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Poincaré Inequalities and Neumann Problems for the p-Laplacian

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Abstract. We prove an equivalence between weighted Poincaré inequalities and the existence of weak solutions to a Neumann problem related to a degenerate p-Laplacian. The Poincaré inequalities are formulated in the context of degenerate Sobolev spaces defined in terms of a quadratic form, and the associated matrix is the source of the degeneracy in the p-Laplacian.

1 Introduction

In the study of regularity for elliptic PDEs, the existence of a Poincaré inequality plays a central role. For example, [MRW2] employs a Poincaré inequality to establish a Harnack inequality for a large class of non-linear degenerate elliptic equations with finite-type degeneracies. In many recent works, the existence of a suitable Poincaré inequality is either assumed or must be proved separately.

Given this, it is of interest to give a characterization of the existence of a Poincaré inequality. In this paper we show that this is equivalent to the existence of a regular solution of a Neumann boundary value problem for a degenerate p-Laplacian. We formulate our result in the very general setting of degenerate Sobolev spaces: elliptic operators have been considered in this setting by a number of authors; see, for instance, [CMN, CMR, MR, MRW1, MRW2, SW1, SW2] and the references they contain.

To state our main result we fix some definitions and notation that we will use throughout. For brevity we will defer some more technical definitions to Section 2. Let $\Omega \subset \mathbb{R}^n$ be a fixed domain, and let E be a bounded open set with $\overline{E} \subset \Omega$. Let $\mathcal{S}_n$ denote the collection of all positive, semi-definite $n \times n$ self-adjoint matrices; fix a function $Q\colon \Omega \to S_n$ whose entries are Lebesgue measurable and define the associated quadratic form $\mathcal{Q}(x,\xi) = \xi^t Q(x)\xi$, $x \in \Omega$ a.e. and $\xi \in \mathbb{R}^n$. We define

$$\gamma(x) = \big|Q(x)\big|_{\text{op}} = \sup_{|\xi|=1} |Q(x)\xi|,$$

to be the operator norm of $Q(x)$.

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Let v be a weight on Ω; *i.e.*, v is a non-negative function in $L^1_{\mathrm{loc}}(\Omega)$. Given a function f and a (bounded open) set E, we define the weighted average of f on E by

$$f_E = f_{E,v} = \frac{1}{v(E)} \int_E f(x)v(x)\,dx = ⨍_E f\,dv.$$

We can now give two definitions that are central to our main result. Note that the degenerate Sobolev space $\widetilde{H}^{1,p}_Q(v;E)$ and our precise definition of weak solutions is given in Section 2.

Definition 1.1 Given $1 \le p < \infty$, a quadratic from $\mathcal{Q}$ is said to have the Poincaré property of order p on E if there is a positive constant $C_p = C_p(E)$ such that for all $f \in C^1(\overline{E})$,

$$\text{(1.1)} \qquad \int_E |f(x) - f_E|^p v(x)\,dx \le C_p \int_E \Big|\sqrt{Q(x)}\nabla f(x)\Big|^p\,dx = C_p \int_E \mathcal{Q}(x, \nabla f(x))^{p/2}\,dx.$$

Definition 1.2 Given $1 \le p < \infty$, a quadratic form $\mathcal{Q}$ is said to have the p-Neumann property on E if the following hold:

(i) Given any $f \in L^p(v;E)$, there exists a weak solution $(u,\mathbf{g})_f \in \widetilde{H}^{1,p}_Q(v;E)$ to the weighted homogeneous Neumann problem

$$\text{(1.2)} \qquad \begin{cases} \operatorname{div}\Big(\Big|\sqrt{Q(x)}\nabla u(x)\Big|^{p-2} Q(x)\nabla u(x)\Big) = |f(x)|^{p-2} f(x)v(x) \text{ in } E \\ \mathbf{n}^t \cdot Q(x)\nabla u = 0 \text{ on } \partial E, \end{cases}$$

where $\mathbf{n}$ is the outward unit normal vector of ∂E.

(ii) Any weak solution $(u,\mathbf{g})_f \in \widetilde{H}^{1,p}_Q(E)$ of (1.2) is regular; that is, there is a positive constant $D_p = D_p(v,E)$ such that

$$\text{(1.3)} \qquad \|u\|_{L^p(v;E)} \le C\|f\|_{L^p(v;E)}.$$

Our main result shows that given very weak assumptions on the matrix Q, these two properties are equivalent.

Theorem 1.3 *Given $1 < p < \infty$, suppose that $\gamma^{p/2} \in L^1_{\mathrm{loc}}(E)$. Then the quadratic form $\mathcal{Q}(x,\cdot)$ is p-Neumann on E if and only if $\mathcal{Q}(x,\cdot)$ has the Poincaré property of order p on E.*

Remark 1.4 The regularity of the weak solution $(u,\mathbf{g})_f \in \widetilde{H}^{1,p}_Q(v;E)$ of the Neumann problem (1.2) can also be characterized by the seemingly stronger estimate

$$\text{(1.4)} \qquad \|(u,\mathbf{g})_f\|_{H^{1,p}_Q(v;E)} \le D_p\|f\|_{L^p(v;E)}$$

for a positive constant C independent of u and f. The equivalence of (1.3) and (1.4) arises as part of the proof of Theorem 1.3; see Lemma 3.1.

Remark 1.5 Implicit in Definition 1.2 appears to be the assumption that ∂E is sufficiently regular that the normal derivative exists almost everywhere. This, however, is not the case; see the discussion in Remark 2.10 following the precise definition of a weak solution to the Neumann problem.

Remark 1.6 The assumption that $\gamma^{p/2} \in L^1_{\mathrm{loc}}(E)$ is a technical one because we are dealing with matrix weighted spaces. It guarantees that the matrix weighted spaces we are dealing with are well-behaved Banach spaces: see Lemmas 2.1 and 2.4.

The remainder of the paper is organized as follows. In Section 2 we define and prove the completeness of the matrix weighted space $\mathcal{L}^p_Q(\Omega)$, give the important properties of the degenerate Sobolev spaces associated with the quadratic form $\mathcal{Q}(x,\xi)$, and define a weak solution of the Neumann boundary problem. In Sections 3 and 4 we give the proof of Theorem 1.3; each section contains the proof of one implication. Finally, in Section 5, we give several applications of Theorem 1.3. The first is a model example where we deduce the classical Poincaré inequality in the plane; it is of interest, because this proof is accessible to undergraduates. The second gives the solution of degenerate p-Laplacians, where the degeneracy is controlled by a Muckenhoupt A_p weight. These problems are analogous to the Dirichlet problems considered by Fabes, Kenig, and Serapioni [FKS] and Modica [M]. The third considers solutions to degenerate p-Laplacians where the least eigenvalue of the matrix Q vanishes at the origin as a large power of $|x|$. These examples are obtained by considering two weight Poincaré inequalities. We conclude this example by discussing briefly the application of recent work on two-weight norm inequalities for the fractional integral operator to prove degenerate Poincaré inequalities.

