

TWO-SIDED SHIFT SPACES OVER INFINITE ALPHABETS

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Abstract

Ott, Tomforde and Willis proposed a useful compactification for one-sided shifts over infinite alphabets. Building from their idea, we develop a notion of two-sided shift spaces over infinite alphabets, with an eye towards generalizing a result of Kitchens. As with the one-sided shifts over infinite alphabets, our shift spaces are compact Hausdorff spaces but, in contrast to the one-sided setting, our shift map is continuous everywhere. We show that many of the classical results from symbolic dynamics are still true for our two-sided shift spaces. In particular, while for one-sided shifts the problem about whether or not any M -step shift is conjugate to an edge shift space is open, for two-sided shifts we can give a positive answer for this question.

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1. Introduction

This paper is a continuation of a line of study of symbolic dynamics over infinite alphabets initiated in [21] and further developed in [11–14, 26]. In classical symbolic dynamics one starts with a finite alphabet A and constructs the infinite products $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$. These are compact spaces when given the product topology, and the map σ which shifts all the entries of the sequences one to the left is a continuous map. In the case of $A^{\mathbb{Z}}$ this map is a homeomorphism. A *shift space* (or a *subshift*) is then a closed subspace of $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$ which is invariant under σ . These dynamical systems are fundamental and well studied—see [19] for an excellent reference.

Given the importance of the above situation, it is natural to wonder what happens if we do not insist that the alphabet A is finite. There has been much research into shift

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spaces over infinite alphabets, most of it in the case of countable-state shifts of finite type [8–10, 18]. Exel and Laca [7] defined a C^* -algebra \mathcal{O}_B from a given countable $\{0, 1\}$ -matrix B which is thought of as the incidence matrix of an infinite graph, and proposed that the spectrum of a certain commutative C^* -subalgebra of \mathcal{O}_B is a good candidate for the Markov shift associated to the graph (see also [28] for a C^* -algebra constructed from such a matrix). In [3, 4], Boyle *et al.* studied almost isomorphism for countable-state Markov shifts. Thermodynamical formalism for such shifts has also been well developed; see, for example, [5, 15, 16, 20, 23, 25]. Seminal work of Young [30] defined Markov towers and used them to study hyperbolic systems; see also [31] and see [27] for a survey.

A difficulty one encounters with infinite-alphabet shifts is that if A is infinite then the infinite products $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ are not compact, and indeed not even locally compact. This difficulty is dealt with in various ways in the papers listed above based on the application at hand. Our interest is in a compactification of $A^{\mathbb{N}}$ put forth by Ott, Tomforde and Willis in [21]—this compactification can be identified with the set of all finite sequences (including an empty sequence) and infinite sequences over A , and was motivated by spaces arising from countable graphs [29]. From this, they say that subshifts are closed (hence compact) subsets $\Lambda \subset \Sigma_A^{\mathbb{N}}$ invariant under the shift and with an additional condition which guarantees that the set of infinite sequences in Λ is dense in Λ . As noted in the introduction of [21], compactness of their subshifts is an important feature that allows them to establish results analogous to classical results about finite-alphabet subshifts. To us, the space constructed in [21] seems so natural that it deserves study in its own right. Further to this, in [13] we defined sliding block codes on such spaces and studied to what extent such maps characterize shift-commuting continuous maps between Ott–Tomforde–Willis shifts, and there compactness played a central role. See also [11], where a one-sided $(M + 1)$ -shift which is not conjugate (in the sense of [21]) to a one-sided M -step shift was constructed.

For the work at hand, our main motivation was to generalize the paper of Kitchens [17] to the Ott–Tomforde–Willis infinite-alphabet case. In fact, our work on generalizing Kitchens’ work grew to lead to not only the present work but also [14] and the previously mentioned [13]. We briefly describe Kitchens’ result. If A is a finite group, then the two-sided full shift $A^{\mathbb{Z}}$ is a compact zero-dimensional topological group with the operation of pointwise multiplication, and the shift map is an expansive group automorphism. Kitchens proved a converse to this: [17, Theorem 1.i] says that if X is a compact zero-dimensional topological group and ϕ is an expansive group automorphism of X , then (X, ϕ) is topologically conjugate to a *subshift* over a finite group via a group isomorphism. He also proved [17, Theorem 1.ii] that given the same assumptions, (X, ϕ) is topologically conjugate to $(A^{\mathbb{Z}}, \sigma) \times (F, \tau)$, where F is a finite group and τ is an automorphism. That the topology is compact and zero dimensional, and that the shift is bijective, are crucial in the proofs.

Hence, our goal is to define and study a two-sided compact analogue of the Ott–Tomforde–Willis construction. Our first clue of how to accomplish this comes from

the classical situation, where one can always obtain a two-sided shift from a one-sided shift via the inverse limit construction. Applying this idea to Ott–Tomforde–Willis shifts yields a set which can be identified with the set, which we denote $\Sigma_A^{\mathbb{Z}}$, of all infinite sequences in A , all left-infinite sequences in A ending at an arbitrary integer index and an empty sequence. This is the set which underlies the topological space we define; see Definition 2.1. Since the shift map is not continuous for Ott–Tomforde–Willis shifts, we do not take the topology to be the product topology (as is usual for topological inverse limits) and instead take a suitably modified topology generated by cylinders corresponding to left-infinite words; see Lemma 2.3. This topology makes $\Sigma_A^{\mathbb{Z}}$ zero dimensional, compact and Hausdorff, and in this topology the shift map is a homeomorphism. Using this topology, we were successfully able to obtain results akin to Kitchens’ [17, Theorem 1.ii] in [14, Theorem 5.18] (although we note that we were only able to generalize [17, Theorem 1.i] for the one-sided case; see [14, Propositions 3.2 and 3.5]).

In this paper, we focus on the fundamental properties of $\Sigma_A^{\mathbb{Z}}$ and its subshifts. Namely, we prove that any shift space is determined by a set of forbidden words (Proposition 2.25), and determine to what extent the shift-commuting continuous maps between shift spaces are given by sliding block codes (Theorem 3.13). We also show that our shift spaces are conjugate to their higher block presentations (Proposition 3.15), which implies that every M -step shift is conjugate to the edge shift on some graph (Proposition 3.19). Finally, we show that one can go back and forth from one-sided shifts to two-sided shifts via projection and (set-theoretical) inverse limit (Propositions 4.6 and 4.8), though these processes are not in general inverses of each other.

The article is organized as follows. In Section 2 we recall background and definitions from [21] and define our notion of the two-sided full shift over an infinite alphabet, including a careful description of the topology. We then define our notion of a shift space in analogy to that in [21] and reproduce fundamental results from finite-alphabet symbolic dynamics, adapted for the infinite-alphabet situation. In Section 3 we discuss sliding block codes between our shift spaces and discuss their higher block presentations. In Section 4 we present the relationship between the two-sided shift spaces defined here and Ott–Tomforde–Willis one-sided shift spaces. In Section 5 we provide a final discussion.

2. Two-sided shift spaces over infinite alphabets

2.1. The Ott–Tomforde–Willis one-sided full shift. We briefly recall the construction of [21] of the one-sided full shift over an infinite alphabet. This topology is inspired by spaces associated to infinite graphs [29], and one sees similar topologies associated to semilattices arising from C^* -algebras [2, 6, 22].

Let A be a countably infinite alphabet and define a new symbol \emptyset not in A . We call \emptyset the *empty letter* and let $\tilde{A} := A \cup \{\emptyset\}$. Let

$$\begin{aligned} \Sigma_A^{\mathbb{N}} &= \{(x_i)_{i \in \mathbb{N}} \in \tilde{A}^{\mathbb{N}} : x_i = \emptyset \text{ implies } x_{i+1} = \emptyset\}, \\ \Sigma_A^{\mathbb{N} \text{ fin}} &= A^{\mathbb{N}}, \quad \Sigma_A^{\mathbb{N} \text{ inf}} = \Sigma_A^{\mathbb{N}} \setminus \Sigma_A^{\mathbb{N} \text{ inf}}. \end{aligned}$$

The set $\Sigma_A^{\mathbb{N} \text{ fin}}$ is identified with the finite sequences in A via the identification

$$(x_1 x_2 \dots x_k \emptyset \emptyset \dots) \equiv (x_1 x_2 \dots x_k)$$

and the sequence of all \emptyset is denoted \emptyset and called the *empty sequence*. For a finite set $F \subset A$ and $x \in \Sigma_A^{\mathbb{N} \text{ fin}}$, let

$$Z(x, F) := \{y \in \Sigma_A^{\mathbb{N}} : y_i = x_i \forall 1 \leq i \leq k, y_{k+1} \notin F\}.$$

Sets of this form are called *generalized cylinders*. Endowed with the topology generated by the generalized cylinders, $\Sigma_A^{\mathbb{N}}$ is a compact totally disconnected metrizable Hausdorff space, and in this topology the generalized cylinders are compact and open [21, Proposition 2.5, Theorem 2.15 and Corollary 2.18]. We note that the shift map (the map which shifts every entry one to the left) is continuous everywhere except at \emptyset [21, Proposition 2.23].

2.2. Construction of the two-sided shift—its topology and dynamics. As mentioned in the introduction, we are interested in the two-sided analogue of the above. In the finite-alphabet case, the two-sided full shift is the inverse limit of the one-sided full shift—we take this as our starting point. We first write down the set we obtain from the inverse limit, and then describe the topology we endow this set with. After we do this, in Remark 2.5 we discuss the relation with the inverse limit.

We take A and \tilde{A} as in Section 2.1. Consider the subsets $\Sigma_A^{\mathbb{Z} \text{ inf}}, \Sigma_A^{\mathbb{Z} \text{ fin}} \subset \tilde{A}^{\mathbb{Z}}$ given by

$$\Sigma_A^{\mathbb{Z} \text{ inf}} := A^{\mathbb{Z}} := \{(x_i)_{i \in \mathbb{Z}} : x_i \in A \forall i \in \mathbb{Z}\}$$

and

$$\Sigma_A^{\mathbb{Z} \text{ fin}} := \{(x_i)_{i \in \mathbb{Z}} \in \tilde{A}^{\mathbb{Z}} : \exists k \in \mathbb{Z} \ni x_k = \emptyset \text{ and } x_i = \emptyset \Rightarrow x_{i+1} = \emptyset \forall i \in \mathbb{Z}\}.$$

DEFINITION 2.1. Let A be a countably infinite set and let $\Sigma_A^{\mathbb{Z} \text{ inf}}$ and $\Sigma_A^{\mathbb{Z} \text{ fin}}$ be as above. The *two-sided full shift* over A is the set

$$\Sigma_A^{\mathbb{Z}} := \Sigma_A^{\mathbb{Z} \text{ inf}} \cup \Sigma_A^{\mathbb{Z} \text{ fin}}.$$

Also, given $x \in \Sigma_A^{\mathbb{Z}}$, define the length of x as $l(x) := \sup\{i : x_i \neq \emptyset\}$.

We will refer to the constant sequence $\emptyset := (\dots \emptyset \emptyset \emptyset \dots) \in \Sigma_A^{\mathbb{Z} \text{ fin}}$ as the *empty sequence*—we note that $l(\emptyset) = -\infty$. The elements of $\Sigma_A^{\mathbb{Z} \text{ fin}}$ will be referred to as *finite sequences* and we will identify $\Sigma_A^{\mathbb{Z} \text{ fin}}$ with the set $\{\emptyset\} \cup \bigcup_{i \in \mathbb{Z}} \prod_{-\infty}^i A$ via the identification

$$(\dots x_i x_{i+1} x_{i+2} \dots x_k \emptyset \emptyset \emptyset \dots) \equiv (\dots x_i x_{i+1} x_{i+2} \dots x_k).$$

Following this identification, we will use the notation $x = (x_i)_{i \leq k}$ to refer to a point in $\Sigma_A^{\mathbb{Z} \text{ fin}}$ with length $-\infty < k < \infty$. If needed, we will separate the zeroth entry from the first entry by a period; that is, if $k > 0$,

$$(x_i)_{i \leq k} = (\dots x_{-2} x_{-1} x_0 . x_1 x_2 \dots x_k).$$

We now define a topology on $\Sigma_A^{\mathbb{Z}}$, which will have a basis of clopen sets. First, we notice that the topology of generalized cylinders defined for Ott–Tomforde–Willis one-sided shifts (see [21, Definition 2.8]) cannot be straightforwardly adapted to our case, since cylinders which are defined using only finitely many entries will not be closed sets. To deal with this, we define generalized cylinder sets of $\Sigma_A^{\mathbb{Z}}$ as sets obtained by specifying infinitely many coordinates to the left. This definition is aimed towards making our space manageable while keeping its geometric origins intact¹. We are more precise below.

DEFINITION 2.2. Let $x = (x_i)_{i \leq k} \in \Sigma_A^{\mathbb{Z} \text{ fin}}$, $x \neq \emptyset$, and let $F \subset A$ be a finite set. Define

$$Z(x, F) := \{y \in \Sigma_A^{\mathbb{Z}} : y_i = x_i \ \forall i \leq k, y_{k+1} \notin F\}.$$

If F is the empty set, we will shorten $Z(x, F)$ to $Z(x)$.

Let $B_{\Sigma_A^{\mathbb{Z}}}$ be the collection of all generalized cylinders $Z(x, F)$, together with sets that are complements of finite unions of generalized cylinders of the form $Z(x)$. We endow $\Sigma_A^{\mathbb{Z}}$ with the topology $\tau_{\Sigma_A^{\mathbb{Z}}}$, which is generated by $B_{\Sigma_A^{\mathbb{Z}}}$.

