend – to *continue* – our knowledge of the amplitude '*under the cut*'. It is clear that perturbation theory would be of no help here. Nevertheless, there is a way to get under the cut.

Let us recall the unitarity condition:



$$A(s+i\epsilon,t) - A(s-i\epsilon,t) \;=\; i\int \frac{d^4k}{(2\pi)^2}\delta\big(m_3^2-k^2\big)\delta\big(m_4^2-(p_1+p_2-k)^2\big)$$
$$\cdot A(p_1,p_2,k)A^*(p_5,p_6,k). \qquad (3.1)$$

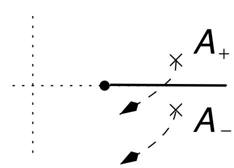
Since each of the block amplitudes A, A^* depends in fact on two invariants, it is convenient to rewrite the integral in terms of the Lorentz invariant momentum transfers,

$$A(s+i\epsilon,t) - A(s-i\epsilon,t)$$
$$= \int\!\!\int dt_1\, dt_2 K(s,t_1,t_2)\cdot A(s+i\epsilon,t_1)A(s-i\epsilon,t_2), \qquad (3.2)$$

where we have introduced K as the corresponding Jacobian transformation factor.

Now take $A(s-i\epsilon)$ to the r.h.s. and try to look upon (3.2) as an integral equation for $A(s+i\epsilon,t)$ with the kernel $\int dt_2 K(s,t_1,t_2)A(s-i\epsilon,t_2)$ and an inhomogeneity $A(s-i\epsilon,t)$. Imagine that we learned how to calculate the integrals and managed to solve the equation. What would have been the gain? We would have expressed the analytic function on the *upper* side of the cut, $A(+)$, in terms of that on the *lower* side of the cut:

$$A(s+i\epsilon) \;=\; F\left(A(s-i\epsilon)\right) \qquad (3.3)$$



Till now we kept s real and used $i\epsilon$ to separate the points at the two sides of the cut. Let us now give an imaginary part to s itself, a negative one to be definite. Then the argument of $A(s-i\epsilon)$ would simply move onto the lower half-plane of the physical sheet, while the 'upper' function $A(s+i\epsilon)$ whose argument is tightly linked with that of $A(s-i\epsilon)$, will cross the cut and occur on the lower half-plane too, but on another – *unphysical* – sheet!

Under such continuation the relation (3.3) has acquired a new meaning: the value of the amplitude at a given point on the unphysical sheet is

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now (functionally) determined by the value of the physical amplitude. Along this way we would have solved a remarkable problem, namely, with the help of the unitarity condition we would have examined the content of the unphysical sheet and, in particular, found the singularities of the amplitude there (which is what concerns us in the first place).

3.2 Partial waves and two-particle unitarity

The programme that we have described is easy to carry out for the *first* unphysical sheet related to the two-particle unitarity condition (which holds for $4\mu^2 < s < 9\mu^2$).

Recall the partial wave expansion

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell(\cos\Theta)\,. \tag{3.4}$$

It is clear that in these terms the unitarity condition (3.1) will simplify greatly. Indeed, ℓ – the total angular momentum in the cms – is a conserved quantity. Therefore, if we choose an initial state with a given ℓ, the intermediate state will be uniquely determined and the integration on the r.h.s. of (3.1) will have to disappear. Let us see how it actually happens.

First we attend to the momentum integration in (3.1):

$$d^4k = |\mathbf{k}|^2\, d|\mathbf{k}|\, dk_0\, d\Omega = \frac{1}{2}|\mathbf{k}|\, dk^2\, dk_0\, d\Omega\,, \quad \delta\big(m_3^2 - k^2\big) dk^2 = 1\,.$$

In the cms we have $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$, $p_{10} + p_{20} = \sqrt{s}$, so that

$$\delta\big(m_4^2 - (p_1+p_2-k)^2\big) = \delta\big(m_4^2 - m_3^2 - s + 2k_0\sqrt{s}\big).$$

Taking off the integration over k_0 we arrive at

$$\begin{aligned}\operatorname{Im} A(s,t) &= \frac{1}{2}\int \frac{d\Omega}{(2\pi)^2}\cdot\frac{1}{2}|\mathbf{k}|\cdot\frac{1}{2\sqrt{s}} A(s,t_1)A^*(s,t_2)\\ &= \frac{|\mathbf{k}|}{8\pi\sqrt{s}}\int \frac{d\Omega}{4\pi} A(s,\cos\Theta_1)A^*(s,\cos\Theta_2).\end{aligned} \tag{3.5}$$