2 Preliminaries

In this section, we bring together the basic definitions of the objects (Sobolev spaces, weak solutions, etc.) used in the statement and proof of Theorem 1.3.

Let $Q\colon \Omega \to \mathcal{S}_n$ be a matrix-valued function whose entries are Lebesgue measurable functions. By [RS, Lemma 2.3.1] there exists a measurable unitary matrix function $U(x)$ on Ω that diagonalizes $Q(x)$; that is,

$$Q(x) = U^t(x)D(x)U(x) \text{ a.e.},$$

where $D(x) = \operatorname{diag}(\lambda_1(x), \dots, \lambda_n(x))$ is a diagonal matrix with measurable functions on the diagonal. With a fixed choice for $U(x)$, we can define positive powers of $Q(x)$: given $r > 0$, we set

$$Q^r(x) = U^t(x)D^r(x)U(x), \text{ where } D^r(x) = \operatorname{diag}\big(\lambda_1^r(x), \dots, \lambda_n^r(x)\big).$$

Sobolev Spaces

We define our underlying degenerate Sobolev spaces as the completions of C^1 functions with respect to norms related to the quadratic form $\mathcal{Q}(x,\xi)$. For $1 \le p < \infty$ define $\mathcal{L}^p_Q(E)$ to be the collection of all measurable $\mathbb{R}^n$ valued functions $\mathbf{f} = (f_1, \dots, f_n)$ that satisfy

$$\|\mathbf{f}\|_{\mathcal{L}^p_Q(E)} = \Big(\int_E \mathcal{Q}(x,\mathbf{f}(x))^{p/2}\,dx\Big)^{1/p} = \Big(\int_E \Big|\sqrt{Q(x)}\mathbf{f}(x)\Big|^p\,dx\Big)^{1/p} < \infty. \tag{2.1}$$

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More properly we define $\mathcal{L}^p_Q(E)$ to be the normed vector space of equivalence classes under the equivalence relation

$$\mathbf{f} \equiv \mathbf{g} \quad \text{iff} \quad \|\mathbf{f} - \mathbf{g}\|_{\mathcal{L}^p_Q(E)} = 0.$$

Note that if $\mathbf{f}(x) = \mathbf{g}(x)$ a.e., then $\mathbf{f} \equiv \mathbf{g}$, but the converse need not be true, depending on the degeneracy of Q.

In [SW2, chapter 3], the space $\mathcal{L}^2_Q(E)$ was shown to a Hilbert space whenever $|Q(x)|_{\mathrm{op}} \in L^1_{\mathrm{loc}}(E)$. Here we generalize this result by showing that $\mathcal{L}^p_Q(E)$ is complete for all $1 \le p < \infty$.

Lemma 2.1 *Given $1 \le p < \infty$ and a measurable matrix function $Q\colon E \to \mathcal{S}_n$, if $\gamma^{p/2} \in L^1_{\mathrm{loc}}(E)$, then $\mathcal{L}^p_Q(E)$ is a separable Banach space. If $p > 1$, then it is reflexive.*

Proof As above, denote by $\lambda_1(x), \dots, \lambda_n(x)$ the measurable eigenvalues of $Q(x)$. Fix a measurable function $\mathbf{f}\colon E \to \mathbb{R}^n$. We can now argue as in [SW2, Remark 5]. Choose measurable unit eigenvectors $v_j(x)$, $j = 1 \dots n$, and write $\mathbf{f}$ as

$$\mathbf{f}(x) = \sum_{i=1}^{n} \widetilde{f}_j(x) v_j(x),$$

where $\widetilde{f}_j$ is the j-th component of $\mathbf{f}$ with respect to the basis $\{v_j\}$. Since eigenvectors are orthogonal, the action of our quadratic form on $\mathbf{f}$ can be written as

$$\mathcal{Q}(\mathbf{f}(x), x) = \Big(\sum_{j=1}^{n} \widetilde{f}_j(x) v_j(x) \Big) \cdot \Big(\sum_{j=1}^{n} Q(x) \widetilde{f}_j(x) v_j(x) \Big) = \sum_{j=1}^{n} |\widetilde{f}_j(x)|^2 \lambda_j(x).$$

Using this we can rewrite the norm (2.1) as an equivalent sum of weighted norms:

$$\|\mathbf{f}\|_{\mathcal{L}^p_Q(E)} \approx \sum_{j=1}^{n} \Big(\int_E |\widetilde{f}_j(x)|^p \lambda_j(x)^{p/2}\, dx \Big)^{1/p} = \sum_{j=1}^{n} \|\widetilde{f}_j\|_{L^p(\lambda_j^{p/2}; E)}.$$

Since $\lambda_j^{p/2}(x) \le \gamma(x)^{p/2} \in L^1_{\mathrm{loc}}(E)$ a.e., the spaces $L^p(\lambda_j^{p/2}; E)$ are separable Banach spaces that are reflexive if $p > 1$, and so $\mathcal{L}^p_Q(E)$ is as well. ∎

We now define the corresponding degenerate Sobolev spaces as collections of equivalence classes of Cauchy sequences of $C^1(\overline{E})$ functions.

Definition 2.2 For $1 \le p < \infty$, the Sobolev space $H^{1,p}_Q(v; E)$ is the abstract completion of $C^1(\overline{E})$ with respect to the norm

$$\|f\|_{H^{1,p}_Q(E)} = \|f\|_{L^p(v;E)} + \|\nabla f\|_{\mathcal{L}^p_Q(E)}. \tag{2.2}$$

Because of the degeneracy of Q, we cannot represent $H^{1,p}_Q(v; E)$ as a space of functions except in special situations. Since $L^p(v; E)$ and $\mathcal{L}^p_Q(E)$ are complete, given an equivalence class of $H^{1,p}_Q(v; E)$ there exists a unique pair

$$\vec{\mathbf{f}} = (f, \mathbf{g}) \in L^p(v; E) \times \mathcal{L}^p_Q(E)$$

that we can use to represent it. Such pairs are unique and so we refer to elements of $H^{1,p}_Q(v; E)$ using their representative pair. However, because of the famous example of Fabes, Kenig, and Serapioni in [FKS], the vector function $\mathbf{g}$ need not be uniquely determined by the first component f of the pair. If we think of $\mathbf{g}$ as the "gradient" of f, then we have that there exist non-constant functions f whose gradient is 0. Nevertheless, if we consider constant sequences we see that the pair $\vec{\mathbf{f}} = (f, \nabla f)$ is in $H^{1,p}_Q(v;E)$ whenever $f \in C^1(\overline{E}) \cap H^{1,p}_Q(v;E)$ or, more simply, if $f \in C^1(\overline{E})$ and $\gamma^{p/2} \in L^1(E)$. We refer the interested reader to [CMN, CMR, CRW, MR, MRW1, MRW2, SW1, SW2] for definitions and discussions of these and similar spaces, including examples where the gradient is uniquely defined.

