SUPERSOLUBLE IMMERSION

REINHOLD BAER

Supersoluble immersion of a normal subgroup K of a finite group G shall be defined by the following property:

If σ is a homomorphism of G, and if the minimal normal subgroup J of G^σ is part of K^σ, then J is cyclic (of order a prime).

Our principal aim in the present investigation is the proof of the equivalence of the following three properties of the normal subgroup K of the finite group G:

(i) K is supersolubly immersed in G.

(ii) $K/\phi K$ is supersolubly immersed in $G/\phi K$.

(iii) If θ is the group of automorphisms induced in the p-subgroup U of K by elements in the normalizer of U in G, then $\theta' \theta^{p-1}$ is a p-subgroup of θ.

Though most of our discussion is concerned with the proof of this theorem, some of our concepts and results are of independent interest. In § 1 we investigate groups G such that $G' G^{p-1}$ is a p-group. In § 2 some new and useful characterizations of supersoluble groups are obtained. In § 3 we substitute for supersoluble immersion the concept of a supersoluble pair which consists of a group G and a group θ of automorphisms of G meeting the following requirement:

If L is a θ-admissible normal subgroup of G, then every minimal θ-admissible normal subgroup of G/L is cyclic (of order a prime).

These supersoluble pairs are somewhat easier to handle than supersoluble immersion, though their investigation is, for all practical purposes, equivalent to that of supersoluble immersion.

Notations

G' = commutator subgroup of G.

ZG = centre of G.

ϕG = Frattini subgroup of G = intersection of all the maximal subgroups of G.

p-elements and p-groups are elements and groups of order a power of the prime p.

G^k = subgroup of G, generated by all the kth powers of elements in G.

G is a group of exponent e, if $G^e = 1$.

G is p-closed, if products of p-elements are p-elements.

If U is a subgroup of G, then NU is the normalizer and CU the centralizer of U in G.

Received October 14, 1958.

θ is an irreducible group of automorphisms of the group G, if 1 and G are the only θ-admissible subgroups of G.

All groups considered are *finite*.

0. We begin with a survey of the salient facts of the theory of finite supersoluble groups; for details cf. (**1**; **2**, § 11) and (**5**). A group G is termed *supersoluble*, if every epimorphic image, not 1, of G possesses a cyclic normal subgroup different from 1. This implies the apparently stronger fact that the minimal normal subgroups of the epimorphic images of supersoluble groups are cyclic of order a prime. Subgroups, epimorphic images, and direct products of supersoluble groups are likewise supersoluble. Extensions of supersoluble groups by supersoluble groups are, in general, not supersoluble; but extensions of cyclic groups by supersoluble groups and central extensions of supersoluble groups by supersoluble groups are supersoluble.

HUPPERT'S THEOREM: *The following three properties of G are equivalent: G is supersoluble; $G/\phi G$ is supersoluble; every maximal subgroup of G has index a prime.*

If G is supersoluble, then its commutator subgroup G' is nilpotent; and G has the

Sylow Tower Property of supersoluble groups: If H is an epimorphic image of G and p is a maximal prime divisor of the order of H, then the totality P of p-elements in H is a characteristic p-subgroup of H; in other words: H is p-closed.

1. In this section we are going to discuss a very special class of supersoluble groups which, however, will prove important in the sequel.

We recall that it is customary to term exponent of a group G the l.c.m. of the orders of the elements in G. For our purpose it will be more convenient to say that *G is a group of exponent e whenever $G^e = 1$*, that is, whenever e is some common multiple of the orders of the elements in G. If p is a prime, then the group G is termed *p-closed*, whenever products of elements of order a power of p are again elements of order a power of p. This is equivalent to requiring the existence of one and only one p-Sylow subgroup which is then a characteristic p-subgroup of index prime to p; and this characteristic p-subgroup of G shall be termed the p-component of G.

Definition. If the group G is p-closed, and if G/P is abelian of exponent $p - 1$, where P is the p-component of G, then G is strictly p-closed.

If G is strictly p-closed, then its commutator subgroup G' and the subgroup G^{p-1} generated by the $(p - 1)$th powers of elements in G are characteristic p-subgroups. If conversely G' and G^{p-1} are p-subgroups, then $G' G^{p-1}$ is a characteristic p-subgroup of G such that $G/G'G^{p-1}$ is abelian of exponent $p - 1$. Hence *G is strictly p-closed if, and only if, G' and G^{p-1} are p-subgroups of G.*

It is easy to verify that subgroups, epimorphic images, and direct products of strictly p-closed groups are again strictly p-closed. Likewise, extensions of p-groups by strictly p-closed groups are strictly p-closed.

Consider a strictly p-closed group G. Suppose that M is a minimal normal subgroup of an epimorphic image H of G. Then H is likewise strictly p-closed. Hence $H'H^{p-1}$ is a characteristic p-subgroup of H and at the same time the p-Sylow subgroup of H. If the order of M is prime to p, then

$$[M, H'H^{p-1}] \leqslant M \cap H'H^{p-1} = 1;$$

and if the order of M is divisible by p, then M is part of the p-component $H'H^{p-1}$ of H. Application of a well-known property of p-groups shows that in this case $M \cap Z(H'H^{p-1}) \neq 1$; and this implies $M \leqslant Z(H'H^{p-1})$ because of the minimality of M. Thus we have shown again that $[M, H'H^{p-1}] = 1$. Hence condition (viii) of (**2**, p. 184, Theorem 1) is satisfied by G; and this implies that *strictly p-closed groups are supersoluble.* This important fact has various consequences.

THEOREM 1.1. *G is a cyclic group of order p if, and only if, G is a p-group, not* 1, *possessing an irreducible and strictly p-closed group of automorphisms.*

Proof. If G is cyclic of order p, then its group of automorphisms is cyclic of order $p - 1$, proving the necessity of our condition. If conversely, $G \neq 1$ is a p-group and θ is an irreducible and strictly p-closed group of automorphisms of G, then we recall that G is a normal subgroup of its own holomorph and that we may form consequently the subgroup $G\theta$ of the holomorph of G. Since θ is irreducible, G is a minimal normal subgroup of $G\theta$. Since G is a p-group and θ is strictly p-closed, $G\theta$ is likewise strictly p-closed. Hence $G\theta$ is in particular supersoluble; and this implies that its minimal normal subgroup G is cyclic. Thus G is of order p.