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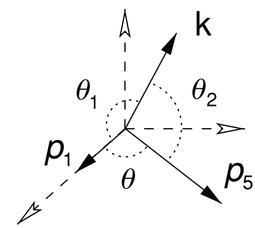
The modulus of the intermediate state momentum $\mathbf{k}$ is fixed by the on-mass-shell conditions. The partial wave expansion will help us to perform the remaining integration over its direction angles. Using (3.4) for the amplitudes on the l.h.s. and r.h.s. of (3.5) we have



$$\begin{aligned}\operatorname{Im} A(s,t) &= \sum_{\ell=0}^{\infty}(2\ell+1)\operatorname{Im} f_\ell(s)P_\ell(\cos\Theta) = \frac{|\mathbf{k}|}{8\pi\sqrt{s}}\sum_{\ell_1,\ell_2} f_{\ell_1}(s)f^*_{\ell_2}(s)\\ &\quad\times(2\ell_1+1)(2\ell_2+1)\int\frac{d\Omega}{4\pi}P_{\ell_1}(\cos\Theta_1)P_{\ell_2}(\cos\Theta_2) \qquad (3.6)\\ &= \frac{|\mathbf{k}|}{8\pi\sqrt{s}}\sum_{\ell_1=0}^{\infty} f_{\ell_1}(s)f^*_{\ell_1}(s)(2\ell_1+1)P_{\ell_1}(\cos\Theta),\end{aligned}$$

where we used the well-known orthogonality relation for spherical functions (Legendre polynomials),

$$\int\frac{d\Omega}{4\pi}P_n(\cos\Theta_1)P_m(\cos\Theta_2) = \frac{\delta_{nm}}{2n+1}P_n(\cos\Theta).$$

Comparing the two sides of the equation, we retrive the unitarity condition for a partial wave with angular momentum ℓ,

$$\operatorname{Im} f_\ell(s) = \tau f_\ell(s)f^*_\ell(s), \qquad (3.7a)$$

$$\tau = \tau(s) \equiv \frac{k_c}{8\pi\sqrt{s}} = \frac{1}{16\pi}\frac{k_c}{\omega_c}, \qquad (3.7b)$$

with k_c the modulus of the intermediate state momentum in the cms:

$$k_c \equiv |\mathbf{k}| = \frac{\sqrt{s^2-2s(m_3^2+m_4^2)+(m_3^2-m_4^2)^2}}{2\sqrt{s}}, \quad \omega_c = \frac{\sqrt{s}}{2}. \qquad (3.8a)$$

For the case of equal masses, $m_3 = m_4 = m$,

$$k_c = \frac{\sqrt{s-4m^2}}{2}\,. \qquad (3.8b)$$

The solution of the elastic unitarity condition (3.7a) reads

$$f_\ell(s) = \frac{1}{2i\tau(s)}\left[\mathrm{e}^{2i\delta_\ell}-1\right] = \frac{\sin\delta_\ell}{\tau}\,\mathrm{e}^{i\delta_\ell}, \qquad (3.9)$$

with $\delta_\ell = \delta_\ell(s)$ the scattering phase in a given angular momentum state.

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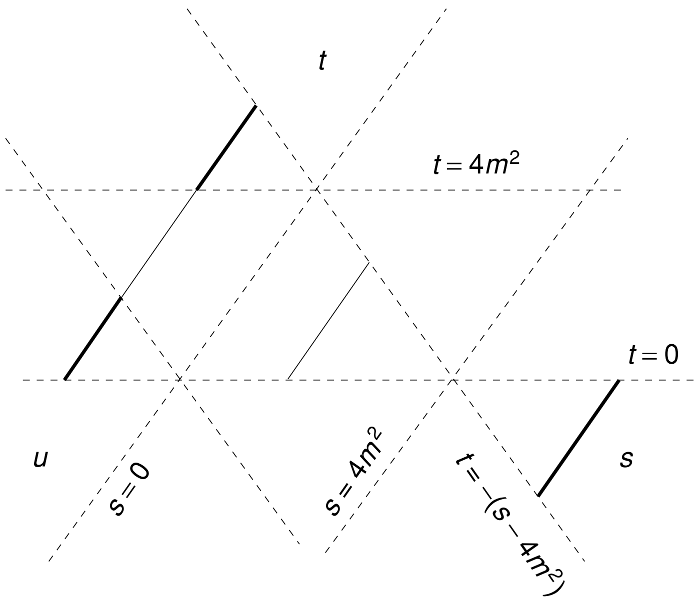

Fig. 3.1 Three integration intervals on the Mandelstam plane. The amplitude is complex at $s > 4m^2$ and $s < 0$ (on the thick lines).

3.3 Analytic properties of partial waves and resonances

We are now going to discuss analytic properties of $f_\ell(s)$. We will need the expression for the partial wave which is complementary to (3.4):

$$f_\ell(s) = \frac{1}{2}\int_{-1}^{1} d\cos\Theta\; P_\ell(\cos\Theta) A(s, t(\cos\Theta))\,. \tag{3.10}$$

The cosine of the scattering angle,

$$\cos\Theta = 1 + \frac{2\,t}{s - 4m^2},$$

varies between -1 and $+1$. On the Mandelstam plane this corresponds to integration over t from $t = -(s - 4m^2)$ up to $t = 0$ (Fig. 3.1). For $s > 0$ the partial wave $f_\ell(s)$ mirrors analytic properties of the amplitude A: it is real for $s < 4m^2$ (since the integration interval then lies inside the triangle where A is real) and is complex above the threshold, $s > 4m^2$. For $s < 0$ $f_\ell(s)$ becomes complex again; this time because of the integration contour hitting t- and u-channel thresholds.