The next lemma follows from the above discussion and Lemma 2.1.

Lemma 2.3 *For $1 \le p < \infty$, $H^{1,p}_Q(v;E)$ is a separable Banach space; if $p > 1$ it is reflexive.*

Since we are considering Neuman boundary problems and Poincaré estimates, it is important to restrict our attention to the "mean-zero" subspace of $H^{1,p}_Q(v;E)$. More precisely, we define

$$\widetilde{H}^{1,p}_Q(v;E) = \Big\{ (u,\mathbf{g}) \in H^{1,p}_Q(v;E) : \int_E u(x)v(x)\,dx = 0 \Big\}.$$

Lemma 2.4 *Given a measurable matrix function $Q\colon E \to \mathcal{S}_n$ and $1 \le p < \infty$, if $\gamma^{p/2} \in L^1_{\mathrm{loc}}(\Omega)$, then $\widetilde{H}^{1,p}_Q(v;E)$ is a separable Banach space. If $p > 1$, then it is reflexive.*

Proof By Lemma 2.3, it will suffice to show that $\widetilde{H}^{1,p}_Q(v;E)$ is a closed subspace of $H^{1,p}_Q(v;E)$. We first consider the case $p > 1$. Let $\{(u_j, \mathbf{g}_j)\}$ be a Cauchy sequence in $\widetilde{H}^{1,p}_Q(v;E)$. By the completeness of $H^{1,p}_Q(v;E)$, there is an element $(u,\mathbf{g}) \in H^{1,p}_Q(v;E)$ such that $u_j \to u$ in $L^p(v;E)$ and $\mathbf{g}_j \to \mathbf{g}$ in $\mathcal{L}^p_Q(E)$. Moreover, we have that

$$\begin{aligned} \Big| \int_E u(x)v(x)dx \Big| &= \Big| \int_E (u(x) - u_j(x))v(x)\,dx \Big| \\ &\le \int_E |u(x) - u_j(x)| v^{1/p}(x) v^{1/p'}(x)\,dx \\ &\le v(E)^{1/p'} \|u - u_j\|_{L^p(v;E)}. \end{aligned}$$

Since v is locally integrable, $v(E) < \infty$. Thus, since the last term on the right goes to zero as $j \to \infty$, we conclude that $\int_E u(x)v(x)\,dx = 0$, so $(u,\mathbf{g}) \in \widetilde{H}^{1,p}_Q(v;E)$.

The case $p = 1$ is similar and is left to the reader. ∎

Below, we will need the following density result.

Lemma 2.5 *Given a measurable matrix function $Q\colon E \to \mathcal{S}_n$ and $1 \le p < \infty$, the set $C^1(\overline{E}) \cap \widetilde{H}^{1,p}_Q(v;E)$ is dense in $\widetilde{H}^{1,p}_Q(v;E)$.*

Proof Fix $(u,\mathbf{g}) \in \widetilde{H}^{1,p}_Q(v;E)$. Since $C^1(\overline{E})$ is dense in $\widetilde{H}^{1,p}_Q(v;E) \subset H^{1,p}_Q(v;E)$, there exists a sequence of functions $u_k \in C^1(\overline{E})$ such that $(u_k, \nabla u_k)$ converges to $(u,\mathbf{g})$. Let $v_k = u_k - (u_k)_E \in C^1(\overline{E}) \cap \widetilde{H}^{1,p}_Q(v;E)$. Then $\nabla v_k = \nabla u_k$, so to prove that $(v_k, \nabla v_k)$ converges to $(u,\mathbf{g})$ it will suffice to prove that $u_k - v_k$ converges to 0

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in $L^p(v;E)$. Since $u_E = 0$, we have that

$$\|u_k - v_k\|_{L^p(v;E)} = |(u_k)_E - u_E| v(E)^{1/p} \le v(E)^{-1/p'} \int_E |u_k - u| v\,dx \le \|u_k - u\|_{L^p(v;E)},$$

and since the right-hand term goes to 0, we get the desired convergence. ∎

Remark 2.6 The space $H_Q^{1,p}(1;E)$ is not in general the same as $W_Q^{1,p}(E)$, defined in [MRW1, MRW2] as the completion with respect to the norm (2.2) (with $v = 1$) of $\mathrm{Lip}_{\mathrm{loc}}(E)$, unless E has some additional boundary regularity. The next result, an amalgam of [CMR, Theorems 5.3, 5.6], gives an example where these spaces coincide; it also requires further regularity on the matrix Q in terms of the matrix $\mathcal{A}_p$ condition and we refer the reader to [CMR] for complete definitions.

Theorem 2.7 *Let $E \subset \mathbb{R}^n$ be a domain whose boundary is locally a Lipschitz graph. If $1 \le p < \infty$, $W = Q^{p/2}$ is a matrix $\mathcal{A}_p$ weight and $v = |W|_{op} = \gamma^{p/2}$, then $H_Q^{1,p}(v;E) = \mathscr{H}_W^{1,p}(E)$, where $\mathscr{H}_W^{1,p}(E)$ is the completion of $C^\infty(E)$ with respect to the norm*

$$\|f\|_{\mathscr{H}_W^{1,p}} = \|f\|_{L^p(\gamma;E)} + \Big(\int_E |W^{1/p}\nabla f|^p\,dx\Big)^{1/p}.$$

Weak Solutions

We can now define a weak solution to the degenerate p-Laplacian in Definition 1.2. Given $f \in L^p(v;E)$, we say that a pair $(u,\mathbf{g})_f \in \widetilde{H}_Q^{1,p}(v;E)$ is a weak solution to the weighted homogeneous Neumann problem

$$\begin{cases} \operatorname{div}\Big(\Big|\sqrt{Q(x)}\nabla u(x)\Big|^{p-2} Q(x)\nabla u(x)\Big) = |f(x)|^{p-2} f(x) v(x) \text{ in } E \\ \mathbf{n}^t \cdot Q(x)\nabla u = 0 \text{ on } \partial E, \end{cases} \tag{2.3}$$

if for all test functions $\varphi \in C^1(\overline{E}) \cap \widetilde{H}_Q^{1,p}(v;E)$,

$$\int_E \big|\sqrt{Q(x)}\mathbf{g}(x)\big|^{p-2} (\nabla\varphi)^t Q(x)\mathbf{g}(x)\,dx = -\int_E |f(x)|^{p-2} f(x)\varphi(x)v(x)\,dx. \tag{2.4}$$