LEMMA 2.3. *The topology $\tau_{\Sigma_A^{\mathbb{Z}}}$ is Hausdorff and $B_{\Sigma_A^{\mathbb{Z}}}$ is a clopen basis for it.*

PROOF. We first show that all elements of $B_{\Sigma_A^{\mathbb{Z}}}$ are clopen in $\tau_{\Sigma_A^{\mathbb{Z}}}$. Since $B_{\Sigma_A^{\mathbb{Z}}}$ generates $\tau_{\Sigma_A^{\mathbb{Z}}}$, generalized cylinders of the form $Z(x)$ and sets that are complements of finite unions of generalized cylinders of the form $Z(x)$ are clopen. Furthermore, generalized cylinders of the form $Z(x, F)$ are open sets by assumption. Thus, what remains to be shown is that sets of the form $Z(x, F)$ are closed. This easily follows by noticing that, for $x = (\dots, x_{k-1}, x_k) \in \Sigma_A^{\mathbb{Z} \text{ fin}}$ and finite $F \subset A$,

$$Z(x, F) = Z(\dots, x_{k-1}, x_k) \cap \left[\bigcup_{f \in F} Z(\dots, x_{k-1}, x_k, f) \right]^c,$$

a finite intersection of closed sets.

Let us show that $B_{\Sigma_A^{\mathbb{Z}}}$ is a basis for $\tau_{\Sigma_A^{\mathbb{Z}}}$. Note that given any $x \in \Sigma_A^{\mathbb{Z}}$, $x \neq \emptyset$, and any $k \leq l(x)$, we have that x belongs to $Z(\dots, x_{k-1}, x_k)$. On the other hand, \emptyset belongs to the sets that are complements of finite unions of generalized cylinders of the form $Z(x)$. Now let $x, y, x^1, \dots, x^p, y^1, \dots, y^q \in \Sigma_A^{\mathbb{Z} \text{ fin}} \setminus \{\emptyset\}$ and let $F, G \subset A$ be finite sets. We have that

$$\left[\bigcup_{i=1}^p Z(x^i) \right]^c \cap \left[\bigcup_{i=1}^q Z(y^i) \right]^c = \left[\bigcup_{i=1}^p Z(x^i) \cup \bigcup_{i=1}^q Z(y^i) \right]^c,$$

while, supposing that $l(x) \leq l(y)$,

$$Z(x, F) \cap Z(y, G) = \begin{cases} Z(y, G) & \text{if } l(x) < l(y), \ x_i = y_i \ \forall i \leq l(x) \text{ and } y_{l(x)+1} \notin F, \\ Z(x, F \cup G) & \text{if } l(x) = l(y) \text{ and } x_i = y_i \ \forall i \leq l(x), \\ \emptyset & \text{otherwise.} \end{cases}$$

¹This type of approach is not uncommon in the literature; see for example [24, Definition 3.2].

Thus, $[\bigcup_{i=1}^p Z(x^i)]^c \cap [\bigcup_{i=1}^q Z(y^i)]^c \in B_{\Sigma_A^{\mathbb{Z}}}$, while $Z(x, F) \cap Z(y, G) \in B_{\Sigma_A^{\mathbb{Z}}}$ whenever it is not empty.

On the other hand,

$$Z(x, F) \cap \left[\bigcup_{i=1}^q Z(y^i) \right]^c = Z(x, F) \cap \bigcap_{i=1}^q Z(y^i)^c \tag{2.1}$$

and, therefore, if the above set is nonempty, then $Z(x, F) \cap Z(y^i)^c \neq \emptyset$ for all i . Note that, if $l(y^i) \leq l(x)$, then $Z(x, F) \cap Z(y^i)^c$ is either \emptyset or $Z(x, F)$. Furthermore, if $l(y^i) > l(x)$, then $Z(x, F) \cap Z(y^i)^c = \emptyset$ implies that $Z(x, F) \cap Z(y^i)^c = Z(x, F)$, while, if $Z(x, F) \cap Z(y^i) \neq \emptyset$, then $y_j^i = x_j$ for all $j \leq l(x)$ and $y_{l(x)+1}^i \notin F$. Let i_1, i_2, \dots, i_r be all the indices such that $l(y^{i_\ell}) > l(x)$ and $Z(x, F) \cap Z(y^{i_\ell}) \neq \emptyset$, taken so that $l(y^{i_\ell}) \leq l(y^{i_{\ell+1}})$. Then we can rewrite (2.1) as

$$Z(x, F) \cap \left[\bigcup_{i=1}^q Z(y^i) \right]^c = Z(x, F) \cap \bigcap_{\ell=1}^r Z(y^{i_\ell})^c.$$

Now let $z \in Z(x, F) \cap [\bigcup_{i=1}^r Z(y^{i_\ell})]^c$. If $l(y^{i_\ell}) > l(z)$ for all $\ell = 1, \dots, r$, then let $H := \{y_{l(z)+1}^{i_\ell} : \ell = 1, \dots, r\}$ and notice that $z \in Z(\{z_j\}_{j \leq l(z)}, F \cup H) \subseteq Z(x, F) \cap [\bigcup_{i=1}^r Z(y^{i_\ell})]^c$. If $l(y^{i_\ell}) \leq l(z)$ for some ℓ , then let $L := \max\{\ell : l(y^{i_\ell}) \leq l(z)\}$. We have that

$$\begin{aligned} z_j &= x_j && \text{for } j \leq l(x), \\ z_{l(x)+1} &\notin F, \\ (z_j)_{j=l(x)+1, \dots, l(y^{i_\ell})} &\neq (y_j^{i_\ell})_{j=l(x)+1, \dots, l(y^{i_\ell})} && \text{for all } \ell \leq L. \end{aligned}$$

Hence, setting $H := \{y_{l(y^{i_\ell})+1}^{i_\ell} : \ell > L\}$,

$$z \in Z(\dots, z_{l(x)}, \dots, z_{l(y^{i_L})}, H) \subset Z(x, F) \cap \left[\bigcup_{i=1}^q Z(y^i) \right]^c.$$

Finally, if $x, y \in \Sigma_A^{\mathbb{Z}}$ are distinct points, then they differ at some entry, so we can clearly separate them with elements of $B_{\Sigma_A^{\mathbb{Z}}}$. Hence, this topology is Hausdorff. \square

Calculations in the proof of the above lemma imply the following, which is analogous to [21, Theorem 2.16].

LEMMA 2.4. *Suppose that $x \in \Sigma_A^{\mathbb{Z}}$, $x \neq \emptyset$. If $l(x) = \infty$, then the family of sets*

$$\{Z(\dots, x_{n-1}, x_n) : n \in \mathbb{Z}\} \subset B_{\Sigma_A^{\mathbb{Z}}}$$

is a neighborhood basis for x . If $l(x) < \infty$, then the family of sets

$$\{Z(x, F) : F \text{ is a finite subset of } A\} \subset B_{\Sigma_A^{\mathbb{Z}}}$$

is a neighborhood basis for x .

We note that while for one-sided shifts the topology of generalized cylinders coincides on $\Sigma_A^{\mathbb{N} \text{ inf}}$ with the product topology, the same does not hold for two-sided shifts. In fact, our topology on $\Sigma_A^{\mathbb{Z} \text{ inf}}$ is strictly finer than the product topology.

REMARK 2.5. The set $\Sigma_A^{\mathbb{Z}}$ can be identified with the inverse-limit dynamical system of the Ott–Tomforde–Willis one-sided shift $\Sigma_A^{\mathbb{N}}$,

$$(\Sigma_A^{\mathbb{N}})^{\sigma} := \{(\mathcal{X}_i)_{i \in \mathbb{Z}} : \forall i \in \mathbb{Z} \mathcal{X}_i \in \Sigma_A^{\mathbb{N}} \text{ and } \sigma(\mathcal{X}_i) = \mathcal{X}_{i+1}\},$$

via the map $p : (\Sigma_A^{\mathbb{N}})^{\sigma} \rightarrow \Sigma_A^{\mathbb{Z}}$ given by $p((\mathcal{X}_i)_{i \in \mathbb{Z}}) = (x_i)_{i \in \mathbb{Z}}$, where x_i is the first symbol of the sequence \mathcal{X}_i . In fact, p is a bijection whose inverse is given by $p^{-1}((x_i)_{i \in \mathbb{Z}}) = (\mathcal{X}_i)_{i \in \mathbb{Z}}$, where $\mathcal{X}_i = (x_{i+j-1})_{j \in \mathbb{N}}$.

However, note that this is only a bijection and not a homeomorphism, since the natural topology considered for inverse-limit dynamical systems is the product topology defined from the topology of $\Sigma_A^{\mathbb{N}}$.

While we have seen that our topology is Hausdorff, the next proposition shows that $\Sigma_A^{\mathbb{Z}}$ is not metrizable.

PROPOSITION 2.6. *Given the topology above, $\Sigma_A^{\mathbb{Z}}$ is not first countable.*

PROOF. Suppose that $\{V_i\}$ is a countable basis at \emptyset . Then, for each i , there exists a $U_i := [\bigcup_{1 \leq j \leq n_i} Z(x^j)]^c$ such that $\emptyset \in U_i \subseteq V_i$. Since $\Sigma_A^{\mathbb{Z}}$ is Hausdorff, we have that $\bigcap_{i=1}^{\infty} U_i = \{\emptyset\}$, so $\bigcup_{i=1}^{\infty} U_i^c = \Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$ and hence $\Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$ can be written as a countable union of cylinder sets, say $\Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\} = \bigcup_{i=1}^{\infty} Z(x^i)$.

Take $y_{l(x^1)}$ to be different from $(x^1)_{l(x^1)}$, let $k_1 = \min\{l(x^1) - 1, l(x^2)\}$ and take $y_{k_1} \neq (x^2)_{k_1}$. For each $n \geq 1$, suppose that y_{k_n} is defined, let $k_{n+1} = \min\{k_n - 1, l(x^{n+1})\}$ and take $y_{k_{n+1}} \neq (x^{n+1})_{k_{n+1}}$. For each $n \in \mathbb{Z}$ not of the form k_i for some i , let y_n be equal to a constant letter different from \emptyset . It follows that $y \in \Sigma_A^{\mathbb{Z} \text{ inf}}$, but y is not in any cylinder set $Z(x^i)$. We have a contradiction and so $\Sigma_A^{\mathbb{Z}}$ is not first countable. \square

The above proof uses the point \emptyset to prove that first countability fails. We now show that if we remove this point, the space becomes first countable, but neither second countable nor separable.

PROPOSITION 2.7. *The space $\Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$ is first countable, but it is neither second countable nor separable.*

PROOF. $\Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$ is first countable since Lemma 2.4 gives a countable neighborhood basis for each point of $\Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$. To see that $\Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$ is neither second countable nor separable, notice that the collection of open sets $\{Z((x_i)_{i \leq 0}) : (x_i) \text{ is a sequence of length zero}\}$ is pairwise disjoint and uncountable. \square

REMARK 2.8. Since there is no countable neighbourhood basis for \emptyset , sequences are not adequate to describe continuity at \emptyset (though they can be used to show that a map is not continuous at \emptyset). Moreover, convergence of a sequence to \emptyset can be tricky. For example, a sequence (x^j) such that $l(x^j) \rightarrow -\infty$ converges to \emptyset . Also, sequences where, for all $j \in \mathbb{N}$ and $n \in \mathbb{Z}$, there exist $k > j$ and $i < n$ such that $x_i^k \neq x_i^j$, will converge to \emptyset .

Indeed, take $a, b \in A$ and, for all $j \in \mathbb{N}$, let $x^j = (x_i^j)_{i \in \mathbb{Z}} \in \Sigma_A^{\mathbb{Z}}$ be defined as $x_i^j = a$ for all $i \neq -j$ and $x_{-j}^j = b$ —the sequence (x^j) converges to \emptyset .

More generally, a sequence (x^j) converges to \emptyset if and only if for any $Z(y)$ there are only a finite number of points x^j which belong to $Z(y)$.

REMARK 2.9. Despite the oddities of our space implied by Propositions 2.6 and 2.7, we include \emptyset in our space for two reasons. Firstly, since the space is built from the inverse limit of $\Sigma_A^{\mathbb{N}}$, \emptyset is naturally included as the point corresponding to the infinite sequence which is constantly $\emptyset \in \Sigma_A^{\mathbb{N}}$. Secondly, our main application is to generalize work of Kitchens [17] to shift spaces which have *inverse semigroup* operations. Inverse semigroups are typically assumed to have a zero element, and the point \emptyset naturally fills this role [14].

The characterization of the sequences that converge to \emptyset given in Remark 2.8 above can be used to prove the following result.

COROLLARY 2.10. *Let $X \subset \Sigma_A^{\mathbb{Z}}$. Then \emptyset is an accumulation point of X if and only if $X \setminus \{\emptyset\}$ is not contained in a finite union of generalized cylinders.*

The two-sided full shift $\Sigma_A^{\mathbb{Z}}$ can be projected onto the positive coordinates to yield the one-sided full shift $\Sigma_A^{\mathbb{N}}$. Let $\pi : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{N}}$ denote this projection map. Then clearly we have that if $x \in \Sigma_A^{\mathbb{Z}}$ and $l(x) \geq 1$, then $l(\pi(x)) = l(x)$ (where the l in the left-hand side stands for the length of a sequence in the one-sided shift).

PROPOSITION 2.11. *Let $\pi : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{N}}$ be as above. Then π is continuous at $x \in \Sigma_A^{\mathbb{Z}}$ if and only if $l(x) \geq 0$.*

PROOF. Let $x = (x_i)_{i \leq l(x)} \in \Sigma_A^{\mathbb{Z}}$. First suppose that $l(x) \geq 1$. Then $\pi(x) = (x_i)_{1 \leq i \leq l(x)}$. Given $k \in \mathbb{N}$, $1 \leq k \leq l(x)$, let $Z(x_1 x_2 \dots x_k, F)$ be a generalized cylinder of $\Sigma_A^{\mathbb{N}}$ containing $\pi(x)$. Then $Z((x_i)_{i \leq k}, F)$ is a generalized cylinder of $\Sigma_A^{\mathbb{Z}}$ which contains x and such that $\pi(Z((x_i)_{i \leq k}, F)) = Z(x_1 x_2 \dots x_k, F)$. In the case $l(x) = 0$, we have that $\pi(x) = \emptyset$ and, hence, given a neighborhood $Z(\emptyset, F)$ of the empty sequence in $\Sigma_A^{\mathbb{N}}$, we have that $Z(x, F)$ is a neighborhood of x in $\Sigma_A^{\mathbb{Z}}$ such that $\pi(Z(x, F)) = Z(\emptyset, F)$. Thus, we have that if $l(x) \geq 0$ then π is continuous.