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Since $f_\ell(s)$ is real on a finite interval in the s-plane, it assumes complex conjugated values on the sides of the cut:

$$(f(s))^* = f(s^*).$$

Therefore we can represent (3.7a) in terms of discontinuity as

$$\frac{1}{2i}[\, f_\ell(s+i\epsilon) - f_\ell(s-i\epsilon)\,] = \tau(s) f_\ell(s+i\epsilon) f_\ell(s-i\epsilon).$$

The formula

$$f_\ell(s+i\epsilon) = \frac{f_\ell(s-i\epsilon)}{1 - 2i\tau(s) f_\ell(s-i\epsilon)} \tag{3.11}$$

solves the problem of analytic continuation of the amplitude (to be precise, of each of its partial wave components) onto the first unphysical sheet related to the two-particle threshold.

So, what are the singularities on the unphysical sheet? Obviously, $f_\ell(+)$ has the same cuts as $f_\ell(-)$. In addition, it acquires new singularities – *poles* – where the denominator in (3.11) vanishes, that is in the points where

$$f_\ell(-) = \frac{1}{2i\tau(s)}\,. \tag{3.12}$$

These poles on the unphysical sheet(s) are called *resonances*. The position of such a pole depends essentially on the value of the coupling constant. If the coupling is small, so is the physical scattering amplitude. The amplitude can reach a finite value which is required by the resonance condition (3.12) only if interaction is strong enough.

We can reverse the statement:

> It is the resonance states that bear additional essential information about interaction that we talked about in the beginning of this lecture.

What is the reason that we have not met singularities more complicated than simple poles?

To answer the question we return to the integral equation in the general form of (3.2). The t_1, t_2 integrations run over finite intervals; moreover, the *kernel* of the equation,

$$\phi(s,t,t_1) = \int dt_2 K(s,t_1,t_2) A(s-i\epsilon,t_2),$$

is a smooth regular function since it is determined by the amplitude on the physical sheet where the amplitude is regular. Therefore, our equation is a standard integral equation of the Fredholm type whose solutions may have only poles (at the points where the Fredholm determinant vanishes).

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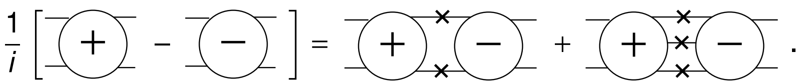
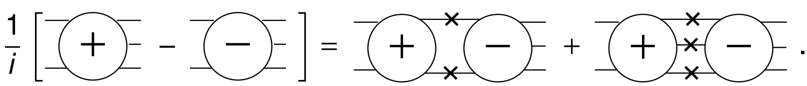
3.4 Three-particle unitarity condition

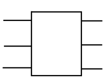
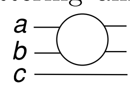
Producing only poles is an intrinsic property of the simplest – two-particle – unitarity. The continuation of the three-particle unitarity relation is technically much more involved. We shall outline the main plan of the corresponding analysis. The sketch that follows will suffice for us to grasp the qualitative features of the answer.

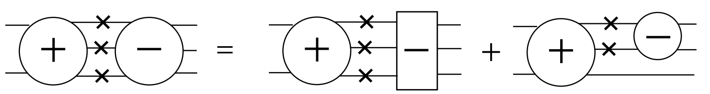
Below the four-particle threshold the unitarity condition has the following schematic structure:

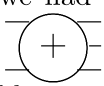

(3.13)

A new object entered namely, the amplitude $A_{2\to 3}$. For the latter we have its proper unitarity equation,

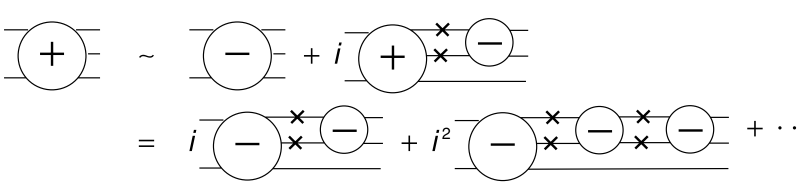

(3.14)

The last kernel $A_{3\to 3}(-)$ contains an *irreducible* scattering amplitude,  but also various *reducible* pieces of the type  so that symbolically

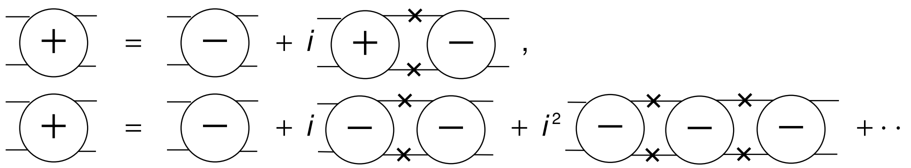

(3.15)

If the last contribution were not present, the result would have been similar to what we had in the two-particle case. That is, an analytic continuation of  beneath the cut would have produced only poles, namely the old two-particle and new three-particle resonances. The new piece, however, is not of the Fredholm type. The corresponding kernel is *singular* as it contains $\delta(p_c - p'_c)$ – one of the particles did not scatter.