For the convenience of the reader we insert here some well-known facts concerning automorphisms of p-groups.

LEMMA. *The automorphism σ of the p-group G is of order a power of p, if it satisfies one of the following conditions:*

(a) *σ induces the identity automorphism in $G/\phi G$ or*

(b) *there exists a σ-admissible normal chain of G in whose factors the identity automorphism is induced by σ.*

Proof. The sufficiency of condition (a) is contained in a result due to P. Hall (**4**, p. 38). Assume next the existence of σ-admissible subgroups $U(i)$ of G such that

$$1 = U(0),\ U(i) \text{ is a normal subgroup of } U(i+1),\ U(k) = G,$$

$$\sigma \text{ induces the identity automorphism in every } U(i+1)/U(i).$$

Then σ certainly induces a p-automorphism in $U(0)$. We may therefore make the inductive hypothesis that σ induces a p-automorphism in $U(i)$ for some

https://doi.org/10.4153/CJM-1959-036-2 Published online by Cambridge University Press

$i < k$. There exists consequently a positive integer n such that σ^{p^n} induces the identity automorphism in $U(i)$. Hence σ^{p^n} induces the identity automorphism both in $U(i)$ and in $U(i+1)/U(i)$. It is well known (and may be verified by a simple computation) that σ^{p^n} induces a p-automorphism in $U(i+1)$. Thus we have shown that σ induces a p-automorphism in $U(i+1)$, completing our inductive argument. Hence σ is a p-automorphism of $U(k) = G$.

THEOREM 1.2. *A group G with $ZG = 1$ is strictly p-closed if, and only if, maximal subgroups of G are either normal or else have index p in G.*

Proof. If G is strictly p-closed, then $G'G^{p-1}$ is a p-subgroup of G. If the maximal subgroup S of G is not normal, then in particular G' is not part of S. Hence $G = SG'$. Since G' is a p-subgroup, this implies that $[G{:}S]$ is a power of p. But strictly p-closed groups are supersoluble; and the maximal subgroups of supersoluble groups have index a prime. Hence $[G{:}S] = p$, proving the necessity of our condition.

Assume conversely the validity of our condition. Denote by P some p-Sylow subgroup of G and by NP its normalizer in G. If P were not normal, then $NP \neq G$ and there would exist a maximal subgroup S of G containing NP. From $P \leqslant NP \leqslant S$ we conclude that $[G{:}S]$ is prime to p; and this implies by hypothesis that S is a normal subgroup of G. Consequently S contains every p-Sylow subgroup of G as its own p-Sylow subgroup so that p-Sylow subgroups of G are conjugate in S. Application of the Frattini argument shows that $G = S \cdot NP = S < G$, a contradiction proving the normality of P and the p-closure of G. Since every maximal subgroup of G which contains P has index prime to p, these maximal subgroups are, by hypothesis, normal. Consequently every maximal subgroup of G/P is normal; and this implies by Wielandt's Theorem the nilpotency of G/P; see (**7**, p. 108, Satz 13). Application of Schur's Theorem shows the existence of a complement D to P in G, since $[G{:}P]$ is prime to p (**7**, p.125, Satz 25). Since $G/P \simeq D$, this subgroup D of G is nilpotent too. Every maximal subgroup of G has, by hypothesis, a prime index. Application of Huppert's Theorem shows the supersolubility of G; see (**5**, p. 416, Satz 9) or (**2**, p. 184, Theorem 1). Consider next normal subgroups A and B of G satisfying $A < B \leqslant P$ and $[B{:}A] = p$. Since G induces in the cyclic group B/A of order p a cyclic group of automorphisms whose order is a divisor of $p - 1$, it follows in particular that $[B, D'D^{p-1}] \leqslant A$. Since G has been shown to be supersoluble, there exist normal subgroups $A(i)$ of G such that

$$1 = A(0), A(i) < A(i+1), A(k) = P, \quad [A(i+1){:}A(i)] = p.$$

From what we have shown just now it follows that $[A(i+1), D'D^{p-1}] \leqslant A(i)$ for every i.

In other words: every element in $D'D^{p-1}$ induces an automorphism in P which in turn induces the identity automorphism in every $A(i+1)/A(i)$. By Lemma (b) such an automorphism has order a power of p. Consequently every element in $D'D^{p-1}$ induces the identity automorphism in P. If $D'D^{p-1}$ were not 1, then we would deduce $D'D^{p-1} \cap ZD \neq 1$ from the nilpotency of D. Elements in $D'D^{p-1} \cap ZD$ commute with every element in P and every element in D; and they belong consequently to the centre of $PD = G$. But $ZG = 1$ by hypothesis and hence $D'D^{p-1} \cap ZD = 1$, a contradiction which proves that $D'D^{p-1} = 1$. It follows from $G/P \simeq D$ that $G'G^{p-1} \leqslant P$; and this completes the proof of the strict p-closure of G.

Remark. Note that $ZG = 1$ was not needed for the proof of the necessity of our condition. The example of suitably selected nilpotent groups shows that $ZG = 1$ is indispensable for the proof of the sufficiency of our condition.

THEOREM 1.3. *A group G is strictly p-closed if, and only if,*

(a) *elements in G do not induce automorphisms of order p in subgroups of order prime to p and*

(b) *subgroups of order prime to p are abelian of exponent $p - 1$.*

Proof. Assume first the existence of a normal p-subgroup P of G such that G/P is abelian of exponent $p - 1$. Consider a subgroup U of order prime to p. Then $P \cap U = 1$ so that U is isomorphic to the subgroup PU/P of G/P. Since the latter gruop is abelian of exponent $p - 1$, so is U. If furthermore the element g in G induces in U an automorphism of order a power of p, then we may assume without loss of generality that g is a p-element. As such g belongs to P and the commutators $[g, u]$ for u in U belong to $P \cap U = 1$. Hence g commutes with every element in U and so induces the identity automorphism in U. This proves the necessity of (a) and (b).