Now suppose that $-\infty < l(x) \leq -1$. Notice that $\pi(x) = \emptyset$. Find an element $(z^n)_{n \geq 1} \in \Sigma_A^{\mathbb{Z}}$ such that $z_i^n = x_i$ for all $i \leq l(x)$, $z_{l(x)+1}^n \neq z_{l(x)+1}^m$ for all $n \neq m$ and $z_i^n = z_i^m$ for all $i \geq 1$ and $m, n \geq 1$. The sequence z^n converges to x , but $\pi(z^n) = (z_i^n)_{i \in \mathbb{N}} = (z_i^1)_{i \in \mathbb{N}} \neq \emptyset = \pi(x)$ and hence π is not continuous at x .

Finally, suppose that $l(x) = -\infty$, that is, $x = \emptyset$. Let $(z^n)_{n \geq 1} \in \Sigma_A^{\mathbb{Z}}$ be a sequence such that $z_i^n = z_i^m$ for all $i, m, n \geq 1$ and such that $z_i^n \neq z_i^m$ for all $i \leq 0$ and $n \neq m \geq 1$. Then (z^n) converges to \emptyset , but $\pi(z^n) = (z_i^n)_{i \in \mathbb{N}} = (z_i^1)_{i \in \mathbb{N}} \neq \emptyset$ and hence π is not continuous at \emptyset . □

The *shift map* $\sigma : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{Z}}$ is given by $\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$ for all $(x_i)_{i \in \mathbb{Z}} \in \Sigma_A^{\mathbb{Z}}$. In particular, we have that $\sigma(\emptyset) = \emptyset$ and if $x \in \Sigma_A^{\mathbb{Z}} \setminus \{\emptyset\}$ then $\sigma(\dots x_{-2} x_{-1} x_0 \cdot x_1 x_2 \dots x_k) = (\dots x_{-1} x_0 x_1 \cdot x_2 x_3 \dots x_k)$.

Note that the shift map for two-sided shifts is invertible. Furthermore, while the shift map on $\Sigma_A^{\mathbb{N}}$ is not continuous at \emptyset (see [21, Proposition 2.23]), we show below that the shift map on $\Sigma_A^{\mathbb{Z}}$ is continuous everywhere.

PROPOSITION 2.12. *The shift map $\sigma : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{Z}}$ is a homeomorphism.*

PROOF. Since $\Sigma_A^{\mathbb{Z}}$ is compact and Hausdorff and σ is a bijection, we just need to prove that σ is continuous.

Let $x \in \Sigma_A^{\mathbb{Z}, \text{fin}}$, with $l(x) = k > -\infty$, and let $F \subset A$ be a finite subset. Since $l(\sigma^{-1}(x)) = k + 1$, for each cylinder $Z(x, F)$, we have that $\sigma^{-1}(Z(x, F)) = Z(\sigma^{-1}(x), F)$. Also, given $x^1, \dots, x^m \in \Sigma_A^{\mathbb{Z}, \text{fin}}$, let $\mathcal{Z} := [\bigcup_{1 \leq j \leq m} Z(x^j)]^c$ be neighborhood of \emptyset . We calculate

$$\begin{aligned} \sigma^{-1}(\mathcal{Z}) &= \sigma^{-1}\left(\left[\bigcup_{1 \leq j \leq m} Z(x^j)\right]^c\right) = \sigma^{-1}\left(\bigcap_{1 \leq j \leq m} Z(x^j)^c\right) \\ &= \bigcap_{1 \leq j \leq m} [\sigma^{-1}(Z(x^j))]^c = \bigcap_{1 \leq j \leq m} Z(\sigma^{-1}(x^j))^c \\ &= \left[\bigcup_{1 \leq j \leq m} Z(\sigma^{-1}(x^j))\right]^c \end{aligned}$$

and hence σ is continuous. □

The following proposition will be useful to us later, and provides another link between our construction and that of [21].

PROPOSITION 2.13. *For all $x \in \Sigma_A^{\mathbb{Z}, \text{fin}} \setminus \{\emptyset\}$, the cylinder $Z(x) \subset \Sigma_A^{\mathbb{Z}}$ with the relative topology is homeomorphic to the Ott–Tomforde–Willis full shift $\Sigma_A^{\mathbb{N}}$.*

PROOF. Take $x \in \Sigma_A^{\mathbb{Z}, \text{fin}} \setminus \{\emptyset\}$ and define a map $f_x : \Sigma_A^{\mathbb{N}} \rightarrow Z(x)$ by

$$(f_x(z))_i = \begin{cases} x_i & \text{if } i \leq l(x), \\ z_{i-l(x)} & \text{if } i \geq l(x) + 1. \end{cases}$$

We claim that f_x is a homeomorphism. This map is clearly bijective and, since $\Sigma_A^{\mathbb{N}}$ is compact and $Z(x)$ is Hausdorff, our claim will be proven if we show that f_x is continuous. By Lemma 2.4, every open set in $Z(x)$ is a union of open sets of the form $Z(x) \cap Z(y, G)$, where $y \in \Sigma_A^{\mathbb{Z}, \text{fin}}$ and $G \subset A$ is finite. A short calculation gives that

$$f_x^{-1}(Z(x) \cap Z(y, G)) = \begin{cases} Z(y_{l(x)+1} \dots y_{l(y)}, G) & \text{if } l(x) < l(y) \text{ and} \\ & x_i = y_i \ \forall i \leq l(x), \\ Z(\emptyset, G) & \text{if } l(x) = l(y) \text{ and} \\ & x_i = y_i \ \forall i \leq l(x), \\ \Sigma_A^{\mathbb{N}} & \text{if } l(x) > l(y), x_i = y_i \\ & \forall i \leq l(y) \text{ and } x_{l(y)+i} \notin G, \\ \emptyset & \text{otherwise.} \end{cases}$$

In any case, $f_x^{-1}(Z(x) \cap Z(y, G))$ is open in $\Sigma_A^{\mathbb{N}}$ and so f_x is continuous. □

Now, in a similar fashion to [21], we show that our construction yields a compact space.

PROPOSITION 2.14. $\Sigma_A^{\mathbb{Z}}$ is compact and sequentially compact.

PROOF. Let $\{V_\alpha\}$ be an open cover for $\Sigma_A^{\mathbb{Z}}$. Without loss of generality, we may assume that each V_α is either a cylinder or the complement of a finite union of cylinders. Since \emptyset does not belong to any cylinder set, there exists α_0 such that $V_{\alpha_0} = [\bigcup_{1 \leq j \leq m} Z(x^j)]^c$, where $x^j \in \Sigma_A^{\mathbb{Z} \text{ fin}}$. Notice that $\{V_\alpha : \alpha \neq \alpha_0\}$ covers $\bigcup_{1 \leq j \leq m} Z(x^j)$, and so compactness of each $Z(x^j)$ (Proposition 2.13) implies we can find a finite subset $\{V_{\alpha_i}\}_{i=1}^K$ of $\{V_\alpha\}$ which covers $\bigcup_{1 \leq j \leq m} Z(x^j)$. Hence, $\{V_{\alpha_i}\}_{i=1}^K \cup \{V_{\alpha_0}\}$ covers $\Sigma_A^{\mathbb{Z}}$ and so $\Sigma_A^{\mathbb{Z}}$ is compact.

To check sequential compactness, let $(y^i)_{i \in \mathbb{N}}$ be any sequence in $\Sigma_A^{\mathbb{Z}}$. First, suppose that there does not exist any cylinder $Z(x) \subset \Sigma_A^{\mathbb{Z}}$ such that $y^i \in Z(x)$ for infinitely many $i \in \mathbb{N}$. In such a case, we have that y^i converges to \emptyset as i goes to infinity. On the other hand, if there exists $Z(x) \subset \Sigma_A^{\mathbb{Z}}$ such that $y^i \in Z(x)$ for infinitely many $i \in \mathbb{N}$, then we can take a subsequence $(y^{i_\ell})_{\ell \in \mathbb{N}}$ such that $y^{i_\ell} \in Z(x)$ for all $\ell \in \mathbb{N}$. Since $Z(x)$ is homeomorphic to $\Sigma_A^{\mathbb{N}}$, and $\Sigma_A^{\mathbb{N}}$ is sequentially compact (because it is a compact metric space), $Z(x)$ is also sequentially compact and there exists a subsequence of $(y^{i_\ell})_{\ell \in \mathbb{N}}$ which converges in $Z(x)$. □

2.3. Two-sided shift spaces. In this section we define shift spaces in our context. We then show that any given shift space can be characterized by a set of forbidden words. The set of forbidden words ends up containing some left-infinite words—this seems only natural since our topology also depends on such words.

Given a subset $X \subseteq \Sigma_A^{\mathbb{Z}}$, let

$$X^{\text{fin}} := X \cap \Sigma_A^{\mathbb{Z} \text{ fin}} \quad \text{and} \quad X^{\text{inf}} := X \cap \Sigma_A^{\mathbb{Z} \text{ inf}}$$

be the set of all finite sequences in X and the set of all infinite sequences in X , respectively. Given any set D , we let $D^{\mathbb{Z}^-} := \{(x_i)_{i \leq 0} : x_i \in D\}$ and let

$$B_{\text{inf}}(X) := \{(a_i)_{i \leq 0} \in (\tilde{A})^{\mathbb{Z}^-} : \exists x \in X, k \in \mathbb{Z} \text{ such that } x_{i+k} = a_i \forall i \leq 0\}$$

be the set of left-infinite subblocks of X . For $1 \leq n < \infty$, let

$$B_n(X) := \{(x_1, \dots, x_n) \in \tilde{A}^n : x_1, \dots, x_n \text{ is a subblock of some } x \in X\}$$

be the set of all blocks of length n in X . Note that blocks in $B_n(X)$ may contain the empty letter.

We single out the blocks of length one, and use the notation $L_A := B_1(X) \setminus \{\emptyset\}$ —this is the set of all symbols of A used by sequences of X , or the letters of X . The language of X is

$$B(X) := \bigcup_{n \in \mathbb{N}} B_n(X).$$

Before we define shift spaces, we need the following definitions.

DEFINITION 2.15. Let $X \subset \Sigma_A^{\mathbb{Z}}$ and $k \geq 1$. Given $1 \leq n < \infty$ and $a \in B_n(X)$, the k th follower set of a in X is the set

$$\mathcal{F}_k(X, a) := \{b \in B_k(X) : ab \in B_{k+n}(X)\},$$

and the k th predecessor set of a in X is

$$\mathcal{P}_k(X, a) := \{b \in B_k(X) : ba \in B_{k+n}(X)\}.$$

Similarly, the k th follower set of $a \in B_{\text{inf}}(X)$ in X is defined as

$$\mathcal{F}_k(X, a) := \{b \in B_k(X) : ab \in B_{\text{inf}}(X)\}.$$

DEFINITION 2.16. Let $X \subset \Sigma_A^{\mathbb{Z}}$. We say that X satisfies the infinite extension property if $x \in X^{\text{fin}} \setminus \{\emptyset\}$ implies that $|\mathcal{F}_1(X, (x_{i+l(x)})_{i \leq 0})| = \infty$.

REMARK 2.17. Note that for one-sided shift spaces the definition of the infinite extension property includes the empty sequence (see [21, Definition 3.1]), while here there is no condition required on the empty sequence (indeed, it is not even clear what it would mean to ‘follow’ the empty sequence).

LEMMA 2.18. Let $X \subseteq \Sigma_A^{\mathbb{Z}}$ be a set such that $\sigma(X) = X$. Then X^{inf} is dense in X if and only if $X^{\text{fin}} \setminus \{\emptyset\}$ satisfies the infinite extension property. Furthermore, \emptyset is an accumulation point of X^{inf} if and only if $|X^{\text{inf}}| = \infty$.

PROOF. Suppose that X^{inf} is dense. If $X^{\text{fin}} \setminus \{\emptyset\} = \emptyset$, then there is nothing to do. So, assume that there exists $x \in X^{\text{fin}}$, $x \neq \emptyset$.

From Lemma 2.4, the family of sets $\{Z(x, F) : F \text{ is a finite subset of } A\}$ is a neighborhood basis for x . Let $(a_i)_{i \in \mathbb{N}}$ be a sequence of symbols from A without repetitions and, for each $n \in \mathbb{N}$, define $F_n := \{a_i : i \leq n\}$. Since X^{inf} is dense in X , we have that $Z(x, F_n) \cap X^{\text{inf}} \neq \emptyset$ for all $n \in \mathbb{N}$. Hence, we can take a sequence $(y^n)_{n \in \mathbb{N}}$ in X^{inf} , where $y^n \in Z(x, F_n) \cap X^{\text{inf}}$ for all n . It follows that $(y_i^n)_{i \leq l(x)} = (x_i)_{i \leq l(x)}$ for all $n \in \mathbb{N}$, and $y_{l(x)+1}^m \neq y_{l(x)+1}^n$ for all $m \neq n$, which means that $\{y_{l(x)+1}^n\}_{n \in \mathbb{N}}$ is an infinite subset of $\mathcal{F}_1(X, x^*)$, where $x^* = (x_{i+k})_{i \leq 0}$.