With our assumptions we have *a priori* that both sides of (2.4) are finite. Since $\mathbf{g} \in \mathcal{L}_Q^p(E)$ and $\varphi \in C^1(\overline{E})$, by Hölder's inequality we get

$$\begin{aligned} \Big|\int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-2} (\nabla\varphi)^t Q(x)\mathbf{g}(x)\,dx\Big| &\le \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-1} |\sqrt{Q}(x)\nabla\varphi|\,dx \\ &\le \|\mathbf{g}\|_{\mathcal{L}_Q^p(E)}^{p-1} \|\nabla\varphi\|_{\mathcal{L}_Q^p(E)} < \infty; \end{aligned} \tag{2.5}$$

similarly, since $f \in L^p(v;E)$,

$$\begin{aligned} \Big|\int_E |f(x)|^{p-2} f(x)\varphi(x)v(x)\,dx\Big| &\le \int_E |f(x)|^{p-1} v^{1/p'}(x) |\varphi(x)| v^{1/p}(x)\,dx \\ &\le \|f\|_{L^p(v;E)}^{p-1} \|\varphi\|_{L^p(v;E)} < \infty. \end{aligned}$$

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Remark 2.8 If $(u,\mathbf{g})_f$ is a weak solution of (2.3), then by an approximation argument, (2.4) also holds for all test "functions" $(w,\mathbf{h}) \in \widetilde{H}^{1,p}_Q(v;E)$. That is, the relation holds if we replace φ with w and $\nabla\varphi$ with $\mathbf{h}$.

Remark 2.9 Our definition of a weak solution to the Neumann problem (2.3) is somewhat different from that used in the classical setting (*i.e.*, $v = 1$, $Q = \mathrm{Id}$); we do not impose a compatibility condition on the initial data f, and instead require that our test functions have mean zero. Thus our definition of a weak solution is weaker than the one used in the classical setting. Our definition is motivated by the connection with the Poincaré inequality, as will be clear from the proof of Theorem 1.3.

Remark 2.10 The definition of a weak solution of equation (2.3) actually makes no assumptions on the regularity of the boundary of the set E. It is the case, however, that if ∂E is such that its normal vector $\mathbf{n}$ exists almost everywhere and the Divergence Theorem holds, then our definition is equivalent to assuming that $\mathbf{n}^t \cdot Q(x)\nabla u = 0$ almost everywhere.

3 The Proof of Theorem 1.3: p-Neumann Implies p-Poincaré

We begin by proving the alternate characterization of the regularity of a weak solution mentioned in Remark 1.4.

Lemma 3.1 *Given $1 < p < \infty$ and $f \in L^p(v;E)$, if $(u,\mathbf{g})_f \in \widetilde{H}^{1,p}_Q(v;E)$ is a weak solution of the Neumann problem (2.3), then $\|(u,\mathbf{g})\|_{H^{1,p}_Q(v;E)} \lesssim \|f\|_{L^p(v;E)}$ if and only if $\|u\|_{L^p(v;E)} \lesssim \|f\|_{L^p(v;E)}$.*

Proof Since

$$\|u\|_{L^p(v;E)} \le \|(u,\mathbf{g})\|_{H^{1,p}_Q(v;E)},$$

one direction is immediate. To prove the converse, suppose $(u,\mathbf{g}) \in \widetilde{H}^{1,p}_Q(v;E)$ is a weak solution of (2.3) that satisfies $\|u\|_{L^p(v;E)} \lesssim \|f\|_{L^p(v;E)}$. Then by Remark 2.8 we can take the pair $(u,\mathbf{g})$ as our test "function" in (2.4) to get

$$\begin{aligned}\|\mathbf{g}\|^p_{\mathcal{L}^p_Q(E)} &= \int_E \big|\sqrt{Q}\mathbf{g}\big|^{p-2} Q\mathbf{g}\cdot\mathbf{g}\,dx\\ &\le \int_E |f|^{p-1}|u|v\,dx \le \|f\|^{p-1}_{L^p(v;E)}\|u\|_{L^p(v;E)} \le C\|f\|^p_{L^p(v;E)}.\end{aligned}$$

Hence,

$$\|(u,\mathbf{g})\|_{H^{1,p}_Q(v;E)} = \|u\|_{L^p(v;E)} + \|\mathbf{g}\|_{\mathcal{L}^p_Q(E)} \le C\|f\|_{L^p(v;E)}. \qquad \blacksquare$$

The proof that the p-Neumann property implies the p-Poincaré property essentially follows from Lemma 3.1. Suppose the quadratic form $\mathcal{Q}(x,\xi)$ is p-Neumann on E. Fix $f \in C^1(\overline{E})$, $\|f\|_{L^p(v;E)} \ne 0$ and assume for the moment that

$$f_E = \int_E f(x)v(x)\,dx = 0.$$

Then we need to prove that $\|f\|_{L^p(v;E)} \lesssim \|\nabla f\|_{\mathcal{L}^p_Q(E)}$.

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Let $(u, \mathbf{g})_f \in \widetilde{H}^{1,p}_Q(v; E)$ be a weak solution of (2.3) corresponding to this function f. Since f itself is a valid test function, we can apply the definition of a weak solution, estimate exactly as in (2.5), and then apply Lemma 3.1, to get

$$\begin{aligned}\|f\|^p_{L^p(v;E)} &= \Big| -\int_E |f(x)|^{p-2} f(x) f(x) v(x)\, dx \Big| \\ &= \Big| \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-2} (\nabla f(x))^t Q(x) \mathbf{g}(x)\, dx \Big| \\ &\leq \|\mathbf{g}\|^{p-1}_{\mathcal{L}^p_Q(E)} \|\nabla f\|_{\mathcal{L}^p_Q(E)} \leq C \|f\|^{p-1}_{L^p(v;E)} \|\nabla f\|_{\mathcal{L}^p_Q(E)}.\end{aligned}$$

Since $\|f\|^{p-1}_{L^p(v;E)} \neq 0$, we can divide by this quantity to get the desired inequality.

To complete the proof, fix an arbitrary $f \in C^1(\overline{E})$ and let $k = f - f_E$. Then k has mean zero and $\nabla k = \nabla f$, so applying the previous argument to k yields the desired Poincaré inequality. ■

4 Proof of Theorem 1.3: p-Poincaré Implies p-Neumann

Assume that Q has the Poincaré property of order p on E. Fix an arbitrary function $f \in L^p(v; E)$; we will show that there exists a weak solution $(u, \mathbf{g})_f \in \widetilde{H}^{1,p}_Q(v; E)$ to the Neumann problem (1.2) and that it satisfies the regularity estimate (1.3).