If strict p-closure were not a consequence of (a) and (b), then there would exist a group G of minimal order satisfying (a), (b), without being strictly p-closed. Every subgroup of G meets requirements (a) and (b). Because of the minimality of G it follows that

(1) every proper subgroup of G is strictly p-closed.

Since G is not strictly p-closed, it is certainly not a p-group. Consequently there exists a prime $q \neq p$ dividing the order of G. Denote by Q a q-Sylow subgroup of G. By (b), $Q \neq 1$ is abelian of exponent $p - 1$. If g is an element in the normalizer NQ of Q, then g is the product $g = g'g''$ of an element g' of order prime to p and an element g'' of order a power of p both of which belong to NQ. Since $Q\{g'\}$ is of order prime to p, it is by (b) abelian so that g' belongs to the centralizer of Q. It is a consequence of (a) that g'' belongs to the centralizer of Q. Thus NQ is the centralizer of the q-Sylow subgroup Q. Hence we may apply Burnside's Theorem asserting the existence of a normal subgroup T of G complementary to Q, (**7**, p. 133, Satz 4). T is a proper subgroup of G, since $Q \neq 1$. Hence T is, by (1), strictly p-closed. Consequently the totality P of p-elements in T is a characteristic p-subgroup of T whose index $[T:P]$ is prime to p. Since P is a characteristic subgroup of the normal subgroup T, P is a normal subgroup of G. Since $[G:T]$ is a power

https://doi.org/10.4153/CJM-1959-036-2 Published online by Cambridge University Press

of q, namely the order of Q, the index $[G:P]$ is prime to p. Application of Schur's Theorem shows the existence of a complement C of P in G. Since $C \simeq G/P$ is of order prime to p, it is by (b) abelian of exponent $p - 1$. Hence G is strictly p-closed, a contradiction proving our theorem.

2. In this section we derive a number of properties of supersoluble groups. Some of them are of independent interest and all of them will be needed in the sequel.

THEOREM 2.1. *The following properties of the group G are equivalent.*

(i) *G is supersoluble.*

(ii) *NU/CU is, for every p-subgroup U of G, strictly p-closed.*

(iii) *The Sylow Tower Property of supersoluble groups is satisfied by G; and NP/CP is, for every p-Sylow subgroup P of G, strictly p-closed.*

Proof. Assume first the supersolubility of G; and consider a p-subgroup U of G. The group θ of automorphisms, induced in U by elements in NU, is essentially the same as NU/CU. Since G is supersoluble, so is its subgroup NU. Since U is a normal p-subgroup of the supersoluble group NU, there exist normal subgroups $U(i)$ of NU such that

$$1 = U(0),\ U(i) < U(i + 1),\ [U(i + 1):U(i)] = p,\ U(k) = U.$$

Normal subgroups of NU which are part of U are θ-admissible. Thus every $U(i)$ is θ-admissible. Denote by θ^* the totality of those automorphisms in θ which induce the identity automorphism in every $U(i + 1)/U(i)$. Clearly θ^* is a normal subgroup of θ. An immediate application of § 1, Lemma (b) shows that every automorphism in θ^* is a p-automorphism. Thus θ^* is a normal p-subgroup of θ. Since every $U(i + 1)/U(i)$ is cyclic of order p, its group of automorphisms is cyclic of order $p - 1$. The automorphisms in $\theta^{p-1}\theta'$ induce consequently the identity automorphism in every $U(i + 1)/U(i)$. Hence $\theta^{p-1}\theta' \leqslant \theta^*$. The isomorphic groups θ and NU/CU are therefore strictly p-closed, proving that (ii) is a consequence of (i).

Assume next the validity of (ii). Consider a subgroup S of G and a minimal prime divisor p of the order of S. If U is a p-subgroup of S, then NU/CU is, by (ii), strictly p-closed. It follows that $[NU \cap S]/[CU \cap S]$ is likewise strictly p-closed. But p is a minimal prime divisor of the order of S. Hence $[NU \cap S]/[CU \cap S]$ is a p-group. Thus p-automorphisms only are induced in U by elements in S. Consequently we may apply a result that we derived elsewhere assuring the validity of the Sylow Tower Property of supersoluble groups in G (**3**, Theorem 6.2). It follows that (iii) is a consequence of (ii).

Assume finally the validity of (iii). If K is a normal subgroup of G and p a maximal prime divisor of $[G:K]$, then the totality P^* of p-elements in $G^* = G/K$ is a characteristic p-subgroup of index prime to p. If P is a p-Sylow subgroup of G, then $P^* = KP/K$; and we note that KP is a normal subgroup of G, since P^* is a characteristic subgroup of G^*. Application of the Frattini argument shows therefore $G = (KP)NP = K \cdot NP$; and now one sees without

https://doi.org/10.4153/CJM-1959-036-2 Published online by Cambridge University Press

difficulty that the group of automorphisms induced in P^* by elements in G^* is an epimorphic image of the group of automorphisms induced in P by elements in NP. The latter group is essentially the same as NP/CP. Since P is a p-Sylow subgroup of G, we deduce strict p-closure of NP/CP from (iii). Consequently a strictly p-closed group θ of automorphisms is induced in P^* by elements in G^*. There exists a minimal normal subgroup M^* of G^* which is part of P^*. The group θ^* of automorphisms which are induced in M^* by elements in G^* is an epimorphic image of θ. Hence θ^* is strictly p-closed and, because of the minimality of M^*, irreducible. Application of Theorem 1.1 shows that M^* is cyclic of order p. Hence G is supersoluble so that (i) is a consequence of (iii), q.e.d.