Conversely, suppose that $X^{\text{fin}} \setminus \{\emptyset\}$ satisfies the infinite extension property. Take $x \in X^{\text{fin}} \setminus \{\emptyset\}$ and let $F \subset A$ be a finite set. Since sets of the type $Z(x, F)$ form a neighborhood basis for x , we just need to prove that $Z(x, F) \cap X^{\text{inf}}$ is nonempty. Due to the infinite extension property, we have that $|\mathcal{F}_1(X, x^*)| = \infty$, which implies that we can find $y \in X$ such that $y \neq x$ and $y \in Z(x, F) \cap X$. If $y \in X^{\text{inf}}$, then we are done. Otherwise, we have that $l(y) > l(x)$ and $Z(y) \cap X \subsetneq Z(x, F) \cap X$. Set $y^1 := y$ and use the infinite extension property again to pick $y^2 \in Z(y^1) \cap X$ such that $y^2 \neq y^1$. Again, if $y^2 \in X^{\text{inf}}$, then we are done. Otherwise, we have that $l(y^2) > l(y^1)$ and $Z(y^2) \cap X \subsetneq Z(y^1) \cap X$ and we can proceed recursively. So, either we will find some infinite sequence in some step proving the result, or we will define a sequence $(y^i)_{i \in \mathbb{N}}$ in X^{inf} such that $Z(y^{i+1}) \cap X \subsetneq Z(y^i) \cap X \subsetneq Z(x, F) \cap X$ for all $i \in \mathbb{N}$. In the last case, since the cylinders are closed, it follows that there exists an infinite sequence $z \in \bigcap_{i \in \mathbb{N}} [Z(y^i) \cap X] \subset Z(x, F) \cap X$.

Finally, for the last statement, notice that if $|X^{\text{inf}}| < \infty$, then \emptyset is not an accumulation point of X^{inf} . Conversely, if $|X^{\text{inf}}| = \infty$, then the fact that $\sigma(X) = X$ precludes the possibility that X^{inf} is contained in a finite union of generalized cylinders. Therefore, by Corollary 2.10, we have that \emptyset is an accumulation point of X^{inf} . \square

We now define two-sided shift spaces.

DEFINITION 2.19. A set $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ is said to be a *two-sided shift space over A* if the following three properties hold:

- (1) Λ is closed with respect to the topology of $\Sigma_A^{\mathbb{Z}}$;
- (2) Λ is invariant under the shift map, that is, $\sigma(\Lambda) = \Lambda$;
- (3) Λ^{inf} is dense in Λ .

If there is no possibility of confusion, we will simply call such a subset a *shift space*. Given a shift space $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$, we endow it with the subspace topology from $\Sigma_A^{\mathbb{Z}}$.

REMARK 2.20. Ott–Tomforde–Willis one-sided shift spaces are closed subsets of $\Lambda \subseteq \Sigma_A^{\mathbb{N}}$ such that $\sigma(\Lambda) \subset \Lambda$ and such that all elements of Λ^{fin} satisfy the infinite extension property (see [21, Definition 3.1]). In their setting, the infinite extension property is a condition equivalent to the density of Λ^{inf} in Λ (see [21, Proposition 3.8]).

REMARK 2.21. According to the above definition of two-sided shift spaces, even if $|L_\Lambda| < \infty$, we have that \emptyset is an accumulation point of the shift space whenever $|\Lambda| = \infty$. Hence, two-sided shift spaces over finite alphabets, defined according to Definition 2.19, do not coincide with classical shift spaces over finite alphabets.

PROPOSITION 2.22. Let $X \subset \Sigma_A^{\mathbb{Z}}$ be a set such that $\sigma(X) = X$. Then \bar{X} is a shift space, where \bar{X} denotes the closure of X in $\Sigma_A^{\mathbb{Z}}$.

PROOF. It is clear that \bar{X} is closed and shift invariant. Furthermore, $X \subset \bar{X}^{\text{inf}}$ and, since X is dense in \bar{X} , the result follows. \square

REMARK 2.23. Note that in the previous proposition we can have X being a proper subset of \bar{X}^{inf} . For example, if $a, b \in A$ are two distinct letters and $X := \{(x_i)_{i \in \mathbb{Z}} : \exists k \in \mathbb{Z} \text{ such that } x_i = a \forall i \leq k \text{ and } x_i = b \forall i > k\}$, then we have that $\bar{X}^{\text{inf}} = X \cup \{(x_i)_{i \in \mathbb{Z}} : x_i = a \forall i \in \mathbb{Z}\}$.

Two-sided shift spaces can also be alternatively defined from a set of forbidden words in an analogous way to [21, Definition 3.11]. However, in the two-sided case, in addition to forbidding subblocks, we need to consider the possibility that entire pasts are forbidden.

DEFINITION 2.24. Given $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$, let

$$X_{\mathbf{F}}^{\text{inf}} := \{x \in \Sigma_A^{\mathbb{Z}} : [B_{\text{inf}}(\{x\}) \cup B(\{x\})] \cap \mathbf{F} = \emptyset\}$$

and

$$X_{\mathbf{F}}^{\text{fin}} := \begin{cases} \emptyset & \text{if } |X_{\mathbf{F}}^{\text{inf}}| < \infty, \\ \{x \in \Sigma_A^{\mathbb{Z}} : |\mathcal{F}_1(X_{\mathbf{F}}^{\text{inf}}, (x_{i-l(x)})_{i \leq 0})| = \infty\} \cup \{\emptyset\} & \text{if } |X_{\mathbf{F}}^{\text{inf}}| = \infty. \end{cases} \tag{2.2}$$

Define $X_{\mathbf{F}} := X_{\mathbf{F}}^{\text{inf}} \cup X_{\mathbf{F}}^{\text{fin}}$.

PROPOSITION 2.25. *Given $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$, $X_{\mathbf{F}}$ is a shift space. Conversely, if $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ is a shift space, then $\Lambda = X_{\mathbf{F}}$ for some $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$.*

PROOF. We notice that the definition of $X_{\mathbf{F}}$ implies that $\sigma(X_{\mathbf{F}}) = X_{\mathbf{F}}$. Furthermore, by Lemma 2.18, $X_{\mathbf{F}}^{\text{inf}}$ is dense in X . Hence, to prove the first claim, we just need to show that $(X_{\mathbf{F}})^c$ is open, which is clear when $|X_{\mathbf{F}}^{\text{inf}}| < \infty$. So, assume that $|X_{\mathbf{F}}^{\text{inf}}| = \infty$ and let $x \notin X_{\mathbf{F}}$. We now have two cases.

- (1) If $l(x) = \infty$, then either:
 - (a) there exist $f \in \mathbf{F} \cap A^{\mathbb{Z}^-}$ and $n \in \mathbb{Z}$ such that $f_i = x_i$, for all $i \leq n$, in which case we have that $x \in Z(f) \subseteq (X_{\mathbf{F}})^c$; or
 - (b) there exists $f \in \mathbf{F} \cap \bigcup_{k \in \mathbb{N}} A^k$ such that $f_i = x_i$, for all $m \leq i \leq n$, for some $m, n \in \mathbb{Z}$, in which case we have that $x \in Z(\dots x_m \dots x_n) \subset (X_{\mathbf{F}})^c$.
- (2) If $-\infty < l(x) < \infty$, then we again have to consider two options:
 - (a) if $x \notin B_{\text{linf}}(X_{\mathbf{F}}^{\text{inf}})$, then we have that $x \in Z(x) \subset (X_{\mathbf{F}})^c$;
 - (b) if instead we have that $x \in B_{\text{linf}}(X_{\mathbf{F}}^{\text{inf}})$, but $|\mathcal{F}_1(X_{\mathbf{F}}^{\text{inf}}, x)| < \infty$, then, setting $F := \mathcal{F}_1(X_{\mathbf{F}}^{\text{inf}}, x)$, we obtain that $x \in Z(x, F) \subset (X_{\mathbf{F}})^c$.

This shows that $(X_{\mathbf{F}})^c$ is open and so $X_{\mathbf{F}}$ is closed.

To prove the second part, given a shift space Λ , we must find a set \mathbf{F} such that $\Lambda = X_{\mathbf{F}}$. To this end, take

$$\mathbf{F} = \left[A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k \right] \setminus [B_{\text{linf}}(\Lambda) \cup B(\Lambda)].$$

We first show that $X_{\mathbf{F}}^{\text{inf}} = \Lambda^{\text{inf}}$. The inclusion $\Lambda^{\text{inf}} \subset X_{\mathbf{F}}^{\text{inf}}$ is trivial. To prove the other inclusion, take $y \in X_{\mathbf{F}}^{\text{inf}}$ and, by contradiction, suppose that $y \in \Lambda^c$. Since Λ is closed, we can find an open set containing y disjoint from Λ . From Lemma 2.4, we can find a prefix z of y such that $Z(z)$ is disjoint from Λ . This implies that z cannot be in $B_{\text{linf}}(\Lambda)$ and so $z \in \mathbf{F}$, which contradicts $y \in X_{\mathbf{F}}^{\text{inf}}$. Hence, $X_{\mathbf{F}}^{\text{inf}} = \Lambda^{\text{inf}}$. Now, by Lemma 2.18, we have that $\Lambda = \overline{\Lambda^{\text{inf}}} = \overline{X_{\mathbf{F}}^{\text{inf}}} = X_{\mathbf{F}}$. □

The last proposition directly implies the following corollary.

COROLLARY 2.26. *Let $\Lambda, \Gamma \subseteq \Sigma_A^{\mathbb{Z}}$ be shift spaces. Then $\Lambda = \Gamma$ if and only if $[B_{\text{linf}}(\Lambda) \cup B(\Lambda)] = [B_{\text{linf}}(\Gamma) \cup B(\Gamma)]$.* □

Note that for standard shift spaces over finite alphabets, as well as for Ott–Tomforde–Willis one-sided shift spaces, two shift spaces are equal if and only if they have the same language. Corollary 2.26 above gives an analogue of this result for our two-sided shift spaces, but we need to take into account not only the language, but also restrictions on infinitely many coordinates to the left.

Unlike the one-sided case, in our setting the shift map is continuous everywhere. In fact, $\sigma : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{Z}}$ is a homeomorphism. Hence, in contrast to the one-sided case, $(\Sigma_A^{\mathbb{Z}}, \sigma)$ may be analyzed as a topological dynamical system.

DEFINITION 2.27. Two shift spaces $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ and $\Gamma \subseteq \Sigma_B^{\mathbb{Z}}$ are said to be topologically conjugate if the dynamical systems (Λ, σ) and (Γ, σ) are topologically conjugate, that is, if there exists a continuous invertible map $\varphi : \Lambda \rightarrow \Gamma$ such that $\varphi \circ \sigma = \sigma \circ \varphi$.

Definition 2.27 is the standard definition of equivalence in dynamical systems and should be contrasted with [21, Definition 4.1], where two one-sided shift spaces are said to be conjugate if there is a bijective continuous shift-commuting map between them which preserves length. It should be noted that we do not require that a conjugacy preserves length.

We will say that a given nonempty shift Λ is:

- a SHIFT OF FINITE TYPE: if $\Lambda = X_{\mathbf{F}}$ for some finite \mathbf{F} ;
- a FINITE-STEP SHIFT: if $\Lambda = X_{\mathbf{F}}$ for some $\mathbf{F} \subseteq \bigcup_{k \geq 1} A^k$. In the case that $\Lambda = X_{\mathbf{F}}$ for some $\mathbf{F} \subseteq A^{M+1}$, we will say that Λ is an M -step shift and, in particular, if $M = 1$ we also say that Λ is a *Markov* (or *Markovian*) shift;
- an INFINITE-STEP SHIFT: if it is not a finite-step shift;
- an EDGE SHIFT: let $G = (E, V, i, t)$ be a directed graph with no sources and sinks and define $\Lambda(G) \subseteq \Sigma_E^{\mathbb{Z}}$ as the closure of the set of all possible bi-infinite walks on G , that is,

$$\Lambda(G) := \overline{\{(e_i)_{i \in \mathbb{Z}} : t(e_i) = i(e_{i+1}) \ \forall i \in \mathbb{Z}\}}. \tag{2.3}$$

By Proposition 2.22, it follows that $\Lambda(G)$ is a shift space. In particular, one can check that $\Lambda(G)^{\text{inf}}$ coincides with the set of all possible bi-infinite walks on G , while $\Lambda(G)^{\text{fin}}$ can be identified with all left-infinite walks on G which end on a vertex of infinite order (that is, a vertex which emits infinitely many edges). Furthermore, $\Lambda(G) = X_{\mathbf{F}}$, where $\mathbf{F} := \{ef : t(e) \neq i(f)\}$;

- a ROW-FINITE SHIFT: if for all $a \in L_{\Lambda}$ we have that $\mathcal{F}_1(\Lambda, a)$ is a finite set;
- a COLUMN-FINITE SHIFT: if for all $a \in L_{\Lambda}$ we have that $\mathcal{P}_1(\Lambda, a)$ is a finite set.

For standard shift spaces (with finite alphabet) the class of finite-step shifts only contains M -step shifts, and the classes of shifts of finite type, finite-step shifts and edge shifts all coincide (of course, every subshift over a finite alphabet is both row finite and column finite). However, this is not true when the alphabet is infinite [21, Remark 5.19]. Furthermore, standard shift spaces and Ott–Tomforde–Willis shift spaces cannot be infinite-step shifts [21, Theorem 3.16].

REMARK 2.28. If $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ is a row-finite shift space, then $\Lambda^{\text{fin}} \subset \{\emptyset\}$ but, unlike the one-sided shift setting, the converse may not be true. For example, the shift space $X_{\mathbf{F}}$ defined in Examples 4.4(b) and (c) are not row finite but the only finite sequence contained in them is \emptyset .

3. Higher block shifts

3.1. Sliding block codes. We now turn to the natural problem of identifying the continuous shift-commuting maps between our shift spaces. In [13], we defined and classified such maps between one-sided shifts over infinite alphabets, and we follow a similar construction here. For this, we shall use the notion of *finitely defined sets*.