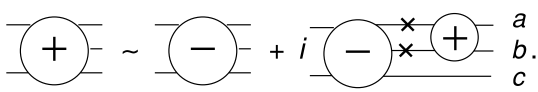
How would this complicate the answer? To understand the key features let us keep only the singular term in (3.14) and iterate the equation for $A_{2\to 3}(+)$:


(3.16)

Comparing the chain of the $A_{2\to2}(-)$ blocks in (3.16) with that of the iterated two-particle unitarity condition,



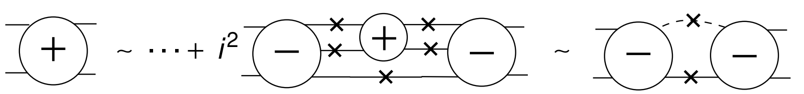
we observe that



(3.17)

In the course of the analytic continuation, the $(-)$ amplitudes stay on the physical sheet while the amplitude $A_{2\to2}(+)$ moves to the first unphysical sheet where, as we know, it has resonance poles in the pair energy s_{ab}.

How will this affect analytic properties of the two-particle scattering amplitude on the l.h.s. of (3.13)? Let us substitute the resonance pole term into the r.h.s. of (3.17) and then into the unitarity condition (3.13):



This is a typical diagram for a threshold branch cut due to the exchange of two poles, one of which is a normal particle and the other – a resonance.

Thus, on the *second* unphysical sheet related to the three-particle cut $9\mu^2 < s < 16\mu^2$ we find, apart from poles with complex masses, also *cuts* – creation thresholds of pairs of a particle with a resonance (from the first unphysical sheet).

In perturbation theory poles have led to the appearance of threshold cuts on the physical sheet. Analogously, on the other sheets there emerge particle–resonance, resonance–resonance, etc. thresholds.

Now we are in a position to formulate the qualitative answer for the analytic structure of the amplitude:

> The full analyticity image is poles – particles and resonances – all other singularities being the unitarity driven consequence of the existence of these poles.

3.5 Properties of resonances

Both theoretically and experimentally resonances are as important as ordinary particles. Therefore we need to learn how to describe them, in the first place.

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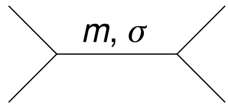
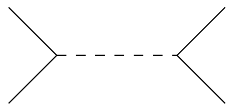
Let us draw an analogy. A particle is characterized by its mass and spin. A usual pole amplitude we describe in terms of a diagram



(3.18)

Does it make sense to draw an analogous diagram for a resonance?



(3.19)

If the series (3.4) converges,

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(s)\, P_\ell(\cos\Theta), \tag{3.20}$$

the full amplitude will but repeat all the poles of the partial waves. The contribution to the amplitude of the resonance in the ℓ-wave,

$$f_\ell(s) \;=\; \frac{r}{m_{\rm res}^2 - s}, \tag{3.21}$$

will be

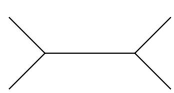
$$A^{\rm pol} \;=\; (2\ell+1)\frac{r\, P_\ell(\cos\Theta)}{m_{\rm res}^2 - s}. \tag{3.22}$$

Now we should check that (3.22) coincides indeed with the Born amplitude of the s-channel exchange of a particle with spin $\sigma = \ell$.

3.5.1 Angular dependence

How would we write a Feynman diagram for a particle with a given spin σ? In the case of a *scalar*, $\sigma = 0$, the amplitude

$$A^{\rm pol} \;=$$



$$=\; \frac{g^2}{m^2 - s}$$

does not depend on angles.

The propagator of a *vector* particle, $\sigma = 1$, contains vector indices and the exchange diagram takes the form

$$A \;=\; g^2\, \Gamma_\mu D^{\mu\nu}(k) \Gamma_\nu\,; \tag{3.23}$$

$$D^{\mu\nu}(k) \;=\; \frac{-g^{\mu\nu} + b\cdot k^\mu k^\nu}{m^2 - k^2}\,. \tag{3.24}$$

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In QED we dealt with a vector particle – a photon. There we were allowed to omit the term $b \cdot k^\mu k^\nu$. Remember why? Because due to the conservation of the electromagnetic current the interaction was insensitive to this piece in the photon Green function. Not so in a general case, and we ought to determine the coefficient b in (3.24).

A vector particle has three polarization states that are 'propagated' by the propagator

$$D^{\mu\nu}(k) \;=\; \sum_{\lambda=1}^{3} \frac{e_\lambda^\mu(k) e_\lambda^\nu(k)}{m^2 - k^2}\,. \tag{3.25}$$

How to choose three out of four independent vectors? To this end we have to invoke an additional condition

$$k_\mu e_\lambda^\mu = 0\,. \tag{3.26}$$

In the rest frame of the particle this condition turns into $m e_\lambda^0 = 0$,

$$e_\lambda^\mu \;=\; (0, \mathbf{e}_\lambda)\,, \qquad \lambda = 1, 2, 3.$$

These are usual space vectors describing three possible spin projections. Taking into account the transversality condition (3.26), the Green function becomes

$$D^{\mu\nu}(k) \;=\; \frac{\frac{k^\mu k^\nu}{m^2} - g_{\mu\nu}}{m^2 - k^2}\,. \tag{3.27}$$

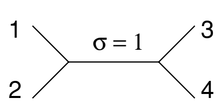
Now, what to write for the vertex function? Possible vector structures are

$$\Gamma^\mu \;=\; a p_1^\mu + b p_2^\mu \;=\; \alpha(p_1 + p_2)^\mu + \beta(p_1 - p_2)^\mu\,.$$

