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MOST INFINITELY DIFFERENTIABLE FUNCTIONS ARE NOWHERE ANALYTIC

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1. **Introduction.** We define a natural metric, d, on the space, C^∞, of infinitely differentiable real valued functions defined on an open subset U of the real numbers, R, and show that C^∞ is complete with respect to this metric. Then we show that the elements of C^∞ which are analytic near at least one point of U comprise a first category subset of C^∞.

2. First, there exists a sequence $\{U_i\}_{i=1}^\infty$ of segments (a_i, b_i) in R which satisfy:

(i) the closure, V_i, of U_i is a compact subset of U;

(ii) $\bigcup_i U_i = U$;

and

(iii) if $x \in U$ and W is an open set containing x, then there exists a positive integer i for which $x \in U_i \subset V_i \subset W$.

Denote by (S, ρ) the metric space of sequences of real numbers, where the distance $\rho(x, y)$ between an element $x=\{x_i\}_{i=0}^\infty$ of S and an element $y=\{y_i\}_{i=0}^\infty$ of S is defined by

$$\rho(x, y) = \sum_{i=0}^{\infty} 2^{-i} \frac{|x_i - y_i|}{1+|x_i - y_i|}.$$

Let T be the metric space of S valued functions defined on U. For $u, v \in T$, let d_i be the semi-metric defined by

$$d_i(u, v) = \sup_{x \in V_i} \rho(u(x), v(x)),$$

and then let

$$\delta(u, v) = \sum_{i=1}^{\infty} 2^{-i} d_i(u, v).$$

Then (T, δ) is a complete metric space.

Next, we define a linear map, ϕ, of C^∞ into T by

$$(\phi f)(x) = (f(x), f^{(1)}(x), \ldots)$$

Finally, we define $d(f, g) = \delta(\phi(f), \phi(g))$, where f and $g \in C^\infty$. In view of Theorem 7.17 of [3] we see that (C^∞, d) is a complete metric space.

Let $A = \{f \in C^\infty : \exists\, x \in U \text{ such that } f \text{ is analytic at } x\}$. We wish to show that A is a first category subset of C^∞. To this end, we remind the reader (cf. [1, p. 26, Example 4]) that if f is analytic at a point x of U, then f is analytic on a neighborhood of some V_i and there exists a positive integer n such that $f \in A_{i,n}$, where

$$A_{i,n} = \left\{ f \in C^\infty : \sup_{x \in V_i} |f^{(k)}(x)| \leq n^k \cdot k!,\ k = 0, 1, \ldots \right\}.$$

Consequently, it suffices to show that $A_{i,n}$ is a closed and nowhere dense subset of C^∞. We leave it to the reader to check that $A_{i,n}$ is closed. Suppose $f \in C^\infty$ and $\varepsilon > 0$. Denote by N the set of elements g of C^∞ satisfying $d(f, g) < \varepsilon$. If $f \notin A_{i,n}$, then $N \not\subset A_{i,n}$. Suppose $f \in A_{i,n}$. Let $x_0 \in U_i$. Then choose λ to be a positive number which is so small that the restriction of $f + \lambda e^{-1/(x-x_0)^2}$ to U is in $N - A_{i,n}$.

In closing, we remind the reader that May showed in [2] that A is a proper subset of C^∞; we have shown that A is a rather small subset of C^∞.

References

1. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968.
2. W. Rudin, *Principles of mathematical analysis*, 2nd ed., McGraw-Hill, New York, 1964.
3. L. E. May, *On C^∞ functions analytic on sets of small measure*, Canad. Math. Bull. **12** (1969), 25–30.

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