DEFINITION 3.1. Let $X \subset \Sigma_A^{\mathbb{Z}}$, let $k \leq \ell \in \mathbb{Z}$ be two integers and let $b = (b_1 \dots b_{-k+\ell+1}) \in B(\Sigma_A^{\mathbb{Z}})$. A *pseudocylinder* of X is a set

$$[b]_k^\ell := \{(x_i)_{i \in \mathbb{Z}} \in X : (x_k \dots x_\ell) = b\}.$$

We say that the pseudocylinder $[b]_k^\ell$ has *memory* K and *anticipation* L if $-K \leq \min\{0, k\}$ and $L \geq \max\{0, \ell\}$. The *least memory* of $[b]_k^\ell$ is $-\min\{0, k\}$, while the *least anticipation* of $[b]_k^\ell$ is $\max\{0, \ell\}$.

We will adopt the convention that the empty set is a pseudocylinder of X whose memory and anticipation are zero.

We remark that the definition of a pseudocylinder of X corresponds to the definition of cylinders around the position 0 for standard shift maps over finite alphabets (for one-sided shifts, $k = 0$). On the other hand, for a one-sided shift space Λ , cylinders of the form $Z(x) \cap \Lambda$, with $x \neq \emptyset$, satisfy Definition 3.1.

PROPOSITION 3.2. Let $[b]_k^\ell$ and $[c]_m^n$ be two pseudocylinders in $X \subset \Sigma_A^{\mathbb{Z}}$. Then $[b]_k^\ell \cap [c]_m^n$ is a union of pseudocylinders in X .

PROOF. Suppose that $[b]_k^\ell \cap [c]_m^n \neq \emptyset$, since if not, then the result is proved. Without loss of generality, we can assume that $\ell \leq n$ and therefore we have three cases.

$m \leq k \leq \ell \leq n$: In this case $[c]_m^n \subset [b]_k^\ell$ and then $[b]_k^\ell \cap [c]_m^n = [c]_m^n$.

$k < m \leq \ell \leq n$: In this case $[b]_k^\ell \cap [c]_m^n = [d]_k^n$, where $d = b$ if $\ell = n$, while in the case $\ell < n$ the word d is obtained by concatenating the word $b = (b_1 \dots b_{\ell-k+1})$ with $(c_{\ell-m+2} \dots c_{n-m+1})$.

$k \leq \ell < m \leq n$: In this case, if $m - \ell = 1$, then $[b]_k^\ell \cap [c]_m^n = [d]_k^n$, where d is the concatenation of b and c . On the other hand, if $m - \ell > 1$, then taking $\Omega := \{f \in B_{m-\ell-1}(\Sigma_A^{\mathbb{Z}}) : bfc \in B(\Sigma_A^{\mathbb{Z}})\}$, it follows that

$$[b]_k^\ell \cap [c]_m^n = \bigcup_{f \in \Omega} [bfc]_k^n. \quad \square$$

PROPOSITION 3.3. Let $\{[b^i]_{k_i}^{\ell_i}\}_{i \in \mathbb{N}}$ be a family of pseudocylinders in $X \subset \Sigma_A^{\mathbb{Z}}$ such that, for all $b^i = (b_1^i \dots b_{-k_i+\ell_i+1}^i)$, we have that $b_{-k_i+\ell_i+1}^i \neq \emptyset$. Then:

- (i) $\bigcup_{i \in \mathbb{N}} [b^i]_{k_i}^{\ell_i}$ is an open subset of X ;
- (ii) if there exists $L \in \mathbb{N}$ such that $\ell_i \leq L$ for all $i \in \mathbb{N}$, then $\bigcap_{i \in \mathbb{N}} [b^i]_{k_i}^{\ell_i}$ is an open subset of X .

PROOF. Item (i) is direct, since each pseudocylinder $[b^i]_{k_i}^{\ell_i}$ is an open subset of X .

To prove (ii), notice that if $(\bigcap_{i \in \mathbb{N}} [b^i]_{k_i}^{\ell_i}) \cap X = \emptyset$, we are done. So, suppose that $\bigcap_{i \in \mathbb{N}} [b^i]_{k_i}^{\ell_i} \neq \emptyset$. Then, for all $x \in \bigcap_{i \in \mathbb{N}} [b^i]_{k_i}^{\ell_i}$, it follows that

$$Z((x_n)_{n \leq \max\{\ell_i\}}) \cap X \subset \bigcap_{i \in \mathbb{N}} [b^i]_{k_i}^{\ell_i}. \quad \square$$

We now define the subsets of a shift space from which we will construct our sliding block codes.

DEFINITION 3.4. Given $C \subset X \subset \Sigma_A^{\mathbb{Z}}$, we will say that C is *finitely defined in X* if there exist two collections of pseudocylinders of X , $\{[b^i]_{k_i}^{\ell_i}\}_{i \in I}$ and $\{[d^j]_{m_j}^{n_j}\}_{j \in J}$, such that

$$\begin{aligned} C &= \bigcup_{i \in I} [b^i]_{k_i}^{\ell_i}, \\ C^c &= \bigcup_{j \in J} [d^j]_{m_j}^{n_j}. \end{aligned} \tag{3.1}$$

We recall that Definition 3.4 above is equivalent to [13, Definition 3.1] by taking $k = 0$. In other words, C is a finitely defined set of X if and only if for each $x \in X$ there exist $k, \ell \in \mathbb{N}$ such that the knowledge of $(x_{-k} \dots x_{\ell})$ allows us to know whether x belongs or not to C . In particular, X and \emptyset are finitely defined sets of X . We note that, in contrast to the one-sided case of [13], the set $\{\emptyset\}$ is *not* finitely defined, since knowing that a finite number of entries of a sequence are all \emptyset is not enough to guarantee that it is equal to \emptyset .

Note that if C is a finitely defined set of X , then there exist infinitely many ways to write C and C^c as unions of pseudocylinders of X .

DEFINITION 3.5. Let C be a finitely defined set in X . Then we say that C has *memory k* and *anticipation ℓ* if there exists a collection $\{[b^i]_{k_i}^{\ell_i}\}_{i \in I}$, which satisfies (3.1), with

$$\begin{aligned} -k &\leq \inf\{0, k_i : i \in I\}, \\ \ell &\geq \sup\{0, \ell_i : i \in I\}. \end{aligned} \tag{3.2}$$

We say that C has *least memory k* and *least anticipation ℓ* in the case that k and ℓ are the least integers for which (3.2) holds for some collection of pseudocylinders satisfying (3.1).

If for any collection $\{[b^i]_{k_i}^{\ell_i}\}_{i \in I}$ of pseudocylinders satisfying (3.1) we have that $\inf_{i \in I} k_i = -\infty$ or $\sup_{i \in I} \ell_i = \infty$, then we say that C has *unbounded memory* or *unbounded anticipation*, respectively.

We now show that the class of finitely defined sets is closed under taking finite unions and intersections.

PROPOSITION 3.6. *A finite union or intersection of finitely defined sets is also a finitely defined set.*

PROOF. Let $\{C_p\}_{p=1, \dots, n} \subset X \subset \Sigma_A^{\mathbb{Z}}$ be a collection of finitely defined subsets of X .

Let us prove that $K := \bigcup_{p=1}^n C_p$ is a finitely defined set in X . For each $p = 1, \dots, n$, let $\{[b^{p,i}]_{k_i^p}^{\ell_i^p}\}_{i \in I_p}$ and $\{[d^{p,j}]_{m_j^p}^{n_j^p}\}_{j \in J_p}$ be two collections of pseudocylinders for which

$C_p = \bigcup_{i \in I_p} [b^{p,i}]_{k_i^p}^{l_i^p}$ and $C_p^c = \bigcup_{j \in J_p} [d^{p,j}]_{m_j^p}^{n_j^p}$. Clearly, K is a union of pseudocylinders. On the other hand,

$$K^c = \left(\bigcup_{p=1}^n C_p \right)^c = \bigcap_{p=1}^n C_p^c = \bigcap_{p=1}^n \bigcup_{j \in J_p} [d^{p,j}]_{m_j^p}^{n_j^p} = \bigcup_{j \in J_p} \bigcap_{p=1}^n [d^{p,j}]_{m_j^p}^{n_j^p}$$

and, since Proposition 3.2 assures that a finite intersection of pseudocylinders can be written as a union of pseudocylinders, we have that K^c is a union of pseudocylinders, proving that K is a finitely defined set.

To prove that $\bigcap_{p=1}^n C_p$ is finitely defined, we just need to observe that $(\bigcap_{p=1}^n C_p)^c = \bigcup_{p=1}^n C_p^c$ and, since each C_p^c is a finitely defined set, the result follows from the first part of this proof. \square

PROPOSITION 3.7. *A nonempty shift space $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ is never a finitely defined set of $\Sigma_A^{\mathbb{Z}}$.*

PROOF. Suppose that $\Lambda = X_{\mathbf{F}}$ for some $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$. To check whether some point $x \in \Sigma_A^{\mathbb{Z}}$ belongs or not to Λ , we need to check if any subblock of x belongs to \mathbf{F} , which we cannot do with only knowledge of $(x_{-k} \dots x_{\ell})$ for some $k, \ell \in \mathbb{N}$. \square

Using Proposition 3.6, we can now define a natural class of shift-commuting maps between two-sided shift spaces.

DEFINITION 3.8. Let A and B be alphabets and $\Lambda \subset \Sigma_A^{\mathbb{Z}}$ and $\Gamma \subset \Sigma_B^{\mathbb{Z}}$ be shift spaces. Suppose that $\{C_a\}_{a \in L_{\Gamma} \cup \{\emptyset\}}$ is a pairwise disjoint partition of Λ such that:

- (1) for each $a \in L_{\Gamma} \cup \{\emptyset\}$, the set C_a is finitely defined in Λ ; and
- (2) C_{\emptyset} is shift invariant (that is, $\sigma(C_{\emptyset}) \subset C_{\emptyset}$).

We will say that a map $\Phi : \Lambda \rightarrow \Gamma$ is a *sliding block code* if

$$(\Phi(x))_n = \sum_{a \in L_{\Gamma} \cup \{\emptyset\}} a \mathbf{1}_{C_a} \circ \sigma^n(x), \quad \forall x \in \Lambda, \forall n \in \mathbb{Z}, \tag{3.3}$$

where $\mathbf{1}_{C_a}$ is the characteristic function of the set C_a and \sum stands for the symbolic sum.

For a sliding block code Φ , let k_a and ℓ_a be the memory and anticipation of each C_a , respectively, and let $k := \sup_{a \in L_{\Gamma} \cup \{\emptyset\}} k_a$ and $\ell := \sup_{a \in L_{\Gamma} \cup \{\emptyset\}} \ell_a$. If $k, \ell < \infty$, we will say that Φ is a $(k + \ell + 1)$ -block code with memory k and anticipation ℓ , while, if $k = \infty$ or $\ell = \infty$, we will say that Φ has unbounded memory or anticipation, respectively.

Intuitively speaking, $\Phi : \Lambda \rightarrow \Gamma$ is a sliding block code if, for all $n \in \mathbb{Z}$ and $x \in \Lambda$, there exist $k, \ell \geq 0$ such that we only need to know $(x_{n-k} x_{n-k+1} \dots x_{n+\ell})$ to determine $(\Phi(x))_n$. We remark that k and ℓ above do not depend on the value of n , but do depend on the configuration of x around x_n .

We will also say that sliding block codes are maps given by a *local rule* (which is implicitly given by (3.3) and, in some cases, can be given explicitly). Also, note that the sets C_a in the definition of sliding block codes can be written as $C_a = \Phi^{-1}([a]_0^0)$,

where for each $a \in L_\Gamma \cup \{\emptyset\}$, $[a]_0^0$ is the pseudocylinder of Γ consisting of all elements that have the letter a at coordinate zero.

We now give some examples and nonexamples of sliding block codes.

EXAMPLE 3.9.

- (a) If Λ is any shift space, then the shift map $\sigma : \Lambda \rightarrow \Lambda$ is a continuous and invertible sliding block code with memory 0 and anticipation 1. Its local rule is given by

$$(\sigma(x))_n = \sum_{a \in L_\Lambda \cup \{\emptyset\}} a \mathbf{1}_{C_a} \circ \sigma^n(x),$$

where $C_a = \{x \in \Lambda : x_1 = a\} = \bigcup_{b \in L_\Lambda} [ba]_0^1$ for all $a \in L_\Lambda \cup \{\emptyset\}$.

- (b) Let $A = \mathbb{N}$ and let $\Phi : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{Z}}$ be given by

$$(\Phi(x))_n := \sup_{j \leq n} x_j \quad \text{for all } x \in \Sigma_A^{\mathbb{Z}}, \quad n \in \mathbb{Z},$$

with the convention that $\emptyset > a$ for all $a \in A$. It is immediate that Φ is not a sliding block code, but Φ is continuous and shift commuting.

- (c) Let $A = \mathbb{Z}$ and consider the shift space $\Lambda := \{(x_i)_{i \in \mathbb{Z}} \in \Sigma_A^{\mathbb{Z}} : x_{i+1} \geq x_i \forall i \in \mathbb{Z}\}$, with the convention that $\emptyset > a$ for all $a \in A$. Define $\Phi : \Lambda \rightarrow \Lambda$ by

$$(\Phi(x))_n := \begin{cases} \emptyset & \text{if } x_n = \emptyset, \\ x_{x_n} & \text{otherwise.} \end{cases}$$

Then Φ is a sliding block code with unbounded memory and anticipation.

- (d) Let $A = \mathbb{N}$ and define $\Phi : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_A^{\mathbb{Z}}$ by

$$(\Phi(x))_n := \begin{cases} (x_n + 1)/2 & \text{if } x_n \text{ is odd,} \\ x_n/2 & \text{if } x_n \text{ is even,} \\ \emptyset & \text{if } x_n = \emptyset. \end{cases}$$

Then Φ is a sliding block code which is onto but not one-to-one. Furthermore, Φ is not continuous. Indeed, consider the sequence $(x^i)_{i \geq 1}$, where $x_{-i}^i = 2$ and $x_n^i = 1$ if $n \neq -i$. Then x^i converges to \emptyset , but $\Phi(x^i) = (\dots 111 \dots)$ for all $i \geq 1$.