It is clear that the first term will not contribute to the pole. Using (3.27) we get

$$(p_1 + p_2)_\mu D^{\mu\nu}(k) \equiv k_\mu D^{\mu\nu}(k) = \frac{k^\nu}{m^2} \cdot \frac{k^2 - m^2}{m^2 - k^2} \Longrightarrow \text{finite.}$$

The pole contribution to the amplitude (3.23) takes the form

$A^{\rm pol} \;=$ 

$= \; g'^2 (p_1 - p_2)_\mu D^{\mu\nu}(k)(p_3 - p_4)_\nu.$ (3.28)

To understand the meaning of this expression we go again to the cms of colliding particles (which we take for simplicity to have equal masses). Then the vertex functions

$$(p_1 - p_2)_\mu = (0, 2\mathbf{q}), \qquad (p_3 - p_4)_\nu = (0, 2\mathbf{q}')$$

turn into three-vectors describing relative momenta of initial and final state particles, correspondingly:

$$A^{\text{pol}} = g'^2 \frac{4(\mathbf{q}\cdot\mathbf{q}')}{m^2 - s}.$$

As compared with the case $\sigma = 0$, a scalar product appeared in the numerator,

$$(\mathbf{q}\cdot\mathbf{q}') \propto \cos\Theta = P_1(\cos\Theta). \tag{3.29}$$

It is straightforward to verify that for an arbitrary spin σ the Feynman diagram describing s-channel particle exchange gives an expression proportional to

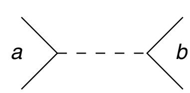
$$P_\sigma(\cos\Theta_{\text{cms}}),$$

as we have checked for the simplest cases $\sigma = 0, 1$.

So, the angular dependence turned out to be the same for particle and resonance exchanges.

3.5.2 Factorization and unitarity

Now we have to check another very important property of 'particles': factorization. What does it mean? When we draw a Feynman graph, the initial and final states enter as a product, that is, they are *factorized*. If the analogy between resonances and particles is to be preserved, the residue r in (3.21) should split into the product of two constants each belonging to the proper vertex,

$A^{\text{res}} =$ 

$$= g_a \cdot \frac{1}{m^2 - s} \cdot g_b\,. \tag{3.30}$$

The fact that it is indeed so is not accidental and is tightly related to unitarity.

Up to now we have considered elastic scattering and did not differentiate between initial- and final-state systems (a and b in (3.30)).

It is straightforward to generalize the notion of partial amplitudes for the case of multi-channel scattering problem by introducing the unitary scattering matrix

$$SS^\dagger = I\,, \qquad \sum_c S_{ac}S^\dagger_{cb} = I_{ab},$$

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where a, b, c mark different channels of the reaction. The generalized expression for the partial wave amplitude describing $a \to b$ transition reads*

$$f_{ab} = \frac{1}{2i\sqrt{\tau_a \tau_b}}\,[\,S_{ab} - 1\,]\,. \tag{3.31}$$

The scattering matrix S can be diagonalized with the help of a unitary tranformation U,

$$S_{ab} = U^{\dagger}_{ac} S_{cd} U_{db},$$

with S a diagonal unitary matrix

$$S_{cd} = S_c I_{cd}\,, \quad S_c = \exp(2i\delta_c).$$

Our resonance is a pole in one of its eigenvalues at some complex value

$$s = M^2 = M_1^2 - iM_2^2.$$

Let it be the first element S_1; we may write in the pole approximation

$$S_1 \;=\; \frac{M^{*2} - s}{M^2 - s}\,\mathrm{e}^{2i\beta} \;=\; \frac{-2i\,\mathrm{Im}\,M^2}{M^2 - s}\,\mathrm{e}^{2i\beta} \;+\; \text{regular}.$$

Here 2β describes the scattering phase away from the resonance:

$$S_1 \simeq \mathrm{e}^{2i\beta} \qquad \text{for} \;\; \left|s - M_1^2\right| \gg M_2^2.$$

The pole contribution to the full scattering matrix becomes

$$S_{ab} \;\simeq\; U_{a1}\left[\frac{-2i\,\mathrm{Im}\,M^2}{M^2 - s}\,\mathrm{e}^{2i\beta}\right] U^{\dagger}_{1b} = U_{a1}\frac{2iM_2^2}{M^2 - s}\,\mathrm{e}^{2i\beta}\,U^*_{b1}; \tag{3.32a}$$

$$f_{ab} \;\simeq\; \frac{U_{a1}}{\sqrt{\tau_a}}\frac{M_2^2}{M^2 - s}\,\mathrm{e}^{2i\beta}\frac{U^*_{b1}}{\sqrt{\tau_b}} \;=\; \frac{g_a\,g_b^*}{M^2 - s}\,\mathrm{e}^{2i\beta}\,, \tag{3.32b}$$

where we have introduced the constants

$$g_a \equiv U_{a1}\cdot\frac{M_2}{\sqrt{\tau_a}} \tag{3.32c}$$

that measure coupling of the resonance to different particle states a and play the rôle of (complex) interaction constants. In fact g_a can be taken to be real. This follows from the invariance of the scattering martix with respect to time reversal. Indeed, in this case the S-matrix is symmetric, $S_{ab} = S_{ba}$. This gives $U_{a1}U^*_{b1} = U^*_{a1}U_{b1}$ so that U_{a1} and U_{b1} have equal

* We don't write ℓ implying that the total angular momentum is included in the channel indices a, b (ed.).