The first step is to show that a solution exists. We will do so by using Minty's theorem [S], which can be thought of as a Banach space version of the classical Lax-Milgram theorem. To state this result we first introduce some notation. Given a reflexive Banach space $\mathcal{B}$ denote its dual space by $\mathcal{B}^*$. Given a functional $\alpha \in \mathcal{B}^*$, write its value at $\varphi \in \mathcal{B}$ as $\alpha(\varphi) = \langle \alpha, \varphi \rangle$. Thus, if $\beta: \mathcal{B} \to \mathcal{B}^*$ and $u \in \mathcal{B}$, then we have $\beta(u) \in \mathcal{B}^*$ and so its value at φ is denoted by $\beta(u)(\varphi) = \langle \beta(u), \varphi \rangle$.

Theorem 4.1 *Let $\mathcal{B}$ be a reflexive, separable Banach space and fix $\Gamma \in \mathcal{B}^*$. Suppose that $\mathcal{T}: \mathcal{B} \to \mathcal{B}^*$ is a bounded operator that has the following properties:*

(i) Monotone*: $\langle \mathcal{T}(u) - \mathcal{T}(\varphi), u - \varphi \rangle \geq 0$ for all $u, \varphi \in \mathcal{B}$;*
(ii) Hemicontinuous*: for $z \in \mathbb{R}$, the mapping $z \to \langle \mathcal{T}(u + z\varphi), \varphi \rangle$ is continuous for all $u, \varphi \in \mathcal{B}$;*
(iii) Almost Coercive*: there exists a constant $\lambda > 0$ so that $\langle \mathcal{T}(u), u \rangle > \langle \Gamma, u \rangle$ for any $u \in \mathcal{B}$ satisfying $\|u\|_{\mathcal{B}} > \lambda$.*

Then the set of $u \in \mathcal{B}$ such that $\mathcal{T}(u) = \Gamma$ is non-empty.

To apply Minty's theorem to find a weak solution, let $\mathcal{B} = \widetilde{H}^{1,p}_Q(v; E)$; if $p > 1$ it is a reflexive, separable Banach space by Lemma 2.4. Given $\vec{\mathbf{u}} = (u, \mathbf{g})$ and $\vec{\mathbf{w}} = (w, \mathbf{h})$ in $\widetilde{H}^{1,p}_Q(v; E)$, define the operator $\mathcal{T}: \widetilde{H}^{1,p}_Q(v; E) \to (\widetilde{H}^{1,p}_Q(v; E))^*$ by

$$\langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{w}} \rangle = \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-2} \mathbf{h}^t(x) Q(x) \mathbf{g}(x)\, dx.$$

Now define $\Gamma \in \mathcal{B}^*$ by

$$\Gamma(\vec{\mathbf{w}}) = \langle \Gamma, \vec{\mathbf{w}} \rangle = -\int_E |f(x)|^{p-2} f(x) w(x) v(x)\, dx;$$

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by Hölder's inequality we have that $\Gamma \in \left(H_Q^{1,p}(v;E)\right)^*$ whenever $f \in L^p(v;E)$. Then $\vec{\mathbf{u}} = (u, \mathbf{g})$ is a weak solution of (1.2) if and only if

$$\left\langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{w}} \right\rangle = \langle \Gamma, \vec{\mathbf{w}} \rangle$$

for all $\vec{\mathbf{w}} \in \widetilde{H}_Q^{1,p}(v;E)$. Assume for the moment that $\mathcal{T}$ is a bounded, monotone, hemicontinuous, almost coercive operator. Then we have proved the following result.

Theorem 4.2 *Suppose $1 < p < \infty$, $\gamma^{p/2} \in L^1_{\text{loc}}(E)$ and that Ω has the Poincaré property of order p on E. If $f \in L^p(v;E)$, then there is a weak solution $\vec{\mathbf{u}} = (u, \mathbf{g})_f \in \widetilde{H}_Q^{1,p}(v;E)$ to the Neumann problem* (1.2).

Remark 4.3 Theorem 4.2 complements [CMN, Theorem 3.14] where they proved the existence of weak solutions to the corresponding Dirichlet problem.

Before establishing that the hypotheses of Theorem 4.1 hold, we will first complete the proof of Theorem 1.3 by showing that inequality (1.3) is a consequence of the Poincaré inequality. Fix $f \in C^1(\overline{E})$ and let $\vec{\mathbf{u}} = (u, \mathbf{g})_f \in \widetilde{H}_Q^{1,p}(v;E)$ be a corresponding weak solution of (1.2). First note that $(u, \mathbf{g})_f$ satisfies the Poincaré inequality $\|u\|_{L^p(v;E)} \lesssim \|\mathbf{g}\|_{\mathcal{L}_Q^p(E)}$. This follows from Lemma 2.5, since we can apply the Poincaré inequality to a sequence of mean zero functions in $C^1(\overline{E})$ that approximate $(u, \mathbf{g})_f$.

Given this inequality, we can now argue as follows; by the definition of a weak solution and Hölder's inequality,

$$\begin{aligned}\|u\|^p_{L^p(v;E)} \leq C\|\mathbf{g}\|^p_{\mathcal{L}_Q^p(E)} &= C\int_E |\sqrt{Q}\mathbf{g}|^{p-2}(\mathbf{g})^t Q\mathbf{g}\,dx \\ &\leq C\Big|\int_E |f|^{p-2} fuv\,dx\Big| \leq C\|u\|_{L^p(v;E)}\|f\|^{p-1}_{L^p(v;E)}.\end{aligned}$$

Rearranging terms, we get

$$\|u\|_{L^p(v;E)} \leq C\|f\|_{L^p(v;E)},$$

which is (1.3), the desired estimate.

Finally, in the next four lemmas we verify all hypotheses of Minty's theorem for the operator $\mathcal{T}$.

Lemma 4.4 *For $1 \leq p < \infty$, the operator $\mathcal{T}$ is bounded on $\widetilde{H}_Q^{1,p}(v;E)$.*

Proof With the same notation as before, fix $\vec{\mathbf{u}}, \vec{\mathbf{w}} \in \widetilde{H}_Q^{1,p}(v;E)$. By the Cauchy–Schwarz inequality,

$$\left|\eta \cdot Q(x)\xi\right| \leq \left|\sqrt{Q(x)}\eta\right| \left|\sqrt{Q(x)}\xi\right|,$$

and so by Hölder's inequality,

$$\left|\langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{w}}\rangle\right| \leq \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-1}|\sqrt{Q(x)}\mathbf{h}(x)|\,dx \leq \|\vec{\mathbf{u}}\|^{p-1}_{H_Q^{1,p}(v;E)}\|\vec{\mathbf{w}}\|_{H_Q^{1,p}(v;E)}.$$

It follows at once that T is bounded. ■

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Lemma 4.5 *For $1 \le p < \infty$, the operator $\mathcal{T}$ is monotone.*