Remark 2.1. It is impossible to omit the first half of condition (iii) as may be seen from the following example. Assume that p and q are primes and that q is a divisor of $p - 1$. Then there exists a group A of order pq which possesses a normal subgroup B of order p such that the elements in A induce in B a group of automorphisms or order q. Clearly A is supersoluble, but not cyclic. Next denote by K an elementary abelian q-group of order q^{pq} and let G be an extension of K by A such that A acts as a regular permutation group on a basis of K (we may choose G as a splitting extension of K by A). The group of automorphisms induced in K by elements in G is isomorphic to A and hence not strictly q-closed. This implies in particular that G, though soluble, is not supersoluble (Theorem 2.1). A q-Sylow subgroup of G is an extension of K by a cyclic group of order q. Since A is an extension of a p-group by a q-group and not cyclic, one sees that q-Sylow subgroups of G are their own normalizers. Hence $NQ/CQ = Q/ZQ$ is, for every q-Sylow subgroup Q of G, a q-group. The p-Sylow subgroups of G are cyclic of order p. Their normalizers may contain elements in K; but these would belong to their centralizers. It follows that NP/CP is cyclic of order q for every p-Sylow subgroup P of G; and such a group is strictly p-closed, since q is a divisor of $p - 1$. Thus G is soluble, but not supersoluble; and G satisfies these cond half of condition (iii), but not the Sylow Tower Property of supersoluble groups.

Remark 2.2. Using results derived by us elsewhere (**3**, Theorem 6.2) one shows the equivalence of the three conditions of Theorem 2.1 with the following property:

If p is a minimal prime divisor of the order of the subgroup S of G, then S is completely p-normal; and NP/CP is, for every p-Sylow subgroup P of G, strictly p-closed.

THEOREM 2.2. *Assume that P is a p-Sylow subgroup of a supersoluble group G.*

(a) *$(NP)'(NP)^{p-1}$ is the direct product of a p-group and a group of order prime to p; and $G = NP$ in case p is the maximal prime divisor of the order of G.*

(b) *$P \cap \phi G \leqslant \phi P$; and $P \cap \phi G = \phi P$ in case p is the maximal prime divisor of the order of G.*

Proof. We note first that NP/CP is essentially the same as the group of automorphisms induced in P by elements in NP. It is a consequence of Theorem 2.1 that this group of automorphisms is strictly p-closed. Consequently $(CP)(NP)'(NP)^{p-1}/CP$ is a p-group. Since the p-Sylow subgroup P of G is a normal subgroup of NP, this implies

$$(NP)'(NP)^{p-1} \leqslant P\cdot CP.$$

Since P is also a normal subgroup of $P\cdot CP$ whose index is prime to p, it follows that $P \cap CP = ZP$ is a normal p-subgroup of CP whose index in CP is prime to p. By Schur's Theorem there exists a complement Q of $P \cap CP$ in CP; see, for instance (**7**, p. 125, Satz 25). Hence

$$P\cdot CP = P[P \cap CP]Q = PQ.$$

Since Q is part of the centralizer of P, $P\cdot CP$ is the direct product of P and Q. Since the elements in Q are of order prime to p, it follows that $P\cdot CP$ is the direct product of a p-group and a group of order prime to p. But this property is subgroup inherited. Hence $(NP)'(NP)^{p-1}$ is the direct product of a p-group and a group of order prime to p. That $G = NP$ in case p is the maximal prime divisor of the order of G, is a consequence of Theorem 2.1 (the Sylow Tower Property of supersoluble groups).

Denote by A the set of all those elements in G whose orders are divisible by primes greater than p only. Because of the Sylow Tower Property of supersoluble groups A is a characteristic subgroup of G whose order is divisible by primes greater than p only. The product AP is likewise a characteristic subgroup of G. It consists of just those elements in G whose orders are not divisible by primes smaller than p. By Schur's Theorem or by P. Hall's characteristic property for soluble groups there exists a complement B of A in G, since $o(A)$ and $[G:A]$ are relatively prime. Since B is isomorphic to G/A, the complement B contains a p-Sylow subgroup of G; and since any two p-Sylow subgroups of G are conjugate in G, we may assume without loss in generality that $P \leqslant B$. Since AP is a characteristic subgroup of G, $P = AP \cap B$ is a normal subgroup of B. The characteristic subgroup ϕP of P is consequently a normal subgroup of B. Let $B^* = B/\phi P$ and $P^* = P/\phi P$. Then P^* is an elementary abelian p-group, the p-Sylow subgroup of B^* and characteristic in B^*. Since B^* is supersoluble, the elements in B^* induce in P^* a strictly p-closed group θ of automorphisms. This group θ is essentially the same as B^*/CP^*. Since P^* is abelian, $P^* \leqslant CP^*$ so that $[B^*:CP^*]$ is prime to p. Hence θ is a strictly p-closed group of order prime to p; in other words θ is abelian of exponent $p - 1$. Since the group θ of order prime to p acts on the elementary abelian p-group P^*, it is completely reducible (Maschke's Theorem) (**6**, p. 81, Theorem 46). This signifies that every θ-admissible subgroup of P^* possesses in P^* a θ-admissible complement. Because of the supersolubility of B^* minimal normal subgroups of B^* which are contained in P^* have order p. It follows that P^* is the direct product of θ-admissible cyclic groups of order p. This in turn implies that the intersection of all maximal θ-admissible subgroups of P^* is equal to 1. Consider now some maximal θ-admissible subgroup M of P^*. Then M has index p in P^* and is a normal subgroup of B^*. There exists, by Schur's Theorem, a complement D of P^* in B^*. It is clear that MD is a maximal subgroup of B^*. It follows now that

$$P^* \cap \phi B^* = 1.$$

Every maximal subgroup of B^* has the form $S/\phi P$ with S a maximal subgroup of B. If J is the intersection of all these maximal subgroups S, then we deduce $P \cap J = \phi P$ from $P^* \cap \phi B^* = 1$. If S is a maximal subgroup of B, then AS is a maximal subgroup of G, since B is a complement of the characteristic subgroup A of G. From $P \cap J = \phi P$ we deduce now that $P \cap \phi G \leqslant \phi P$.