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phases which cancel in (3.32a), (3.32b) and, consequently, we can redefine

$$g_a \implies |U_{a1}| \cdot \frac{M_2}{\sqrt{\tau_a}}. \tag{3.33}$$

The resonance contribution to the full amplitude takes the form very similar to the Born diagram for spin-ℓ particle exchange:

$$A^{\text{res}} = \frac{g_a (2\ell+1) P_\ell(z)\, g_b}{M^2 - s}\, \mathrm{e}^{2i\beta}\,. \tag{3.34}$$

We see that the resonance appears in all channels, $a \to a$, $a \to b$, $b \to b$ etc., and that the transition amplitudes between various channels are factorized. This property followed directly from unitarity.

Thus we have checked that both properties of normal particles – the angular dependence of the amplitude and factorization – hold for complex poles as well.

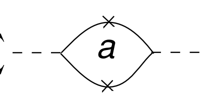
3.5.3 Width of the resonance

For resonances, however, there is one more statement which makes no sense for ordinary particles, namely, the relation between $\operatorname{Im} M^2$ and the coupling constants that also follows from unitarity. Recalling that the matrix U is unitary,

$$\sum_a U_{a1}^* U_{a1} = 1\,,$$

and expressing U_{a1} via (3.33) one relates the set of g_a with the full *width* of the resonance:

$$\sum_a \tau_a g_a^2 = \sum_a$$



$$= M_2^2 \equiv M_1 \Gamma. \tag{3.35}$$

The constant g_a is the transition amplitude between the resonance and ordinary particles with τ_a their phase space. Thus the amount by which the pole is shifted from the real energy axis is determined by the total probability of the resonance decay into real particles.

3.6 A resonance or a particle?

How to establish a generic link between particles and resonances? Let us discuss a simple example of how a resonance emerges.

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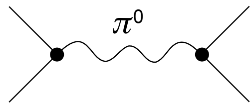
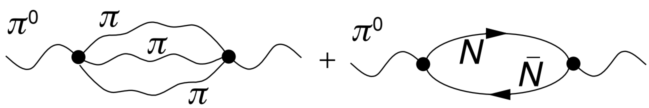
Imagine that in some reaction a π^0 meson appears as a pole:



Now we switch on the electromagnetic interaction with a very small interaction constant. Then a new transition will emerge, $\pi^0 \to \gamma\gamma$. How will this small correction to strong dynamics affect the propagation of π^0?

Pion self-energy is determined (in the hadronic language) by a variety of possible intermediate states such as 3π, $N\bar{N}$, etc.:

$\Sigma_h =$  $+\cdots$ (3.36)

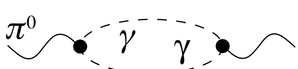
The point in k^2 where the denominator of the Green function

$$G(k) = \frac{1}{m_0^2 - k^2 - \Sigma(k^2)}$$

turns into zero determines the renormalized pion mass:

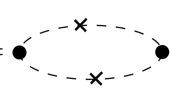
$$m_\pi^2 \;=\; m_0^2 - \Sigma\big(m_\pi^2\big). \tag{3.37}$$

For (3.37) to have a real solution, Σ has to be real. We know that the graphs of (3.36) become complex at $k^2 > (3m_\pi)^2$ and $k^2 > (2M_N)^2$, respectively, which scales are significantly higher than m_π^2. That was the case before we turned QED on. Now, however, we shall also have the graph

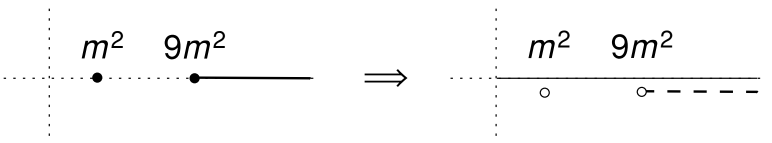
$\Sigma_{\text{e.m.}} =$  , $\quad \Sigma(k^2) = \Sigma_h(k^2) + \Sigma_{\text{e.m.}}(k^2),$ (3.38)

which has a threshold (and therefore a branch cut) at $k^2 = (2m_\gamma)^2 = 0$. As a result, (3.37) will have no real solution and the π^0 Green function will acquire a 'resonance' form

$$G(k^2) \;=\; \frac{1}{m^2 - k^2 - im_2^2},$$

$m_2^2 \;=\; \operatorname{Im}\Sigma_{\text{e.m.}}(m^2) =$  $= g^2_{\pi^0\to\gamma\gamma}\cdot\tau_{\gamma\gamma}\,.$ (3.39)

What happened? After the introduction of a small-mass intermediate state the pion pole occurred right on the cut! This, however, would have contradicted the unitarity condition. Therefore the pole (together with all multi-state cuts that it generates) moves under the two-photon cut onto the unphysical sheet:

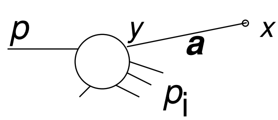


This analysis tells us that the difference between particles and resonances is elusive. If we believe that electromagnetic interaction has no major effect on the basic properties of pions as strongly interacting particles, then it becomes insignificant whether a π-meson is a particle or a resonance, whether the corresponding pole lies on the real axis or slightly below.