Proof Fix $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ as before. Then we have that

$$\begin{aligned}
&\big\langle \mathcal{T}(\vec{\mathbf{u}}) - \mathcal{T}(\vec{\mathbf{w}}), \vec{\mathbf{u}} - \vec{\mathbf{w}} \big\rangle \\
&\quad = \langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{u}} - \vec{\mathbf{w}} \rangle - \langle \mathcal{T}(\vec{\mathbf{w}}), \vec{\mathbf{u}} - \vec{\mathbf{w}} \rangle \\
&\quad = \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-2} (\mathbf{g}(x) - \mathbf{h}(x))^t Q(x) \mathbf{g}(x)\, dx \\
&\qquad - \int_E |\sqrt{Q(x)}\mathbf{h}(x)|^{p-2} (\mathbf{g}(x) - \mathbf{h}(x))^t Q(x) \mathbf{h}(x)\, dx \\
&\quad = \int_E \big\langle |\sqrt{Q}\mathbf{g}|^{p-2}\sqrt{Q}\mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p-2}\sqrt{Q}\mathbf{h},\ \sqrt{Q}\mathbf{g} - \sqrt{Q}\mathbf{h} \big\rangle_{\mathbb{R}^n}\, dx,
\end{aligned}$$

where we suppress dependence on x in the last line and where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is the standard inner product on $\mathbb{R}^n$. The last integrand is of the form

$$\langle |s|^{p-2}s - |r|^{p-2}r, s - r \rangle_{\mathbb{R}^n},$$

where $s, r \in \mathbb{R}^n$, and $p \ge 1$. For such p, s, r an inequality in [L, chapter 10] (also found in [MDJ, §1.2, lemma 1]) shows that this expression is non-negative. Hence, $\mathcal{T}$ is monotone. ■

Lemma 4.6 *For $1 < p < \infty$, the operator $\mathcal{T}$ is hemicontinuous.*

Proof Let $z, y \in \mathbb{R}$ and let $\vec{\mathbf{u}} = (u, \mathbf{g}), \vec{\mathbf{w}} = (w, \mathbf{h})$ be as before. To simplify our notation, set $\psi = \mathbf{g} + z\mathbf{h}$ and $\gamma = \mathbf{g} + y\mathbf{h}$. Then we have that

$$\begin{aligned}
(4.1)\qquad &\big\langle \mathcal{T}(\vec{\mathbf{u}} + z\vec{\mathbf{w}}) - \mathcal{T}(\vec{\mathbf{u}} + y\vec{\mathbf{w}}), \vec{\mathbf{w}} \big\rangle \\
&\quad = \int_E \big|\sqrt{Q}\psi\big|^{p-2} (\mathbf{h})^t Q\psi\, dx - \int_E \big|\sqrt{Q}\gamma\big|^{p-2} (\mathbf{h})^t Q\gamma\, dx \\
&\quad = \int_E \big(\sqrt{Q}\mathbf{h}\big)^t \big[|r|^{p-2}r - |s|^{p-2}s \big]\, dx,
\end{aligned}$$

where $r = \sqrt{Q}\psi$ and $s = \sqrt{Q}\gamma$.

We now consider two cases: $p \ge 2$ and $1 < p < 2$. If $p \ge 2$, then by [L, chapter 10] we have that for $r, s \in \mathbb{R}^n$,

$$\big| |r|^{p-2}r - |s|^{p-2}s \big| \le (p-1)|r - s| \big(|s|^{p-2} + |r|^{p-2} \big).$$

Furthermore, by our choice of r, s we have that $r - s = (z - y)\sqrt{Q}\mathbf{h}$; hence,

$$\|r - s\|_p \le |z - y| \|\vec{\mathbf{w}}\|_{H_Q^{1,p}(v;E)}.$$

If we combine these three inequalities, we get

$$\big| \langle \mathcal{T}(\vec{\mathbf{u}} + z\vec{\mathbf{w}}) - \mathcal{T}(\vec{\mathbf{u}} + y\vec{\mathbf{w}}), \vec{\mathbf{w}} \rangle \big| \le (p-1)|z - y| \int_E |\sqrt{Q}\mathbf{h}|^2 \big[|r|^{p-2} + |s|^{p-2} \big]\, dx.$$

If $p = 2$, it is clear that the integral is finite, so the right-hand side tends to zero as $z \to y$. If $p > 2$, then by Hölder's inequality with exponents $\frac{p}{2}$ and $\frac{p}{p-2} > 1$ we get

$$\int_E |\sqrt{Q}\mathbf{h}|^2 \big[|r|^{p-2} + |s|^{p-2} \big]\, dx \lesssim \big(\|s\|_p^{p-2} + \|r\|_p^{p-2} \big) \|\mathbf{h}\|^2_{\mathcal{L}_Q^p(E)} < \infty,$$

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since $\vec{\mathbf{u}}, \vec{\mathbf{w}} \in \widetilde{H}^{1,p}_Q(v;E)$, $s = \sqrt{Q}\gamma$, and $r = \sqrt{Q}\psi \in L^p(E)$. So again the integral is finite and the right-hand size tends to zero as $z \to y$. Thus, $\mathcal{T}$ is hemicontinuous when $p \geq 2$.

Now suppose $1 < p < 2$. Then again from [L, chapter 10] we have that

$$\big||s|^{p-2}s - |r|^{p-2}r\big| \leq C(p)|s-r|^{p-1}$$

for $r, s \in \mathbb{R}^n$ and a positive constant $C(p)$. But then from this estimate, (4.1), and Hölder's inequality, we get

$$\begin{aligned}\big|\langle \mathcal{T}(\vec{\mathbf{u}} + z\vec{\mathbf{w}}) - \mathcal{T}(\vec{\mathbf{u}} + y\vec{\mathbf{w}}), \vec{\mathbf{w}}\rangle\big| &\leq C(p)\int_E |s-r|^{p-1}|\sqrt{Q}\mathbf{h}|\,dx \\ &\leq C(p)\|s-r\|_p^{p-1}\|\mathbf{h}\|_{\mathcal{L}^p_Q(E)} = C(p)|z-y|\|\mathbf{h}\|^p_{\mathcal{L}^p_Q(E)}.\end{aligned}$$

The final term tends to zero as $z \to y$, so $\mathcal{T}$ is hemicontinuous when $1 < p < 2$. ∎

Lemma 4.7 *Given $1 < p < \infty$, if $\mathcal{Q}$ has the Poincaré property of order $p > 1$ on E, then for any $f \in L^p(v;E)$ in the definition of Γ, $\mathcal{T}$ is almost coercive.*