Suppose finally in particular that p is the maximal prime divisor of the order of G. Then P is a characteristic subgroup of G (Sylow Tower Property of supersoluble groups). Consequently ϕP is a characteristic subgroup of G too. We recall that maximal subgroups of supersoluble groups have index a prime. If the maximal subgroup S of G does not contain P, then consequently $[G:S] = p$. From $G = PS$ we deduce now that $[P:P \cap S] = p$, since P is a characteristic subgroup of G. It follows that $\phi P \leqslant P \cap S \leqslant S$. Thus we have shown $\phi P \leqslant \phi G$. Hence

$$P \cap \phi G \leqslant \phi P \leqslant P \cap \phi G,$$

proving $\phi P = P \cap \phi G$ in case p is the maximal prime divisor of the order of G.

Remark. Consider primes p, q such that q^2 is a divisor of $p - 1$. Then there exists an extension G of a cyclic group P of order p by a cyclic group of order q^2 such that the elements in G induce in P a cyclic group of automorphisms whose order is q^2. Every q-Sylow subgroup of G is cyclic of order q^2 and a maximal subgroup of G; and there exist two q-Sylow subgroups of G with intersection 1. Hence $\phi G = 1$. If Q is a q-Sylow subgroup of G, then ϕQ is cyclic of order q. Hence

$$Q \cap \phi G = 1 < \phi Q$$

showing the impossibility of improving (b).

3. We are now ready to turn to the study of supersoluble immersion.

Definition. A normal subgroup K of G is supersolubly immersed in G if to every homomorphism σ of G with $K^\sigma \neq 1$ there exists a cyclic normal subgroup $A \neq 1$ of G^σ such that $A \leqslant K^\sigma$.

This implies the apparently stronger property mentioned in the introduction: If K is a supersolubly immersed normal subgroup of G, if σ is a homomorphism of G, and if a minimal normal subgroup M of G^{σ} is part of K^{σ}, then M is cyclic of order a prime.

We note that S is a supersoluble subgroup of G if S contains the supersolubly immersed normal subgroup K of G, and if S/K is supersoluble. This important property has two interesting consequences:

(a) The product of all supersolubly immersed normal subgroups of G is a supersolubly immersed characteristic subgroup of G.

(b) Every maximal supersoluble subgroup of G contains every supersolubly immersed normal subgroup of G.

Examples show, however, that the intersection of all maximal supersoluble subgroups of G may actually be greater than the product of all supersolubly immersed normal subgroups of G.

Much use will be made of the following theorem: If the normal subgroup K of G is supersolubly immersed in G, then the elements in G induce in K a supersoluble group of automorphisms (**5**, p. 420, Satz 12).

This leads us to the following companion concept.

Definition. A group G and a group θ of automorphisms form a supersoluble pair if the minimal θ-admissible normal subgroups of G/T, for T a θ-admissible normal subgroup of G, are cyclic (of order a prime).

If, for instance, T is a supersolubly immersed normal subgroup of the group G and Γ is the group of automorphisms, induced in T by elements in G, then T, Γ is clearly a supersoluble pair. In a way the converse is true too. For consider a supersoluble pair G, θ. Then let H be the holomorph of G. This contains G as a normal subgroup and it also contains θ. Their product $G\theta$ is an extension of G by θ which realizes in G in an obvious way the automorphism group θ. We shall refer to this splitting extension of G by θ as to *the product of the group G and its group θ of automorphisms.* Since G, θ is a supersoluble pair, it is quite obvious that G is supersolubly immersed in $G\theta$. It follows in particular that the group Γ of automorphisms induced in G by $G\theta$ is supersoluble. Since $\theta \leqslant \Gamma$, we have shown the supersolubility of θ. Incidentally we have shown that G, Γ is a supersoluble pair where the group Γ of automorphisms of G is the compositum of θ and the group of inner automorphisms of G.

We may summarize the principal results of the preceding discussion as follows:

The following properties of the normal subgroup T of the group G are equivalent.

(i) *T is supersolubly immersed in G.*

(ii) *A supersoluble pair is formed by T and the group Γ of automorphisms induced in T by G.*

(iii) *$T\Gamma$ is supersoluble.*

We mention finally the following easily verified inheritance properties: If G, θ is a supersoluble pair, and if U is a θ-admissible subgroup of G, then (if we denote the group of automorphisms induced in U by θ likewise by θ) U, θ is a supersoluble pair. If J is a θ-admissible normal subgroup of G, then (if we denote the group of automorphisms induced in G/J by θ likewise by θ) $G/J, \theta$ is a supersoluble pair. If, furthermore, θ is a group of automorphisms of the group G, if L is a cyclic θ-admissible normal subgroup, and if $G/L, \theta$ is a supersoluble pair, then G, θ is a supersoluble pair.

If X is a subgroup of G, it will be convenient to denote by θ_X the totality (subgroup) of X-preserving automorphisms in the group θ.

THEOREM 3.1. *If a group θ of automorphisms of the group G contains all the inner automorphisms of G, then the following properties of the pair G, θ are equivalent:*

(i) *G, θ is a supersoluble pair.*

(ii) *If U is a p-subgroup of G, then θ_U induces a strictly p-closed group of automorphisms in U.*

(iii) *G has the Sylow Tower Property of supersoluble groups; and if P is a p-Sylow subgroup of G, then θ_P induces a strictly p-closed group of automorphisms in P.*

(iv) *θ has the Sylow Tower Property of supersoluble groups; and if Σ is a subgroup of θ and P a Σ-admissible p-Sylow subgroup of G, then maximal Σ-admissible subgroups of P have index p in P.*

Proof. If G, θ is a supersoluble pair, then their product $G\theta$ is a supersoluble group. If U is a p-subgroup of G, then the normalizer of U in $G\theta$ induces in U a strictly p-closed group Γ of automorphisms (Theorem 2.1). Since θ_U is by definition part of the normalizer of U in $G\theta$, it follows that θ_U induces in U a subgroup of Γ. Since the latter is strictly p-closed, so is its subgroup induced by θ_U. Hence (ii) is a consequence of (i).