Concluding, we have proved that a resonance as a contribution to the scattering amplitude is identical to a particle with definite quantum numbers and possesses the usual factorization properties that are characteristic for particle exchange. Therefore we can describe resonances with the help of Feynman diagrams as we did for particles, the only difference being a complex mass whose imaginary part is related to the total decay probability of the resonance into stable particles.

3.7 Observation of resonances

This is the last question that we need to address in this lecture. Imagine that a beam hits a target and some particles are produced. The probability amplitude to observe one of these particles a at a given point $\mathbf{r}$ at time t,



$$x = (t, \mathbf{r}),$$

is given by the expression

$$A = \int \frac{d^4k}{(2\pi)^4\, i} \frac{\mathrm{e}^{-ik^\mu (x-y)_\mu}}{m^2 - k^2 - i\epsilon}\, f(y, p, p_i, \ldots), \tag{3.40}$$

where we have explicitly written the propagator of our particle a.

In a real experiment the observation time t is macroscopically large, $t - y_0 \gg m^{-1}$. Therefore the phase factor in (3.40) is oscillating fast with k, and the integral would be exponentially small if not for the singularity of the propagator function. Since $t - y_0 > 0$, we can close the integration contour in k_0 around the pole at $k_0(\mathbf{k}) = \sqrt{m^2 + \mathbf{k}^2}$, that is to put particle

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a on mass shell:

$$A = \int \frac{d^3\mathbf{k}}{2k_0(\mathbf{k})(2\pi)^3}\, \mathrm{e}^{-ik_0(\mathbf{k})(t-y_0)+i(\mathbf{k}\cdot(\mathbf{r}-\mathbf{y}))} f. \tag{3.41}$$

At $t \to \infty$ plane waves in this sum cancel each other everywhere but on a classical trajectory where $\mathbf{r}$ is linearly increasing with t:

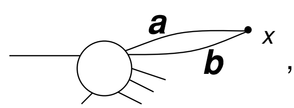
$$\boldsymbol{\nabla}_{\mathbf{k}}[\,k_0(\mathbf{k})(t-y_0) - (\mathbf{k}\cdot(\mathbf{r}-\mathbf{y}))\,] = 0 \;\Rightarrow \mathbf{r} = \mathbf{v}t + \mathbf{r}_0,\;\; \mathbf{v} \equiv \frac{dk_0(\mathbf{k})}{d\mathbf{k}} = \frac{\mathbf{k}}{k_0}.$$

It is easy to verify that the probability of observing the particle at $\mathbf{r}$ falls with the distance as

$$w \propto |A|^2 \propto \frac{1}{\mathbf{r}^2},$$

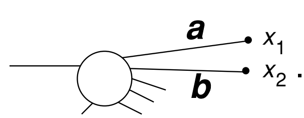
which is in accord with the increasing size of the surface of the observation sphere.

If we were to register *two* particles a and b at the same point,



the probability would fall faster with $|\mathbf{r}|$ because two particles prefer to separate at large distances.

Consider now the general case of the observation of two particles.



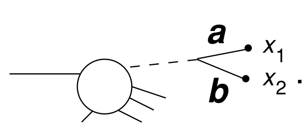
(3.42)

Repeating literally the above analysis we will obtain that the observation probability is concentrated along the trajectories

$$\mathbf{r}_1 = \mathbf{v}_1 t + \mathbf{r}_{10}, \quad \mathbf{r}_2 = \mathbf{v}_2 t + \mathbf{r}_{20}.$$

By measuring the directions of particle momenta we can determine, within the Heisenberg uncertainty, classical trajectories, and verify that the particles indeed originate from the interaction region.

Suppose now that a and b may combine into a resonance state:



(3.43)

How does this graph differ from the usual particle creation in (3.42)? In the case of a complex pole with a small imaginary part $\mathrm{Im}\, M^2 = M_2^2 \ll M_1^2$

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we have

$$k_0 = \sqrt{M^2 + \mathbf{k}^2} \simeq \sqrt{M_1^2 + \mathbf{k}^2} - \frac{iM_2^2}{2\sqrt{M_1^2 + \mathbf{k}^2}} \equiv E - \frac{iM_2^2}{2E}. \tag{3.44}$$

The amplitude then falls exponentially at large times,

$$A \propto \mathrm{e}^{-ik_0 t} \propto \exp\left\{-\frac{1}{2}\frac{M_2^2}{E}\,t\right\} \equiv \exp\left\{-\frac{\Gamma}{2}\cdot\frac{M_1}{E}\,t\right\}, \tag{3.45}$$

and the probability decreases as

$$|A|^2 \propto \exp\left\{-\Gamma\, t\cdot\frac{M_1}{E}\right\}. \tag{3.46}$$

Here the ratio M_1/E is the usual Lorentz time dilatation factor. Introducing the proper life-time τ of the resonance,

$$\tau^{-1} \equiv \Gamma = \frac{M_2^2}{M_1}, \tag{3.47}$$

we conclude that for finite times $t \sim \tau \cdot E/M_1$ our resonance state may propagate as a whole. Therefore, restoring trajectories of its decay products we will see that a and b originate not from the target but fly away from a point at some finite distance from the interaction region. This distance will vary event by event. It is clear, however, that the probability of a large displacement is exponentially small.