Proof Fix a non-zero $\vec{\mathbf{u}} = (u, \mathbf{g}) \in \widetilde{H}^{1,p}_Q(v;E)$; then

$$\langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{u}}\rangle = \int_E |\sqrt{Q}\mathbf{g}|^{p-2}(\mathbf{g})^t Q\mathbf{g}\,dx = \|\mathbf{g}\|^p_{\mathcal{L}^p_Q(E)}. \tag{4.2}$$

Arguing as we did above using Lemma 2.5, we can apply the Poincaré inequality to $(u, \mathbf{g})$. Since $u_E = 0$, we get

$$\|u\|^p_{L^p(v;E)} \leq C_p^p\|\mathbf{g}\|^p_{\mathcal{L}^p_Q(E)} = C_p^p\langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{u}}\rangle,$$

which in turn implies that

$$\|\vec{\mathbf{u}}\|^p_{H^{1,p}_Q(v;E)} \leq (C_p^p + 1)\langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{u}}\rangle.$$

Since $p > 1$, by Hölder's inequality and this inequality, we have

$$\begin{aligned}\big|\langle \Gamma, \vec{\mathbf{u}}\rangle\big| = \Big| -\int_E |f|^{p-2}fuv\,dx\Big| &\leq \int_E |f|^{p-1}v^{\frac{1}{p'}}|u|v^{\frac{1}{p}}\,dx \\ &\leq \|f\|^{p-1}_{L^p(v;E)}\|\vec{\mathbf{u}}\|_{H^{1,p}_Q(v;E)} \\ &\leq (C_p^p + 1)\|f\|^{p-1}_{L^p(v;E)}\|\vec{\mathbf{u}}\|^{1-p}_{H^{1,p}_Q(v;E)}\langle T(\vec{\mathbf{u}}), \vec{\mathbf{u}}\rangle.\end{aligned}$$

Thus, if $f \not\equiv 0$,then $|\langle \Gamma, \vec{\mathbf{u}}\rangle| < \langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{u}}\rangle$ provided that

$$(C_p^p + 1)\|f\|^{p-1}_{L^p(v;E)}\|\vec{\mathbf{u}}\|^{1-p}_{H^{1,p}_Q(v;E)} < 1.$$

If $f \equiv 0$, then $\Gamma = 0 \in (H^{1,p}_Q(v;E))^*$. Hence, by (4.2), if we let $\lambda = 1$, then $|\langle \Gamma, \vec{\mathbf{u}}\rangle| < \langle \mathcal{T}(\vec{\mathbf{u}}), \vec{\mathbf{u}}\rangle$. Thus, we have shown that for any $f \in L^p(v;E)$, $\mathcal{T}$ is almost coercive with constant

$$\lambda = \max\left\{1, (C_p^p + 1)^{1/p-1}\|f\|_{L^p(v;E)}\right\}.$$ ∎

5 Applications of Theorem 1.3

In this section we give several applications of Theorem 1.3, primarily showing that the existence of the Poincaré inequality yields the existence of solutions to the corresponding Neumann problem.

An Undergraduate Problem

We begin, however, with an elementary application; we show that the solution of the Neumann problem on a rectangle in $\mathbb{R}^2$ can be used to deduce the existence of the Poincaré inequality. Nothing in this result is new, but we believe that it has a certain pedagogic value, as it lets us prove the Poincaré inequality using methods accessible to undergraduates.

Let $R = (0, a) \times (0, b)$ be a rectangle in $\mathbb{R}^2$, and for $f \in C^1(\overline{R})$ with $f_R = 0$, consider the Neumann problem for the Poisson equation on R:

$$\begin{cases} \Delta u = f & (x, y) \in R, \\ u_x(0, y) = 0 = u_x(a, y) & 0 < y < b, \\ u_y(x, 0) = 0 = u_y(x, b) & 0 < x < a. \end{cases}$$

We can find a classical solution to this problem using the eigenfunction expansion associated with the regular Sturm–Liouville problem $\Delta u = \lambda u$ with boundary values as above; equivalently, via the cosine expansion of f. We can write

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{mn} \cos\Big(\frac{n\pi x}{a}\Big) \cos\Big(\frac{m\pi y}{b}\Big),$$

where

$$F_{mn} = C(R) \iint_R f(x, y) \cos\Big(\frac{n\pi x}{a}\Big) \cos\Big(\frac{m\pi y}{b}\Big)\, dx dy$$

for $m, n \in \mathbb{N} \cup \{0\}$, $m + n > 0$, and $F_{00} = 0$ as $f_R = 0$. By Sturm–Liouville theory, this series expansion for $f(x, y)$ converges uniformly on $\overline{R}$. If we write

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} \cos\Big(\frac{n\pi x}{a}\Big) \cos\Big(\frac{m\pi y}{b}\Big)$$

and insert this into the equation $\Delta u = f$, we find that for $m + n > 0$,

$$A_{mn} = -\frac{F_{mn}}{\lambda_{mn}} \text{ where } \lambda_{mn} = \pi^2 \Big[\frac{n^2}{a^2} + \frac{m^2}{b^2}\Big].$$

We have that A_{00} is arbitrary, so choose $A_{00} = 0$ to ensure $u_R = 0$. Then the series expansions for $u, u_x, u_y, \Delta u$ each converge uniformly on $\overline{R}$, and the orthogonality of eigenfunctions implies that

$$\|u\|^2_{L^2(R)} = \sum_{\substack{n=0 \\ m+n>0}}^{\infty} \sum_{m=0}^{\infty} \Big(\frac{F_{mn}}{\lambda_{mn}}\Big)^2.$$

Since

$$|F_{mn}| \leq C(R) \|f\|_{L^2(R)} \quad \text{and} \quad \Lambda^2 = \sum_{\substack{n=0 \\ m+n>0}}^{\infty} \sum_{m=0}^{\infty} \lambda_{mn}^{-2} < \infty,$$

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we have that

$$\|u\|_{L^2(R)} \le C(R)\Lambda\|f\|_{L^2(R)}.$$

Hence, we can apply Theorem 1.3; more properly, we can apply the argument in Section 3, which in this case becomes completely elementary. Thus, there is a constant $C(R) > 0$ such that the 2-Poincaré inequality

$$\int_R |f - f_R|^2\,dx \le C(R)\int_R |\nabla f|^2\,dx$$

holds for any $f \in C^1(\overline{R})$.

One-weight Estimates for the Degenerate p-Laplacian

In this section we consider the degenerate p-Laplacian where the degeneracy is controlled by Muckenhoupt A_p weights. We sketch a few basic facts; for more information on weights, see [DUO]. For $1 < p < \infty$, a weight w satisfies the A_p condition, denoted $w \in A_p$, if

$$[w]_{A_p} = \sup_B \fint_B w\,dx\Big(\fint_B w^{1-p'}\,dx\Big)^{p-1} < \infty,$$

where the supremum is taken over all balls B. Given a domain E, the weighted Poincaré inequality is

$$\int_E |f(x) - f_E|^p w(x)\,dx \lesssim \int_E |\nabla f(x)|^p w(x)\,dx. \tag{5.1}$$

This inequality is known for quite general domains. It was first proved for balls in [FKS], and then for bounded domains satisfying the Boman chain condition in Chua [C]. (See this reference for precise definitions; domains that satisfy the Boman chain condition include Lipschitz domains. For this result, see also [DRS].)