If (ii) is satisfied by the pair G, θ then the second part of (iii) is, as a special case of (ii), likewise satisfied. If U is a p-subgroup of G, then the group Γ of automorphisms of U which are induced in U by elements in the normalizer NU of U in G is a subgroup of the group of automorphisms induced in U by automorphisms in θ_U. The latter group of automorphisms of U is strictly p-closed by (ii). Hence Γ is strictly p-closed too. Thus the condition (ii) of Theorem 2.1 is satisfied by G, proving the supersolubility of G. Thus we have shown that (iii) is a consequence of (ii).

Assume next that (iii) is satisfied by the pair G, θ. If K is a θ-admissible normal subgroup of G, then θ induces in G/K a group Σ of automorphisms. Denote by S/K a p-Sylow subgroup of G/K. Then the group of S/K-preserving automorphisms in Σ may be denoted by Σ_S, since it is induced by the automorphisms in θ_S. Denote by P a p-Sylow subgroup of S. Since S/K is a p-Sylow subgroup of G/K, we have $S = KP$ and P is a p-Sylow subgroup of G. Application of (iii) shows that θ_P induces in P a strictly p-closed group of automorphisms. If σ is an automorphism in θ_S, then $KP = S = S^\sigma = KP^\sigma$, since K is θ-admissible. Hence P^σ is a p-Sylow subgroup of KP. Consequently there exist elements a and b in K and P respectively such that $P^\sigma = P^{ba} = P^a$. Since θ contains every inner automorphism, the automorphism σa^{-1} belongs to θ_P. Since a belongs to K, it induces the identity automorphism in G/K. Thus we have shown that the group of automorphisms induced in S/K by elements in θ_S is an epimorphic image of the group of automorphisms induced by θ_P in P. Since the latter is strictly p-closed, so is the former. Hence Σ_S induces a strictly p-closed group of automorphisms in S/K. Since the Sylow Tower Property of supersoluble groups is inherited by quotient groups, we have shown that (iii) implies the following property:

(iii*) *If K is a θ-admissible normal subgroup of G, then the Sylow Tower Property of supersoluble groups is satisfied by $G/K = H$; and if Σ is the group of automorphisms induced in H by θ, and P is a p-Sylow subgroup of H, then Σ_P induces a strictly p-closed group of automorphisms in P.*

It is not difficult to derive (i) from (iii*). For consider a θ-admissible normal subgroup K of G. If p is the maximal prime divisor of the order of $H = G/K$, then the p-Sylow subgroup P of H is a characteristic p-subgroup of H because of the Sylow Tower Property. Clearly $P \neq 1$; and as a characteristic subgroup of H, P is Σ-admissible, if we denote by Σ the group of automorphisms induced in H by θ. There exists a minimal Σ-admissible subgroup M of H which is part of P. Since θ, and hence Σ, contains every inner automorphism of G and H respectively, M is a normal subgroup of H. Because of (iii*) a strictly p-closed group Γ of automorphisms is induced by Σ in P; and the automorphisms in Γ induce a group Γ^* of automorphisms in the Σ-admissible subgroup M which is strictly p-closed as an epimorphic image of Γ. Since M is a minimal Σ-admissible subgroup of H, Γ^* is irreducible. Since M is a p-group, we may apply Theorem 1.1 to see that M is cyclic of order p. Thus we have established the existence of a cyclic normal Σ-admissible subgroup $M \neq 1$ of H; and this shows that G, θ is a supersoluble pair. This completes the proof of the equivalence of conditions (i) to (iii).

If G, θ is a supersoluble pair, then G and θ are both supersoluble groups, as has been mentioned before, so that both G and θ have the Sylow Tower Property of supersoluble groups. Consider now a subgroup Σ of θ and a Σ-admissible p-subgroup U of G. Then U, Σ is a supersoluble pair too so that their product $U\Sigma$ is a supersoluble group. Thus every maximal subgroup of $U\Sigma$ has index a prime. If V is a maximal Σ-admissible subgroup of U, then $V\Sigma$ is a maximal subgroup of $U\Sigma$. Hence $[U\Sigma : V\Sigma]$ is a prime; and this prime is p since $V < U$. It follows that (iv) is a consequence of (i).

Assume finally the validity of (iv). Since θ contains the group of inner automorphisms of G which is essentially the same as G/ZG, and since θ has the Sylow Tower Property of supersoluble groups, G/ZG has likewise this property. But this implies naturally that G itself enjoys the Sylow Tower Property of supersoluble groups. Consider a p-Sylow subgroup P of G; and denote by Γ the group of automorphisms induced in P by θ_P. Since ϕP is a characteristic subgroup of P, automorphisms preserving P will also preserve ϕP. Hence ϕP is θ_P-admissible. Denote by Σ a subgroup of θ_P which induces in $P/\phi P$ a group Σ^* of automorphisms whose order is prime to p. Since Σ^* acts on the elementary abelian p-group $P/\phi P$, it is completely reducible (Maschke's Theorem) (**6**, p. 81, Theorem 46). This signifies that every Σ^*-admissible subgroup of $P/\phi P$ possesses in $P/\phi P$ a Σ^*-admissible complement. By (iv), maximal Σ^*-admissible subgroups of $P/\phi P$ have index p in $P/\phi P$. Consequently $P/\phi P$ is the direct product of cyclic Σ^*-admissible subgroups. Since cyclic subgroups of $P/\phi P$ have order p, and since the group of automorphisms of a cyclic group of order p is cyclic of order $p - 1$, it follows that Σ^* is abelian of exponent $p - 1$. We recall the result of P. Hall that an automorphism of the p-group P has order a power of p in case it induces the identity in $P/\phi P$; (§ 1, Lemma (a)). Combining these results we see that a subgroup of Γ whose order is prime to p is abelian of exponent $p - 1$. If the order of Γ is divisible by p, then p is the maximal prime divisor of the order of Γ. By (iv), the group θ, and consequently Γ too, have the Sylow Tower Property of supersoluble groups. This implies in particular the existence of a characteristic p-Sylow subgroup Γ_P of Γ. By Schur's Theorem there exists a complement Δ of Γ_P in Γ. Since $\Delta \simeq \Gamma/\Gamma_P$ is of order prime to p, Δ and consequently Γ/Γ_P is abelian of exponent $p - 1$. Hence Γ is strictly p-closed; and thus we have shown that (iii) is a consequence of (iv). This completes the proof of the equivalence of conditions (i) to (iv).