In [13, Theorems 3.16 and 3.17], we gave necessary and sufficient conditions under which one can obtain an analogue of the Curtis–Hedlund–Lyndon theorem for Ott–Tomforde–Willis one-sided shift spaces. However, due to the features of the topology assumed here for two-sided shift spaces of $\Sigma_A^{\mathbb{Z}}$, we cannot use the techniques of [13] to obtain a similar result. For instance, in the one-sided case the shift map is only continuous at \emptyset for column-finite shifts (which is explained by [13, Theorem 3.16]), while in the two-sided case the shift map is always continuous at \emptyset . Example 3.9(d) is another example which illustrates the fact that the Curtis–Hedlund–Lyndon theorems obtained in [13] cannot be applied for two-sided shift spaces. In fact, while for the one-sided case a sliding block code $\Phi : \Sigma_A^{\mathbb{N}} \rightarrow \Sigma_A^{\mathbb{N}}$ defined with the same rule as

Example 3.9(d) is continuous everywhere, for the two-sided case it is not continuous at \emptyset .

One can easily prove that sliding block codes are always shift commuting. In fact, the following results and their proofs are analogous to [13, Proposition 3.12 and Corollaries 3.13–3.15].

PROPOSITION 3.10. *Any sliding block code commutes with the shift map.*

COROLLARY 3.11. *If $\Phi : \Lambda \rightarrow \Gamma$ is a sliding block code and $x \in \Lambda$ is a sequence with period $p \geq 1$ (that is, $\sigma^p(x) = x$), then $\Phi(x)$ has also period p .*

COROLLARY 3.12. *If $\Phi : \Lambda \rightarrow \Gamma$ is a sliding block code, then $\Phi(\emptyset)$ is a constant sequence (that is, $\Phi(\emptyset) = (\dots ddd \dots)$ for some $d \in L_\Gamma \cup \{\emptyset\}$). Furthermore, if $\Phi(\emptyset) = \emptyset$, then the image of a finite sequence $x \in \Lambda^{\text{fin}}$ by Φ is a finite sequence of Γ .*

As alluded to already, in the finite-alphabet case a map between shift spaces is shift commuting and continuous if and only if it is a sliding block code—this is the Curtis–Lyndon–Hedlund theorem. The following theorems give sufficient conditions for our sliding block codes to be continuous.

THEOREM 3.13. *Let $\Phi : \Lambda \rightarrow \Gamma$ be a sliding block code such that $\Phi(\emptyset) = \emptyset$, and suppose that there exists $L \in \mathbb{Z}$ such that $\Phi = \sum_{a \in L_\Gamma \cup \{\emptyset\}} a \mathbf{1}_{C_a} \circ \sigma^L(x)$ and such that for each $a \in L_\Gamma$, $C_a = [c^a]_{k_a}^L$ for some $c^a_{-k_a+L+1} \neq \emptyset$. Then Φ is continuous. Additionally, if $C_\emptyset = \bigcup_{\substack{b \in B(\Lambda): \\ b_{-k_b+L+1} = \emptyset}} [b]_{k_b}^L$, then Φ is a homeomorphism onto its image.*

PROOF. Given $y = (y_n)_{n \in \mathbb{Z}} \in \Gamma$, for all $K \leq l(y)$,

$$\begin{aligned} \Phi^{-1}(Z((y_n)_{n \leq K}) \cap \Gamma) &= \Phi^{-1}\left(\bigcap_{n \leq K} [y_n]_n^n\right) = \bigcap_{n \leq K} \Phi^{-1}([y_n]_n^n) \\ &= \bigcap_{n \leq K} \Phi^{-1} \circ \sigma^{-n}([y_n]_0^0) = \bigcap_{n \leq K} \sigma^{-n} \circ \Phi^{-1}([y_n]_0^0) \\ &= \bigcap_{n \leq K} \sigma^{-n}(C_{y_n}) = \bigcap_{n \leq K} \sigma^{-n}([c^{y_n}]_{k_{y_n}}^L) \\ &= \bigcap_{n \leq K} [c^{y_n}]_{k_{y_n}+n}^{L+n}. \end{aligned} \tag{3.4}$$

Thus, $\Phi^{-1}(Z((y_n)_{n \leq K}) \cap \Gamma)$ is a clopen set since it is either the empty set or a generalized cylinder defined on the coordinates less than or equal to $K + L$. Furthermore, if $y^j \in \Gamma$ and $K_j \leq l(y^j)$ for $j = 1, \dots, N$, then

$$\begin{aligned} \Phi^{-1}\left(\left[\bigcup_{j=1}^N Z((y_n^j)_{n \leq K_j}) \cap \Gamma\right]^c\right) &= \Phi^{-1}\left(\bigcap_{j=1}^N [Z((y_n^j)_{n \leq K_j}) \cap \Gamma]^c\right) \\ &= \bigcap_{j=1}^N [\Phi^{-1}(Z((y_n^j)_{n \leq K_j}) \cap \Gamma)]^c \end{aligned}$$

and, since each $\Phi^{-1}(Z((y_n^j)_{n \leq K_j}) \cap \Gamma)$ is a clopen set, we have that $\Phi^{-1}([\bigcup_{j=1}^N Z((y_n^j)_{n \leq K_j}) \cap \Gamma]^c)$ is also a clopen set.

To finish, we observe that for any finite set $F \subset A$, given a cylinder of the form $Z((y_n)_{n \leq K}, F) \cap \Gamma$,

$$Z((y_n)_{n \leq K}, F) \cap \Gamma = Z((y_n)_{n \leq K}) \cap \left[\bigcup_{f \in F} Z(\dots, y_{K-1}, y_K, f) \right]^c,$$

which implies that its inverse image by Φ is also a clopen set. Hence, we conclude that Φ is continuous.

Now suppose additionally that $C_\emptyset = \bigcup_{\substack{b \in B(\Lambda): \\ b_{-k_b+L+1} = \emptyset}} [b]_{k_b}^L$. Since for all $y \in \Phi(\Lambda)$ we have that $\{y\} = \bigcap_{n \in \mathbb{Z}} [y_n]_n^n$, then, by a computation similar to (3.4), $\Phi^{-1}(y)$ will be a singleton. Thus, Φ is invertible on its image and, since Λ and $\Phi(\Lambda)$ are compact, Φ is a homeomorphism. \square

3.2. Higher block presentation of a shift space. Another standard construction in symbolic dynamics is the higher block presentation, where a subshift is recoded to another shift over the alphabet of all possible blocks of a given length. In this section we adapt this construction to our shift spaces.

DEFINITION 3.14. Given $\Sigma_A^{\mathbb{Z}}$ and $M \in \mathbb{N}$, denote $A^{[M]} := B_M(\Sigma_A^{\mathbb{Z}})$. The M th higher block code is the map $\Xi^{[M]} : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_{A^{[M]}}^{\mathbb{Z}}$ given, for all $x \in \Sigma_A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, by

$$(\Xi^{[M]}(x))_i := \begin{cases} \emptyset & \text{if } x_i = \emptyset, \\ [x_{i-M+1} \dots x_i] & \text{if } x_i \neq \emptyset. \end{cases}$$

It is immediate that a higher block code is an invertible M -block code with anticipation 0. We now show that the M th higher block code is continuous, injective and sends shift spaces to shift spaces.

PROPOSITION 3.15. Let $\Lambda \subset \Sigma_A^{\mathbb{Z}}$ be a shift space and let $\Xi^{[M]} : \Sigma_A^{\mathbb{Z}} \rightarrow \Sigma_{A^{[M]}}^{\mathbb{Z}}$ be the M th higher block code. Then:

- (i) $\Xi^{[M]}|_{\Lambda} : \Lambda \rightarrow \Xi^{[M]}(\Lambda)$ is continuous;
- (ii) $\Xi^{[M]}|_{\Lambda} : \Lambda \rightarrow \Xi^{[M]}(\Lambda)$ is invertible and its inverse is a one-block code given by

$$(y_i)_{i \in \mathbb{Z}} \mapsto (\tilde{x}_i)_{i \in \mathbb{Z}},$$

where $\tilde{x}_i = x_i$ if $y_i = [x_{i-M+1} \dots x_i]$ and $\tilde{x}_i = \emptyset$ if $y_i = \emptyset$;

- (iii) $\Xi^{[M]}(\Lambda)$ is a shift space in $\Sigma_{A^{[M]}}^{\mathbb{Z}}$.

PROOF. The statement (i) follows from the fact that Φ is a sliding block code which satisfies the properties of Theorem 3.13, while (ii) is direct from the definition of $\Xi^{[M]}$.

To prove (iii), we notice that, since Φ is a continuous sliding block code, we have that $\Xi^{[M]}(\Lambda)$ is closed in $\Sigma_{A^{[M]}}^{\mathbb{Z}}$ and invariant under σ . Thus, we only need to show that $\Xi^{[M]}(\Lambda)$ satisfies the infinite extension property.

Observe that if $y = ([b_{i-M+1} \dots b_i])_{i \leq L} \in \Xi^{[M]}(\Lambda)^{\text{fin}}$, then it is necessarily the image of some $b = (b_i)_{i \leq L} \in \Lambda^{\text{fin}}$. Since Λ has the infinite extension property, there exists a sequence $x_n = (x_i^n)_{i \in \mathbb{Z}}$ in Λ such that $x_i^n = b_i$, for all $n \geq 1$ and $i \leq L$, and $x_{L+1}^n \neq x_{L+1}^m$ if $n \neq m$ (that is, x_n is a sequence which converges to b).

Define $y_n = (y_i^n)_{i \in \mathbb{Z}} := \Xi^{[M]}(x^n)$. It is clear that $y_n \in \Lambda^{[M]}$. Note that for all $n \geq 1$ and $i \leq L$ we have that $y_i^n = [b_{i-M+1} \dots b_i] = y_i$, and $y_{L+1}^n = [b_{L-M+2} \dots x_{L+1}^n]$. Hence, since $x_{L+1}^n \neq x_{L+1}^m$ if $n \neq m$, it follows that $y_{L+1}^n = [b_{L-M+2} \dots x_{L+1}^n] \neq [b_{L-M+2} \dots x_{L+1}^m] = y_{L+1}^m$ if $m \neq n$, which means that y_n converges to y . Since this holds for any sequence of $\Xi^{[M]}(\Lambda)^{\text{fin}}$, $\Xi^{[M]}(\Lambda)$ satisfies the infinite extension property. \square

Given a shift space $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$, we will call $\Lambda^{[M]} := \Xi^{[M]}(\Lambda)$ its *Mth higher block shift*. Proposition 3.15 above means that any shift space Λ is always topologically conjugate to its *Mth higher block shift* for any $M \geq 2$.

Given a left-infinite block $w = (w_i)_{i \leq 0}$, define

$$\Xi^{[M]}(w) := ([w_{i-M+1} \dots w_i])_{i \leq 0}$$

and, given a finite block $w = (w_i)_{1 \leq i \leq k}$ with $k \geq M$, define

$$\Xi^{[M]}(w) := ([w_i \dots w_{i+M-1}])_{1 \leq i \leq k-M+1}.$$

The next result shows that a higher block presentation of a shift given by a set of forbidden words \mathbf{F} can be given by a set of forbidden words derived from \mathbf{F} , and is proved using the same argument seen in [19, Proposition 1.4.3].

COROLLARY 3.16. *Given a shift space $\Lambda \subset \Sigma_A^{\mathbb{Z}}$, let $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq M} A^k$ be any set such that $\Lambda = X_{\mathbf{F}}$. Then $\Lambda^{[M]} = X_{\mathbf{F}^{[M]}}$, where $\mathbf{F}^{[M]} \subset (A^M)^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} (A^M)^k$ is given by*

$$\mathbf{F}^{[M]} := \{([u_i \dots u_M], [v_1 \dots v_M]) \in (A^M)^2 : u_i \neq v_{i-1}, 2 \leq i \leq M\} \cup \dots \\ \dots \cup \{\Xi^{[M]}(w) : w \in \mathbf{F}\}.$$

We now show that, in contrast to the finite-alphabet case, taking a higher block presentation never results in a shift of finite type.

PROPOSITION 3.17. *Let $\Lambda \subset \Sigma_A^{\mathbb{Z}}$ be a shift space with $|\Lambda| = \infty$. Then, for any $M \geq 2$, the *Mth higher block shift* of Λ is never a shift of finite type.*

PROOF. This follows from the fact that $\Lambda^{[M]} = X_{\mathbf{F}}$ for some \mathbf{F} , which must contain all blocks of length 2 of the form

$$((a_1 \dots a_M)(b_1 \dots b_M)),$$

where $a_{i+1} \neq b_i$ for all $i = 1, \dots, M - 1$. \square

The following results are consequences of Proposition 3.15 and their proofs are analogous to the proofs in the finite-alphabet case (see [19]). We sketch only the proof for Proposition 3.19.