Now fix $1 < p < \infty$ and suppose that $Q\colon \Omega \to \mathcal{S}_n$ satisfies the degenerate ellipticity condition

$$\lambda w(x)^{2/p}|\xi|^2 \le \xi^t \cdot Q(x)\xi \le \Lambda w(x)^{2/p}, \tag{5.2}$$

where $w \in A_p$ and $\Lambda > \lambda > 0$. For instance, we can take $Q(x) = w(x)^{2/p}A(x)$, where A is a uniformly elliptic matrix of measurable functions. Then

$$|\nabla f(x)|^p w(x) \approx |\sqrt{Q(x)}\nabla f(x)|^p$$

and so we see that (5.1) is equivalent to (1.1) with $v = w$. Therefore, we can apply Theorem 1.3 to get the following existence result.

Corollary 5.1 *Given $1 < p < \infty$ and a bounded domain E satisfying the Boman chain condition, suppose the matrix Q satisfies (5.2) for some $w \in A_p$. Then the associated quadratic form $\mathcal{Q}$ has the p-Neumann property on E;* i.e., *(1.2) has a solution for every $f \in L^p(w;E)$.*

Remark 5.2 Corollary 5.1 should be compared to [CMN], where the existence of solutions to the corresponding Dirichlet problem is shown. It would be interesting to determine if their regularity results extend to solutions of the Neumann problem.

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Two-weight Estimates

In this section we consider the Neumann problem for degenerate p-Laplacians where the degeneracy of the quadratic form $\mathcal{Q}$ is controlled by a pair of weights. We first consider the type of degeneracy studied in [CW, CMN].

Fix $p > 1$. Then a pair of weights (w, v) is a p-admissible pair if

(a) $v(x) \geq w(x)$, $x \in \mathbb{R}^n$ almost everywhere.
(b) $w \in A_p$ and v is doubling; *i.e.,* there exists $C > 0$ such that

$$v(B(x,2r)) \leq Cv(B(x,r)) \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0.$$

(c) There are positive constants C and $q > p$ such that given balls $B_1 = B(x,r)$, $B_2 = B(y,s)$, $B_1 \subset B_2$, w and v satisfy the balance condition

$$\frac{r}{s}\Big(\frac{v(B_1)}{v(B_2)}\Big)^{1/q} \leq C\Big(\frac{w(B_1)}{w(B_2)}\Big)^{1/p}.$$

It was shown in [CW, theorem 1.3] that given a p-admissible pair (w, v) and a ball B, the two-weight Poincaré inequality

$$\int_B |f(x) - f_{B_v}|^p v(x)\,dx \leq C(B)\int_B |\nabla f(x)|^p w(x)\,dx \tag{5.3}$$

holds for all $f \in C^1(\overline{B})$ with $C(B)$ independent of f.

Now suppose that $B \Subset \Omega$ and that our matrix function $Q(x)$ satisfies the ellipticity condition

$$w(x)|\xi|^p \leq |\sqrt{Q(x)}\xi|^p \tag{5.4}$$

for every $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. Then (1.1) holds, and we get the following corollary to Theorem 1.3.

Corollary 5.3 *Given $1 < p < \infty$ and a ball $B \subset \Omega$, suppose (w, v) is a p-admissible pair and the matrix Q satisfies (5.4). Then the associated quadratic form $\mathcal{Q}$ has the p-Neumann property on B;* i.e., *(1.2) has a solution for every $f \in L^p(v; B)$.*

We give a specific example of a matrix Q and weights (w, v) by adapting an example from [P]. Fix $n \geq 3$ and choose $p > 1$ so that $n > p'$. Let $\frac{p}{2} < s < p$ and define $w(x) = |x|^{p-s}$, $v(x) = |x|^s$. Since $0 < p - s < s < n(p-1)$, both $w, v \in A_p$ and so v is doubling. The balance condition is easily verified using relation [P, (28)] with $q \in (p, \frac{np}{n+s-p})$. Define $Q(x) = \operatorname{diag}(w(x), \dots, w(x), v(x))$, let $\Omega = B(0,1)$ and let $B \subset \Omega$. Then by Corollary 5.3 we can solve the degenerate p-Laplacian on B.

One drawback to the approach above using p-admissible pairs is that this hypothesis is stronger than we need. In [CW, Theorem 1.3] they actually proved that if (w, v) is a p-admissible pair for some $q > p$, then a two-weight Poincaré inequality with gain holds: for all $f \in C^1(\overline{B})$,

$$v(B)^{-1/q}\|f - f_B\|_{L^q(v;B)} \leq Cr(B)\,w(B)^{-1/p}\|\nabla f\|_{L^p(w;B)}.$$

We only need to assume a (presumably weaker) condition on the weights that implies the weighted (p,p) Poincaré inequality (5.3). By the well-known identity

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(see [GT, Lemma 7.16]), if E is a bounded, convex domain, then for any $f \in C^1(\overline{E})$ and $x \in E$,

$$|f(x) - f_\Omega| \le C(\Omega) I_1(|\nabla f|)(x), \tag{5.5}$$

where I_1 is the Riesz potential,

$$I_1 g(x) = \int_E \frac{f(y)}{|x-y|^{n-1}}\, dy.$$

(Here we assume $\operatorname{supp}(f) \subset E$.) Hence, to prove a two-weight Poincaré inequality, it suffices to find conditions on the weights (w, v) such that

$$I_1 : L^p(w; E) \longrightarrow L^p(v; E). \tag{5.6}$$

(We note in passing that the average in (5.5) is unweighted, but it is easy to pass to weighted averages; see [FKS, p. 88].)

There is an extensive literature on such two-weight norm inequalities for Riesz potentials, and we refer the reader to [CU, CMP] for complete information and references. In particular, we call attention to the so-called A_p bump conditions, as it is straightforward to construct examples of pairs of weights that satisfy these conditions. Here we restrict ourselves to noting that given a pair of weights (w, v) such that (5.6) holds, and given a matrix Q such that (5.4) holds, then we can immediately apply Theorem 1.3 to get the existence of solutions to the associated degenerate p-Laplacian.

Remark 5.4 After this paper was completed, a weaker sufficient condition on $\mathcal{Q}$ for the p-Poincaré property to hold on balls was found in [CIM, Theorem 1.23].

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