THEOREM 3.2. *G, θ is a supersoluble pair if, and only if, $G/\phi G, \theta$ is a supersoluble pair.*

Proof. The necessity of our condition is obvious. Assume that $G/\phi G, \theta$ is a supersoluble pair. Since this condition remains valid, if we adjoin the inner automorphisms of G to θ, we may assume without loss in generality that the inner automorphisms of G belong to θ. Consider a θ-admissible normal subgroup K of G. Then $\phi(G/K) = J/K$ where J is the intersection of all those maximal subgroups of G which contain K. This implies in particular that ϕG is part of J and that consequently $K\cdot\phi G/K \leqslant \phi(G/K)$. Thus $(G/K)/\phi(G/K)$ is an epimorphic image of $G/\phi G$. Since $G/\phi G$, θ is a supersoluble pair, $(G/K)/\phi(G/K), \theta$ is likewise a supersoluble pair. Let $H = G/K$; and denote by Σ the group of automorphisms induced in G/K by θ. Since $H/\phi H, \Sigma$ is a supersoluble pair, $H/\phi H$ (and the group of automorphisms, induced by Σ in $H/\phi H$) are supersoluble. The supersolubility of $H/\phi H$ implies the supersolubility of H; (**7**, p. 418, Satz 10). If p is the maximal prime divisor of the order of H, then H is p-closed and the p-Sylow subgroup P of H is a characteristic p-subgroup of H. Thus P is in particular Σ-admissible. It is a consequence of Theorem 2.2 (b) that $\phi P = P \cap \phi H$. This implies that Σ induces essentially the same group of automorphisms in $P/\phi P = P/[P \cap \phi H]$ and in $P\cdot\phi H/\phi H$. The latter group is the p-Sylow subgroup of $H/\phi H$ and p is the maximal prime divisor of the order of $H/\phi H$. Since $H/\phi H$, Σ is a supersoluble pair, Σ induces in $P\cdot\phi H/\phi H$, and hence in $P/\phi P$, a strictly p-closed group of automorphisms. Denote by Γ the group of automorphisms induced in P by automorphisms in Σ; and denote by Γ^* the subgroup of those automorphisms in Γ which induce the identity automorphism in $P/\phi P$. By § 1, Lemma (a), the normal subgroup Γ^* of Γ is a p-group. Since Γ/Γ^* is essentially the same as the group of automorphisms induced in $P/\phi P$ by Σ, and since the latter group is strictly p-closed, we see that Γ is an extension of a p-group by a strictly p-closed group. Hence Γ itself is strictly p-closed. Since $P \neq 1$ is Σ-admissible, there exists a minimal Σ-admissible subgroup M of H which is part of P. Since θ, and hence Σ, contains every inner automorphism, M is a normal subgroup of H. Because of the minimality of M the group Σ induces in M an irreducible group Λ of automorphisms. Since Λ is likewise induced by Γ (because $M \leqslant P$), Λ is strictly p-closed. Since M is a p-group, Theorem 1.1 is applicable. Hence M is cyclic of order p. Thus we have established the existence of a cyclic, θ-admissible, normal subgroup $M \neq 1$ of G/K. Hence G, θ is a supersoluble pair, q.e.d.

4. The results obtained in § 3 will now be applied to the problem of supersoluble immersion.

THEOREM 4.1. *The following properties of the normal subgroup K of G are equivalent:*

(i) *K is supersolubly immersed in G.*
(ii) *$K/\phi K$ is supersolubly immersed in $G/\phi K$.*
(iii) *If U is a p-subgroup of K, then NU/CU is strictly p-closed.*
(iv) *K has the Sylow Tower Property of supersoluble groups; and if P is a p-Sylow subgroup of K, then NP/CP is strictly p-closed.*
(v) *G/CK has the Sylow Tower Property of supersoluble groups; and if P is a p-Sylow subgroup of K, S a subgroup of NP, then maximal S-normalized subgroups of P have index p in P.*

Proof. Denote by θ the group of automorphisms induced in K by elements in G. Then θ is essentially the same as G/CK. If U is a subgroup of K, then θ_U is just the group of all automorphisms of U which are induced in U by elements in the normalizer NU of U in G; and this shows that θ_U and NU/CU are essentially the same. Note finally that K is supersolubly immersed in G if, and only if, K, θ is a supersoluble pair. Since the inner automorphisms of K are clearly contained in θ, Theorems 3.1 and 3.2 may be applied, and a fairly obvious translation of these results proves the equivalence of properties (i) to (v).

LEMMA 4.2. *If K is a normal subgroup of G, and if the subgroup S of G is minimal with respect to the property $G = KS$, then $K \cap S \leqslant \phi S$, and $S/\phi S$ is an epimorphic image of G/K. In particular S is supersoluble in case G/K is supersoluble.*

https://doi.org/10.4153/CJM-1959-036-2 Published online by Cambridge University Press

Proof. Consider a maximal subgroup T of S. If $K \cap S$ were not part of T, then we could deduce from the maximality of T (and the normality of $K \cap S$ in S) that $S = (K \cap S)T$. Consequently

$$G = KS = K(K \cap S)T = KT.$$

But $T < S$, contradicting the minimality of S. Thus we see that $K \cap S$ is part of every maximal subgroup of S; in other words: $K \cap S \leqslant \phi S$. Next we note the isomorphy $G/K \simeq S/(K \cap S)$. From $K \cap S \leqslant \phi S$ we may deduce therefore that $S/\phi S$ is an epimorphic image of G/K. Thus supersolubility of G/K implies the supersolubility of $S/\phi S$; and the latter implies, by Huppert's Theorem, the supersolubility of S.