PROPOSITION 3.18. *Let $\Lambda \subset \Sigma_A^{\mathbb{Z}}$ be a shift space and let $\Lambda^{[M]}$ be its M th higher block shift.*

- (i) *If Λ is a column-finite (or row-finite) shift, then $\Lambda^{[M]}$ is also a column-finite (or row-finite) shift.*
- (ii) *If Λ is a K -step shift, then $\Lambda^{[M]}$ is an L -step shift, where $L = \max\{1, K - M + 1\}$.*

PROPOSITION 3.19. *Any M -step shift of $\Sigma_A^{\mathbb{Z}}$ is topologically conjugate to some edge shift.*

PROOF. Let Γ be an M -step shift in $\Sigma_A^{\mathbb{Z}}$. If $M = 0$, then Γ is $\Sigma_B^{\mathbb{Z}}$ for some $B \subset A$, which is the edge shift over the graph with one vertex and with edge set equal to B . So, suppose that $M \geq 0$ and, as in [19, Theorem 2.3.2], let $E = B_M(\Gamma)$ and $V = B_{M+1}(\Gamma)$. Furthermore, define maps t and i from V to E by

$$i(v_1v_2 \cdots v_{M+1}) = v_1v_2 \cdots v_M, \quad t(v_1v_2 \cdots v_{M+1}) = v_2v_3 \cdots v_{M+1}.$$

If we let $G = (V, E, t, i)$, then it is clear that $\Gamma^{[M]} = \Lambda(G)$ from (2.3). Hence, by Proposition 3.15, Γ is topologically conjugate to an edge shift. □

Note that Propositions 3.18 and 3.19 say that M -step shifts can always be recoded as L -step shifts for any $L < M$ and as an edge shift in the same way that occurs in the classical theory for shift spaces over finite alphabets. These results establish a remarkable difference with respect to the theory of Ott–Tomforde–Willis one-sided shift spaces [21]. In fact, if one uses the definition of sliding block codes given in [21, Definition 7.1], then it is only possible to prove a version of Proposition 3.15 if Λ is a row-finite shift (see [21, Proposition 6.5]). In such a case Propositions 3.18 and 3.19 will also hold only for row-finite shifts (see [21, Propositions 6.2 and 6.6]). In fact, [21, Example 5.18] presents an example of non-row-finite Ott–Tomforde–Willis one-sided one-step shift spaces that are not conjugate to edge shifts, while [11] presents a non-row-finite Ott–Tomforde–Willis one-sided $(M + 1)$ -step shift space that is not conjugate to any M -step shift (both using [21, Definition 4.1] for conjugacy, which imposes that the length of sequences shall be preserved).

4. Relationships between two-sided and one-sided shift spaces

In this section we flesh out the relationship between two-sided shift space and Ott–Tomforde–Willis one-sided shift spaces hinted at in Remark 2.5 and Proposition 2.11. We begin by showing that shift spaces can be given by a set of words that is *minimal* in a suitable sense.

LEMMA 4.1. *Let $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$ and let $X_{\mathbf{F}}^{\text{inf}} \subset A^{\mathbb{Z}}$ be as in (2.2). If $w \in \bigcup_{k \geq 1} A^k$ does not belong to $B(X_{\mathbf{F}}^{\text{inf}})$, then there exists a subblock u of w such that:*

- (1) $u \notin B(X_{\mathbf{F}}^{\text{inf}})$;
- (2) we have that $v \in B(X_{\mathbf{F}}^{\text{inf}})$ for all proper subblocks v of u ; and
- (3) u is a subblock of some word in \mathbf{F} .

PROOF. Recall that $X_{\mathbf{F}}^{\text{inf}}$ consists of all elements in $A^{\mathbb{Z}}$ which do not contain a subblock which is an element of \mathbf{F} . Suppose that w is a finite block which is not in $B(X_{\mathbf{F}}^{\text{inf}})$. Let u be a subblock of w such that u is not in $B(X_{\mathbf{F}}^{\text{inf}})$ and such that all proper subwords of u belong to $B(X_{\mathbf{F}}^{\text{inf}})$ (note that u is not necessarily unique, and it is possible that $u = w$).

If u is a single letter, say $u = (a)$, then the letter a does not appear in any element of $X_{\mathbf{F}}^{\text{inf}}$. But the only way that this can happen is if there is a word in \mathbf{F} which contains a , and we are done.

So, suppose that u is not a single letter and let $M > 1$ be the length of u . Let $\mathbf{G} := \mathbf{G}_1 \cup \mathbf{G}_2$, where

$$\mathbf{G}_1 := \{v \in A^M : (v_1 \dots v_k) \in \mathbf{F} \text{ for some } 1 \leq k < M\},$$

$$\mathbf{G}_2 := \mathbf{F} \cap \bigcup_{k \geq M} A^k.$$

It is direct that $X_{\mathbf{G}}^{\text{inf}} = X_{\mathbf{F}}^{\text{inf}}$. Furthermore, u is not a prefix of any word in \mathbf{G}_1 , because that would imply existence of a proper subword of u in \mathbf{F} , contradicting that any proper subword of u belongs to $B(X_{\mathbf{F}}^{\text{inf}})$. Since every element of \mathbf{G} has length greater than or equal to M , it follows from Corollary 3.16 that $X_{\mathbf{F}}^{\text{inf}[M]} := \Xi^{[M]}(X_{\mathbf{F}}^{\text{inf}}) = X_{\mathbf{G}^{[M]}}^{\text{inf}}$.

Now $[u] := \Xi^{[M]}(u)$ is a letter which does not appear in any element of $X_{\mathbf{F}}^{\text{inf}[M]}$, so as before $[u]$ is a subword of some word of $\mathbf{G}^{[M]}$ (possibly $[u]$ is itself in $\mathbf{G}^{[M]}$). But this means that u is a subblock of some element of \mathbf{G} . Since \mathbf{G}_1 does not contain any block with u as subblock, u appears as a subword of at least one element of $\mathbf{G}_2 \subset \mathbf{F}$. \square

Note that the above lemma is also true if we replace $X_{\mathbf{F}}^{\text{inf}}$ with the set of the infinite-length elements of an Ott–Tomforde–Willis one-sided shift space (defining $\Xi^{[M]}$ conveniently) and even if we replace it with a classical shift space over a finite alphabet.

We now define the notion of a minimal set of forbidden words and show that, for shifts given by a minimal set of forbidden words, the language of the set of infinite sequences in the shift does not depend on the left-infinite forbidden words (that is, does not depend on left-infinite restrictions).

DEFINITION 4.2. Let $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$. We say that \mathbf{F} is *minimal* if whenever $w \in \mathbf{F}$ and $u \in \bigcup_{k \geq 1} A^k$ is a proper finite subblock of w , then $u \in B(X_{\mathbf{F}})$.

We now show that every shift space can be defined by a minimal set of forbidden words.

PROPOSITION 4.3. *Let Λ be a shift space. Then we can find a minimal set \mathbf{F} such that $\Lambda = X_{\mathbf{F}}$.*

PROOF. By Proposition 2.25, we can find $\mathbf{G} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$ such that $\Lambda = X_{\mathbf{G}}$. We now apply Lemma 4.1 and define \mathbf{F} as the set of all words $u \in A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$ such that either $u \in \mathbf{G}$ and all finite proper subblocks of u belong to $B(\Lambda)$, or u is a finite proper subblock of some $w \in \mathbf{G}$ and all proper subblocks of u are in $B(\Lambda)$. \square

We note that if $\Lambda = X_{\mathbf{F}}$ is a shift space and \mathbf{F} is minimal, then it is not possible to replace restrictions given by left-infinite sequences by restrictions given only by finite words.

We now give examples and nonexamples of minimal sets.

EXAMPLE 4.4. In the following examples, we write our set of forbidden words \mathbf{F} as a union $\mathbf{F} = \mathbf{F}' \cup \mathbf{F}''$, with $\mathbf{F}' \subset \bigcup_{k \geq 1} A^k$ and $\mathbf{F}'' \subset A^{\mathbb{Z}^-}$.

- (a) Let $A = \mathbb{N}$, let $\mathbf{F}' = \emptyset$ and let

$$\mathbf{F}'' = \{(x_i)_{i \leq 0} \in A^{\mathbb{Z}^-} : \inf\{i : x_i \neq 1\} = -\infty\}.$$

Then $\Lambda := X_{\mathbf{F}}$ is a two-sided shift space with $B(\Lambda) = B(\Sigma_A^{\mathbb{Z}})$ such that, for any $x \in \Lambda$, there exists $k \in \mathbb{Z}$ such that $x_i = 1$ for all $i \leq k$. Clearly, \mathbf{F} is minimal and, furthermore, it is not possible to replace the restrictions given by \mathbf{F}'' by restrictions given only by finite words.

- (b) Take $k \in \mathbb{N}$, let $A = \mathbb{N}$, let $\mathbf{F}' = \{11\}$, let

$$\mathbf{F}'' = \{(x_i)_{i \leq 0} \in A^{\mathbb{Z}^-} : x_{i-k} \neq x_i \ \forall i \leq 0\}$$

and let $\Lambda := X_{\mathbf{F}}$. Then $\Lambda^{\text{fin}} = \{\emptyset\}$ and

$$\Lambda^{\text{inf}} = \{x \in \Sigma_A^{\mathbb{Z}} : x \text{ has period } k, \text{ and } x_i x_{i+1} \neq 11 \ \forall i \in \mathbb{Z}\}.$$

It follows that \mathbf{F} is not minimal, since any word of the form $u = u_1 \dots u_{k+1}$ with $u_1 \neq u_{k+1}$ is a subblock of some sequence of \mathbf{F}'' , but $u \notin B(\Lambda)$. However, the set

$$\mathbf{G} := \{11, u_1 \dots u_{k+1} : u_1 \neq u_{k+1}\}$$

is minimal and $\Lambda = X_{\mathbf{G}}$.

- (c) Let $A = \mathbb{N}$, let $\mathbf{F}' = \emptyset$, let

$$\mathbf{F}'' = \{(x_i)_{i \leq 0} \in A^{\mathbb{Z}^-} : \exists k \geq 1, x_{i-k} \neq x_i \ \forall i \leq 0\}$$

and let $\Lambda := X_{\mathbf{F}}$. Then $\Lambda^{\text{fin}} = \{\emptyset\}$ and $\Lambda^{\text{inf}} = \{x \in \Sigma_A^{\mathbb{Z}} : x \text{ is periodic}\}$. Since $B(\Lambda) = B(\Sigma_A^{\mathbb{Z}})$, \mathbf{F} must be minimal.

- (d) Let $A = \mathbb{N}$ and let

$$\begin{aligned} \mathbf{F}' &= \{n2 : n \in A\}, \\ \mathbf{F}'' &= \{(x_i)_{i \leq 0} \in A^{\mathbb{Z}^-} : \exists i \leq 0, x_i = 1\}. \end{aligned}$$

Here, \mathbf{F} is not minimal. In fact, 2 is a proper subblock of all words in \mathbf{F}' and it does not belong to $B(X_{\mathbf{F}})$ and, on the other hand, 1 is a proper subblock of all words in \mathbf{F}'' and it does not belong to $B(X_{\mathbf{F}})$. Here, the set $\mathbf{G} = \{1, 2\}$ is a minimal set such that $X_{\mathbf{F}} = X_{\mathbf{G}} = \Sigma_{A \setminus \{1,2\}}^{\mathbb{Z}}$.

We now show that if \mathbf{F} is minimal, then the language of $X_{\mathbf{F}}$ is not affected by the left-infinite words in \mathbf{F} .

PROPOSITION 4.5. *Let $\mathbf{F} \subset A^{\mathbb{Z}^-} \cup \bigcup_{k \geq 1} A^k$ be a minimal set, where $\mathbf{F} = \mathbf{F}' \cup \mathbf{F}''$ with $\mathbf{F}' \subset \bigcup_{k \geq 1} A^k$ and $\mathbf{F}'' \subset A^{\mathbb{Z}^-}$. Then $B(X_{\mathbf{F}}^{\text{inf}}) = B(X_{\mathbf{F}'}^{\text{inf}})$, that is, the restrictions imposed by \mathbf{F}'' do not affect the language of $X_{\mathbf{F}}^{\text{inf}}$.*

PROOF. Since $\mathbf{F}' \subset \mathbf{F}$, it is straightforward that $B(X_{\mathbf{F}}^{\text{inf}}) \subset B(X_{\mathbf{F}'}^{\text{inf}})$. To prove that $B(X_{\mathbf{F}'}^{\text{inf}}) \subset B(X_{\mathbf{F}}^{\text{inf}})$, let $w \in B(X_{\mathbf{F}'}^{\text{inf}})$ and suppose by contradiction that $w \notin B(X_{\mathbf{F}}^{\text{inf}})$. Lemma 4.1 implies that there exists a subword u of w such that $u \notin B(X_{\mathbf{F}}^{\text{inf}})$ and u is a subword of some word in \mathbf{F} . However, since \mathbf{F} is minimal, we have that u is in \mathbf{F} and, since u is finite, then $u \in \mathbf{F}'$, which contradicts the fact that $w \in B(X_{\mathbf{F}'}^{\text{inf}})$. Hence, $B(X_{\mathbf{F}'}^{\text{inf}}) \subset B(X_{\mathbf{F}}^{\text{inf}})$. \square

The following result gives a relationship between two-sided shift spaces defined in this work and Ott–Tomforde–Willis one-sided shift spaces.

PROPOSITION 4.6. *Let $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ be a shift space such that $\Lambda = X_{\mathbf{F}}$, where \mathbf{F} is minimal. Let $\mathbf{F} = \mathbf{F}' \cup \mathbf{F}''$ with $\mathbf{F}' \subset \bigcup_{k \geq 1} A^k$, $\mathbf{F}'' \subset A^{\mathbb{Z}^-}$ and let $\pi : \Lambda \rightarrow \Sigma_A^{\mathbb{N}}$ be the projection onto the positive coordinates.*

- (i) *If $|L_{\Lambda}| = \infty$, then $\pi(\Lambda)$ is dense in the one-sided Ott–Tomforde–Willis shift space $\hat{X}_{\mathbf{F}'} \subset \Sigma_A^{\mathbb{N}}$.*
- (ii) *If $|L_{\Lambda}| < \infty$, then $\pi(\Lambda \setminus \{\emptyset\})$ is the standard shift space of $\hat{X}_{\mathbf{F}'} \subset (L_{\Lambda})^{\mathbb{N}}$.*

PROOF.

- (i) We will prove that $\pi(\Lambda^{\text{inf}})$ is dense in $\hat{X}_{\mathbf{F}'}^{\text{inf}}$ and that any element of $\pi(\Lambda^{\text{fin}})$ is a finite sequence which satisfies the infinite extension property in $\hat{X}_{\mathbf{F}'}$. Hence,

$$\overline{\pi(\Lambda)} = \overline{\pi(\Lambda^{\text{inf}})} = \overline{\hat{X}_{\mathbf{F}'}^{\text{inf}}} = \hat{X}_{\mathbf{F}'}$$

For this, first note that

$$B(\pi(\Lambda^{\text{inf}})) = B(\Lambda^{\text{inf}}) = B(X_{\mathbf{F}}^{\text{inf}}) = B(\hat{X}_{\mathbf{F}'}^{\text{inf}}),$$

where the first equality is due to $\sigma(\Lambda) = \Lambda$, the second equality is due to Proposition 4.5 and the third equality is due the definitions of $X_{\mathbf{F}}^{\text{inf}}$ and $\hat{X}_{\mathbf{F}'}^{\text{inf}}$.