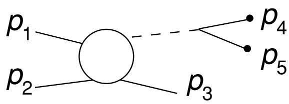
3.7.1 Non-exponential decay?

Sometimes people talk about the ‘non-exponentiality’ of a decay. Calculating concrete integrals I will always find contributions that fall as a power of t (most often as $t^{-3/2}$) in addition to the resonance exponent. Does this imply that the decay law (3.46) is inaccurate?

We register a resonance by observing its decay products. Therefore there will always be a background due to production of particles a and b directly off the target. Let us switch on our measuring devices one hour after irradiating the target; some extremely slow particles could have been produced that crawl and hit the detectors after this immense time is elapsed. This power-behaving ‘tail’ has nothing to do with the decay of the resonance.

3.7.2 Resonance in the invariant mass distribution

Till now we were considering resonances with a small width Γ. For resonances that decay relatively fast (large Γ) a direct visual observation of the resonance path becomes impossible. In this case resonances are extracted by examining energetic characteristics of the process.



$$= f(p_1, p_2, p_3)\frac{1}{M^2 - (p_4 + p_5)^2} g(p_4, p_5). \quad (3.48)$$

The resonance propagator in (3.48) introduces a *peak* in the distribution over the invariant mass s_{45} of the pair of particles 4 and 5:

$$\frac{d\sigma}{ds_{45}} \propto |A|^2 \propto \frac{1}{(s_{45} - M_1^2)^2 + M_2^4}. \quad (3.49)$$

Let us remark that instead of observing decay products it suffices to measure the recoil momentum p_3, since due to momentum conservation $s_{45} = (p_4 + p_5)^2 = (p_1 + p_2 - p_3)^2$.

This method works only when the value of the resonance production amplitude f is sufficiently large. It may turn out, however, that in reactions that are available to experimenters the resonance is produced with a small probability.

3.7.3 Phase analysis

In addition to the invariant mass spectrum there is another method (though less unambiguous) of extracting resonances right from elastic scattering – the phase analysis. Recall the general expression for the partial wave amplitude describing elastic *ab* scattering

$$f_\ell(s) \;=\; \frac{1}{2i\tau(s)}\left[\eta(s)\,\mathrm{e}^{2i\delta_\ell(s)} - 1\right]. \quad (3.50)$$

The scattering phase factor near the resonance is

$$\mathrm{e}^{2i\delta_\ell} \;=\; \frac{M_1^2 - s + iM_2^2}{M_1^2 - s - iM_2^2}\,\mathrm{e}^{2i\beta_\ell}. \quad (3.51)$$

If the amplitude away from the resonance is small (as, for example, for e^+e^- scattering), then $\eta \simeq 1$, $\beta_\ell \simeq 0$. In such a case the resonance is impossible to miss since the amplitude at the peak hits the *maximal* value allowed by unitarity,

$$f_\ell = \frac{i}{\tau} \;=\; f_\ell^{\max} \qquad \text{for } s = M_1^2 \quad (\eta = 1,\ \beta_\ell = 0). \quad (3.52)$$

What to do if the non-resonant scattering is large and the peak does not stick out from the background?

By measuring the shape of the angular distribution, one can extract a few first partial waves by fitting the cross section with an approximate

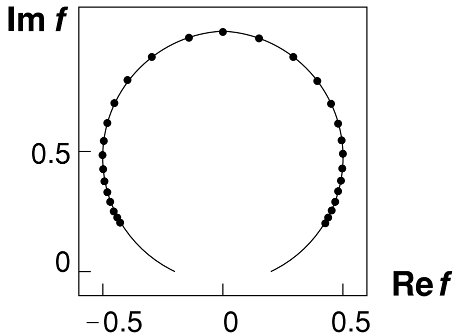


Fig. 3.2 Example of an Argand plot.

expression for the amplitude

$$A(s, \cos\Theta) \simeq \sum_{\ell=0}^{\ell_0} (2\ell+1) f_\ell(s) P_\ell(\cos\Theta).$$

The following remarkable property of $f_\ell(s)$ then comes onto the stage. The resonant factor in (3.51),

$$\frac{M_1^2 - s + iM_2^2}{M_1^2 - s - iM_2^2} \simeq \frac{(M_{\rm res} - E) + i\Gamma/2}{(M_{\rm res} - E) - i\Gamma/2},$$

equals 1 on both sides of the resonance, $|E - M_{\rm res}| \gg \Gamma$, but changes fast in a relatively small energy interval $|E - M_{\rm res}| \sim \Gamma$. If we put a point on a complex plane to mark the value of the amplitude f_ℓ at a given energy, this point will make a *full circle* when the energy variable crosses the mass of the resonance (provided η, β, τ do not change essentially inside the interval of the order Γ), see Fig. 3.2.

Combining the three methods that we have described here, a plethora of hadronic resonances has been established.

The studies of the spectrum of hadrons tells us that the fact of stability/instability of a hadron under consideration, whether it is a particle or a resonance, is not of major importance for strong interaction dynamics. It often looks accidental, depending on an interplay of factors that we rather consider insignificant today.