Remark. This lemma is, naturally, well known. It has been appended for the convenience of the reader. Note that the first part of the lemma has many applications of the type given in its second part, since there exist many group theoretical properties which, when satisfied by a group, are satisfied by its epimorphic images, and which, when satisfied modulo the Frattini subgroup, are satisfied by the group itself; for instance, nilpotency, dispersion, etc.

THEOREM 4.3. *The normal subgroup K of G is supersolubly immersed in G if, and only if,*

(a) *G induces in K a supersoluble group of automorphisms and*

(b) *the supersolubility of the subgroup S of G implies the supersolubility of KS.*

Proof. The necessity of these conditions we have pointed out before. If the conditions (a) and (b) are satisfied by the normal subgroup K of G, then we select among the subgroups X of G satisfying $G = X \cdot CK$ a minimal one, say S. Since the group of automorphisms, induced in K by elements in G, is essentially the same as G/CK, this group is supersoluble by (a). Application of Lemma 4.2 shows the supersolubility of S. Application of (b) shows the supersolubility of KS. Denote now by θ the group of automorphisms induced in K by elements in G. Because of $G = S \cdot CK$ the elements in KS induce in K the same group θ of automorphisms. Since KS is supersoluble, the pair K, θ is a supersoluble pair. But then clearly K is supersolubly immersed in G, q.e.d.

We have pointed out before that *the product $\Sigma_i G$ of all the supersolubly immersed normal subgroups of G* is itself a supersolubly immersed characteristic subgroup of G. If we denote by $\Sigma_0 G$ *the product of all normal subgroups X of G such that XS is supersoluble whenever S is a supersoluble subgroup of G*, then $\Sigma_0 G$ is a characteristic subgroup of G satisfying the same property (b) of Theorem 4.3. Denote finally by ΣG *the intersection of all normal subgroups X of G with supersoluble G/X*. Since direct products and subgroups of supersoluble groups are themselves supersoluble, ΣG is a characteristic subgroup of G with supersoluble quotient group $G/\Sigma G$.

COROLLARY 4.4. $\Sigma_i G = \Sigma_0 G \cap C\Sigma G$.

Proof. If K is a supersolubly immersed normal subgroup of G, then we deduce $K \leqslant \Sigma_0 G$ from Theorem 4.3 (b) and $\Sigma G \leqslant CK$ from Theorem 4.3 (a). The latter inequality implies $K \leqslant C\Sigma G$. Thus we have shown that

$$\Sigma_i G \leqslant \Sigma_0 G \cap C\Sigma G.$$

Let $D = \Sigma_0 G \cap C\Sigma G$. Then D is a characteristic subgroup of G which satisfies $D \leqslant C\Sigma G$ and hence $\Sigma G \leqslant CD$, implying the validity of condition (a) of Theorem 4.3. From $D \leqslant \Sigma_0 G$ we deduce the validity of condition (b) of Theorem 4.3. It follows that D is supersolubly immersed in G. Hence $D \leqslant \Sigma_i G$, completing the proof.

It is worth noting in this context that, in general, $\Sigma_i G < \Sigma_0 G$, and that products of supersoluble normal subgroups will, in general, not be supersoluble.

Slightly generalizing the concept of a supersoluble pair we term the pair G, θ (for θ a group of automorphisms of the group G) *an almost supersoluble pair, if G, Σ is, for every supersoluble subgroup Σ of θ, a supersoluble pair.* If the pair G, θ is almost supersoluble, then G, Σ is a supersoluble pair for every Sylow subgroup Σ of θ. The converse is false, as may be seen from the following

Example. Let p be a prime, q an odd prime divisor of $p - 1$ (for instance, $p = 7$, $q = 3$). Then $2q$ is a factor of $p - 1$. There exists one and essentially only one non-abelian group θ of order $2q$; and θ possesses a normal subgroup of order q and index 2, its only proper normal subgroup. It follows among other things that θ is supersoluble. There exists an elementary abelian p-group A of order p^{2q}; and there exists a group of automorphisms of A which is isomorphic to θ and which we shall denote by θ. Since Sylow subgroups of θ are cyclic of order q or 2, they are strictly p-closed. Hence every pair A, Σ, for Σ a Sylow subgroup of θ, is supersoluble. But θ itself is not strictly p-closed, though it is a group of automorphisms of the p-group A. Hence A, θ is not a supersoluble pair.

The connection between the concept of an almost supersoluble pair and our preceding discussion is effected by the following fairly obvious remark: If K is a normal subgroup of G and θ the group of automorphisms induced in K by the elements in G, then the pair K, θ is almost supersoluble if, and only if, KS is supersoluble whenever S is a supersoluble subgroup of G (Theorem 4.3).

REFERENCES

1. R. Baer, *Supersoluble groups*, Proc. Amer. Math. Soc., *6* (1955), 16–32.
2. ———, *Classes of finite groups and their properties*, Illinois Jour. Math., *1* (1957), 115–187.
3. ———, *Closure and dispersion*, Illinois Jour. Math., *3* (1959), 000–000.
4. P. Hall, *A contribution to the theory of groups of prime power order*, Proc. Lond. Math. Soc., *36* (1932), 29–95.
5. B. Huppert, *Normalteiler und maximale Untergruppen endlicher Gruppen*, Math. Zeit., *60* (1954), 409–34.
6. N. Jacobson, *The Theory of Rings* (New York, 1943).
7. H. Zassenhaus, *Lehrbuch der Gruppentheorie, I* (Leipzig und Berlin, 1937).

Frankfurt am Main