The above means that $\pi(\Lambda^{\text{inf}}) \subset \hat{X}_{\mathbf{F}'}^{\text{inf}}$. Now take $x \in \hat{X}_{\mathbf{F}'}^{\text{inf}}$ and find y such that $x \in Z(y)$. It follows that $y \in B(\hat{X}_{\mathbf{F}'}^{\text{inf}}) = B(\pi(\Lambda^{\text{inf}}))$ and, therefore, there exists $z \in \pi(\Lambda^{\text{inf}})$ such that $z_i = x_i = y_i$ for all $1 \leq i \leq l(y)$. Hence, $z \in Z(y) \cap \pi(\Lambda^{\text{inf}})$, proving that $\pi(\Lambda^{\text{inf}})$ is dense in $\hat{X}_{\mathbf{F}'}^{\text{inf}}$.

Now observe that $\pi(x) = \emptyset$ whenever $l(x) \leq 0$, and π is continuous at $x \in \Lambda^{\text{fin}}$ whenever $l(x) \geq 0$ by Proposition 2.11. The last statement implies that any finite sequence $\pi(x) \in \pi(\Lambda)$ will have the infinite extension property.

- (ii) In this case $\Lambda^{\text{fin}} \subset \{\emptyset\}$. Therefore, $\Lambda^{\text{inf}} = \Lambda \setminus \{\emptyset\}$ and the result follows from the fact that $\pi(\Lambda^{\text{inf}})$ is dense in $\hat{X}_{\mathbf{F}'}^{\text{inf}}$. \square

We apply Proposition 4.6 to each of the examples from Example 4.4.

EXAMPLE 4.7.

- (a) Let $\Lambda = X_{\mathbf{F}}$, where \mathbf{F} is the set given in Example 4.4(a). We have that $L_{\Lambda} = A$ and $\pi(\Lambda) = \hat{X}_{\mathbf{F}'} = \Sigma_A^{\mathbb{N}}$.
- (b) Let \mathbf{F} and \mathbf{G} be as in Example 4.4(b), and let $\Lambda = X_{\mathbf{F}}$. Then $L_{\Lambda} = A$ and $\pi(\Lambda)$ is the set consisting of \emptyset together with all infinite-length elements of $\Sigma_A^{\mathbb{N}}$ with period k which do not contain the word 11. Hence, the closure of $\pi(\Lambda)$ is $\hat{X}_{\mathbf{F}' \cup \mathbf{G}} \subset \Sigma_A^{\mathbb{N}}$, which, aside from the elements of $\pi(\Lambda)$, contains all finite sequences with period k where the word 11 does not appear.
- (c) Let $\Lambda = X_{\mathbf{F}}$, where \mathbf{F} is the set given in Example 4.4(c). Then $L_{\Lambda} = A$, and $\pi(\Lambda) = \{\emptyset\} \cup \{x \in \Sigma_A^{\mathbb{N}} : x \text{ is periodic}\}$. The closure of $\pi(\Lambda)$ is $\hat{X}_{\mathbf{F}'} = \Sigma_A^{\mathbb{N}}$.
- (d) Let $\Lambda = X_{\mathbf{F}}$, where \mathbf{F} is the set given in Example 4.4(d). Here, $L_{\Lambda} = A \setminus \{1, 2\}$ and $\pi(\Lambda) = \hat{X}_{\mathbf{F}'} = \Sigma_A^{\mathbb{N}}$.

Proposition 4.6 gives us a way of obtaining an Ott–Tomforde–Willis one-sided shift space from a two-sided shift space. We now use Remark 2.5 to construct a two-sided shift space from a one-sided shift space, although in general these two operations are not inverses of each other; see Remarks 4.9 and 4.10.

PROPOSITION 4.8. *Let $\mathbf{F} \subset \bigcup_{k \in \mathbb{N}} A^k$ be a minimal set and consider an Ott–Tomforde–Willis one-sided shift $\hat{X}_{\mathbf{F}}$. Consider the set obtained by taking the inverse limit of $\hat{X}_{\mathbf{F}}$,*

$$(\hat{X}_{\mathbf{F}})^{\sigma} := \{(\mathcal{X}_i)_{i \in \mathbb{Z}} : \forall i \in \mathbb{Z} \mathcal{X}_i \in \hat{X}_{\mathbf{F}} \text{ and } \sigma(\mathcal{X}_i) = \mathcal{X}_{i+1}\}$$

and define $\Lambda \in \Sigma_A^{\mathbb{Z}}$ by

$$\Lambda := p((\hat{X}_{\mathbf{F}})^{\sigma}) = \{(x_i)_{i \in \mathbb{Z}} \in \Sigma_A^{\mathbb{Z}} : (x_{i+j-1})_{j \in \mathbb{N}} \in \Lambda \forall i \in \mathbb{Z}\},$$

where p is the map defined in Remark 2.5.

- (i) If either $|L_{\hat{X}_{\mathbf{F}}}| = \infty$ or $|\hat{X}_{\mathbf{F}}| < \infty$, then $\Lambda = X_{\mathbf{F}}$.
- (ii) If $|L_{\hat{X}_{\mathbf{F}}}| < \infty$ and $|\hat{X}_{\mathbf{F}}| = \infty$, then $\Lambda \cup \{\emptyset\} = X_{\mathbf{F}}$.

PROOF. First, let us prove that in both cases we have $\Lambda^{\text{inf}} = X_{\mathbf{F}}^{\text{inf}}$. For this, observe that $\Lambda^{\text{inf}} = p((\Lambda^{\text{inf}})^{\sigma})$. From the definition, elements of Λ^{inf} do not contain any elements of \mathbf{F} as subblocks, which means that $\Lambda^{\text{inf}} \subset X_{\mathbf{F}}^{\text{inf}}$. On the other hand, by Proposition 4.6, it follows that $\pi(X_{\mathbf{F}}^{\text{inf}}) \subset \hat{X}_{\mathbf{F}}^{\text{inf}}$, which means that given $(x_i)_{i \in \mathbb{Z}} \in X_{\mathbf{F}}^{\text{inf}}$, we have that $\mathcal{X}_i = (x_{i+j-1})_{j \in \mathbb{N}} \in \hat{X}_{\mathbf{F}}^{\text{inf}}$ for all $i \in \mathbb{Z}$. Hence, we get that $(x_i)_{i \in \mathbb{Z}} \in \Lambda$ and therefore $X_{\mathbf{F}}^{\text{inf}} \subset \Lambda^{\text{inf}}$.

Before we proceed, note that $L_{\Lambda} = L_{X_{\mathbf{F}}} = L_{\hat{X}_{\mathbf{F}}}$. In fact, since $\Lambda^{\text{inf}} = X_{\mathbf{F}}^{\text{inf}}$, it is straightforward that $L_{\Lambda} = L_{X_{\mathbf{F}}}$. Furthermore, from the definition of Λ , it follows that $L_{\Lambda} \subset L_{\hat{X}_{\mathbf{F}}}$. On the other hand, if by contradiction we suppose that $a \in L_{\Lambda} \setminus L_{\Lambda}$, then there does not exist an element in Λ^{inf} using the letter a . But $\Lambda^{\text{inf}} = X_{\mathbf{F}}^{\text{inf}}$ and, since \mathbf{F} is minimal, then necessarily $a \in \mathbf{F}$, which contradicts that $a \in L_{\hat{X}_{\mathbf{F}}}$.

Now, to complete the proof, we have two cases.

- (i) If $|L_{\hat{X}_F}| = \infty$, then the one-sided empty sequence belongs to \hat{X}_F and therefore the two-sided empty sequence belongs to Λ . Since $L_{X_F} = L_{\hat{X}_F}$, this implies that the two-sided empty sequence belongs to X_F .
 As before, we have that $\Lambda^{\text{fin}} \subset X_F^{\text{fin}}$. On the other hand, to check that $X_F^{\text{fin}} \subset \Lambda^{\text{fin}}$, we take $x \in X_F^{\text{fin}} \setminus \{\emptyset\}$ and use the same argument used for infinite sequences.
 If $|\hat{X}_F| < \infty$, then $|L_{\hat{X}_F}| < \infty$ and hence \hat{X}'_F consists of a finite number of periodic infinite sequences, which implies that \hat{X}_F^{fin} is empty. Therefore, Λ consists only of a finite number of periodic infinite sequences and hence Λ^{fin} is empty. Since $\Lambda^{\text{inf}} = X_F^{\text{inf}}$, we have that X_F^{inf} contains just a finite number of periodic infinite sequences, concluding that X_F^{fin} is also empty.
- (ii) Suppose that $|L_{\hat{X}_F}| < \infty$ and $|\hat{X}_F| = \infty$. Since $|L_{\hat{X}_F}| < \infty$, it follows that \hat{X}_F is a classical shift space over a finite alphabet, which implies that \hat{X}_F^{fin} is empty. Hence, $\Lambda^{\text{inf}} = X_F^{\text{inf}}$ and contains an infinite number of infinite sequences over a finite alphabet. Since Λ^{fin} is empty and $X_F^{\text{fin}} = \{\emptyset\}$, the result follows. \square

So, using Proposition 4.8, we can produce a two-sided shift space from an Ott–Tomforde–Willis one-sided shift space, and using Proposition 4.6 we can produce a one-sided shift space by projecting a two-sided shift space. The following two remarks show that these two operations are not necessarily the inverses of each other.

REMARK 4.9. Given a shift space $\Lambda \subseteq \Sigma_A^{\mathbb{Z}}$ with $|L_\Lambda| = \infty$, take Λ' as the closure of $\pi(\Lambda)$ in $\Sigma_A^{\mathbb{N}}$. Then, in general, $p((\Lambda')^\sigma) \neq \Lambda$. To see this, suppose that $\Lambda = X_F$, where $F = F' \cup F''$ is minimal and $F'' \subset A^{\mathbb{Z}^-}$ nonempty. By Proposition 4.8(i), $\Lambda' = \hat{X}_{F'}$, so we have $p((\Lambda')^\sigma) = X_{F'}$ and $\Lambda \subsetneq X_{F'}$.

REMARK 4.10. Given a one-sided shift space $\Lambda' \subset \Sigma_A^{\mathbb{N}}$, in general $\pi(p((\Lambda')^\sigma)) \neq \Lambda'$. For instance, take $A = \mathbb{N}$ and the nonminimal set $F = \{n1 : n \in A\}$, so that $\Lambda' := \hat{X}_F$ is the one-sided shift space of all sequences where 1 does not appear, except maybe as the first symbol. Hence, $p((\Lambda')^\sigma)$ is the two-sided full shift over the alphabet $B := A \setminus \{1\}$ and, therefore, $\pi(p((\Lambda')^\sigma)) = \Sigma_B^{\mathbb{N}} \subsetneq \Lambda'$.

5. Final discussion

In this work we have proposed a definition of two-sided shift spaces over countably infinite alphabets. The topology we proposed can be viewed as a two-sided version of the space proposed by Ott, Tomforde and Willis [21] for one-sided shift spaces over infinite alphabets.

Although the set of sequences proposed here as the two-sided shift space can be viewed as the inverse-limit system of an Ott–Tomforde–Willis one-sided shift space, we remark that the topology considered was not the product topology (which is the usual topology for inverse-limit systems). We made this choice because the product topology does not have a basis of clopen sets. Thus, instead of the product topology, we considered a topology with a basis consisting of sets defined by specifying infinitely many coordinates and by the complement of such sets.

To contrast the two, notice that the product topology contains sets of the form

$$\{x \in \Sigma_A^{\mathbb{Z}} : x_i = a_i \forall i \in I\}$$

and

$$\{x \in \Sigma_A^{\mathbb{Z}} : x_i \neq a_i \forall i \in I\}$$

for any choice of finite $I \subset \mathbb{Z}$ and $\{a_i\}_{i \in I} \subset A$. On the other hand, the topology we considered in this work contains only sets of the first type.

The key features to note about the shift spaces we introduced here are that they are zero-dimensional, Hausdorff, compact and sequentially compact spaces. Furthermore, the shift map is continuous with respect to this topology (while it was not continuous for Ott–Tomforde–Willis one-sided shift spaces). These features were key for us to generalize work of Kitchens [17] to inverse semigroup shifts over infinite alphabets [14]. It remains open whether the locally compact space obtained by excluding the empty sequence is metrizable.

We also note that because our clopen basis keeps track of an infinite number of entries, the dynamical recurrence properties of the shift map which hold for finite-alphabet shift spaces and for Ott–Tomforde–Willis one-sided shift spaces do not hold in our case. Indeed, in those situations the full shift is topologically chaotic (the set of its periodic orbits is dense, and it is topologically transitive—see [1]) while in our case the empty sequence in the full two-sided shift is an attractor for all nonperiodic orbits. However, we remark that this apparent simplicity of the global dynamics hides the complexity of the local behavior. This can be visualized by taking the projection of the two-sided shift on the positive coordinates, which will result in a dense subset of the Ott–Tomforde–Willis one-sided shift space (which is topologically chaotic).

Interestingly, the problem of finding general conditions to characterize the existence of an analogue of the Curtis–Hedlund–Lyndon theorem remains open. In fact, the class of maps for which we could expect to find a complete characterization in terms of a Curtis–Hedlund–Lyndon-type theorem should be compatible with the topology considered, that is, it should contain all maps $\Phi : \Lambda \rightarrow \Lambda$ for which for all $n \in \mathbb{Z}$ and $x \in \Lambda$ there exists $\ell \geq 0$ which does not depend on the value of n , but does depend on the configuration of x on the coordinates $(\dots x_{n-1}x_n \dots x_{n+\ell})$ to determine $(\Phi(x))_n$.

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