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A Comment on "$\mathfrak{p} < \mathfrak{t}$"

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Abstract. Dealing with the cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ of the continuum, we prove that $\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_2$. In other words, if $\mathbf{MA}_{\aleph_1}$ (or a weak version of this) holds, then (of course $\aleph_2 \leq \mathfrak{p} \leq \mathfrak{t}$ and) $\mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t}$. The proof is based on a criterion for $\mathfrak{p} < \mathfrak{t}$.

Introduction

We are interested in two cardinal invariants of the continuum, $\mathfrak{p}$ and $\mathfrak{t}$. The cardinal $\mathfrak{p}$ measures when a family of infinite subsets of ω with finite intersection property has a pseudo-intersection. A relative is $\mathfrak{t}$, which deals with towers, *i.e.,* families well ordered by almost inclusion. These are closely related classical cardinal invariants. Rothberger [7, 8] proved (stated in our terminology) that $\mathfrak{p} \leq \mathfrak{t}$ and $\mathfrak{p} = \aleph_1 \Rightarrow \mathfrak{p} = \mathfrak{t}$, and he asked if $\mathfrak{p} = \mathfrak{t}$.

Our main result is Corollary 2.5, stating that $\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t}$, where $\mathfrak{m}$ is the minimal cardinal λ such that Martin's Axiom for λ dense sets fails (*i.e.,* $\neg\,\mathbf{MA}_\lambda$). Considering that $\mathfrak{m} \geq \aleph_1$ is a theorem (of ZFC), the parallelism with Rothberger's theorem is clear. The reader may conclude that probably $\mathfrak{m} = \mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{t}$; this is not unreasonable, but we believe that eventually one should be able to show

$$\mathrm{CON}(\mathfrak{m} = \lambda + \mathfrak{p} = \lambda + \mathfrak{t} = \lambda^+).$$

In Section 1 we present a characterization of $\mathfrak{p} < \mathfrak{t}$ that is crucial for the proof of Corollary 2.5, and which also sheds some light on the strategy to approach the question of $\mathfrak{p} < \mathfrak{t}$ presented in [9].

Notation Our notation is rather standard and compatible with that of classical textbooks (like Bartoszyński and Judah [3]). In forcing we keep the older convention that *the stronger condition is the larger one.*

(1) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \dots$) and also by i, j (with possible sub and superscripts).
(2) Cardinal numbers will be called $\kappa, \kappa_i, \lambda$.
(3) A bar above a letter denotes that the considered object is a sequence; usually $\overline{X}$ will be $\langle X_i : i < \zeta \rangle$, where ζ is the length $\ell g(\overline{X})$ of $\overline{X}$. Sometimes our sequences will be indexed by a set of ordinals, say $S \subseteq \lambda$, and then $\overline{X}$ will typically be $\langle X_\delta : \delta \in S \rangle$.

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(4) The set of all infinite subsets of the set ω of natural numbers is denoted by $[\omega]^{\aleph_0}$, and the relation of *almost inclusion* on $[\omega]^{\aleph_0}$ is denoted by $\subseteq^*$. Thus for $A, B \in [\omega]^{\aleph_0}$ we write $A \subseteq^* B$ if and only if $A \setminus B$ is finite.
(5) The relations of *eventual dominance* on the Baire space ${}^\omega\omega$ are called $\leq^*$ and $<^*$. Thus, for $f, g \in {}^\omega\omega$,
- $f \leq^* g$ if and only if $(\forall^\infty n < \omega)(f(n) \leq g(n))$ and
- $f <^* g$ if and only if $(\forall^\infty n < \omega)(f(n) < g(n))$.

1 A Criterion

In this section our aim is to prove Theorem 1.12, stating that $\mathfrak{p} < \mathfrak{t}$ implies the existence of a peculiar cut in $({}^\omega\omega, <^*)$. This also gives the background for our attempts in [9] to make progress on the consistency of $\mathfrak{p} < \mathfrak{t}$.

Definition 1.1 (1) We say that a set $A \in [\omega]^{\aleph_0}$ is *a pseudo-intersection of a family* $\mathcal{B} \subseteq [\omega]^{\aleph_0}$ if $A \subseteq^* B$ for all $B \in \mathcal{B}$.
(2) A sequence $\langle X_\alpha : \alpha < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ is *a tower* if $X_\beta \subseteq^* X_\alpha$ for $\alpha < \beta < \kappa$ but the family $\{X_\alpha : \alpha < \kappa\}$ has no pseudo-intersection.
(3) $\mathfrak{p}$ is the minimal cardinality of a family $\mathcal{B} \subseteq [\omega]^{\aleph_0}$ such that the intersection of any finite subcollection of $\mathcal{B}$ is infinite but $\mathcal{B}$ has no pseudo-intersection, and $\mathfrak{t}$ is the smallest size of a tower.

A lot of results have been accumulated on these two cardinal invariants. For instance:

- Bell [4] showed that $\mathfrak{p}$ is the first cardinal μ for which $\mathbf{MA}_\mu$(σ-centered) fails.
- Szymański proved that $\mathfrak{p}$ is regular (see, *e.g.,* Fremlin [5, Proposition 21K]).
- Piotrowski and Szymański [6] showed that $\mathfrak{t} \leq \operatorname{add}(\mathcal{M})$ (so also $\mathfrak{t} \leq \mathfrak{b}$).

For more results and discussion we refer the reader to [3, §1.3, §2.2].

Definition 1.2 We say that a family $\mathcal{B} \subseteq [\omega]^{\aleph_0}$ *exemplifies* $\mathfrak{p}$ if:

- $\mathcal{B}$ is closed under finite intersections (*i.e.,* $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$), and
- $\mathcal{B}$ has no pseudo-intersection and $|\mathcal{B}| = \mathfrak{p}$.

Proposition 1.3 *Assume $\mathfrak{p} < \mathfrak{t}$ and let $\mathcal{B}$ exemplify $\mathfrak{p}$. Then there are a cardinal $\kappa = \operatorname{cf}(\kappa) < \mathfrak{p}$ and a $\subseteq^*$-decreasing sequence $\langle A_i : i < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ such that*

(a) *$A_i \cap B$ is infinite for every $i < \kappa$ and $B \in \mathcal{B}$, and*
(b) *if A is a pseudo-intersection of $\{A_i : i < \kappa\}$, then for some $B \in \mathcal{B}$ the intersection $A \cap B$ is finite.*

Proof Fix an enumeration $\mathcal{B} = \{B_i : i < \mathfrak{p}\}$. By induction on $i < \mathfrak{p}$ we try to choose $A_i \in [\omega]^{\aleph_0}$ such that

(i) $A_i \subseteq^* A_j$ whenever $j < i$;
(ii) $B \cap A_i$ is infinite for each $B \in \mathcal{B}$;
(iii) if $i = j + 1$, then $A_i \subseteq B_j$.

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If we succeed, then $\{A_i : i < \mathfrak{p}\}$ has no pseudo-intersection, so $\mathfrak{t} \le \mathfrak{p}$, a contradiction. So for some $i < \mathfrak{p}$ we cannot choose A_i. Such an i is easily a limit ordinal; let $\kappa = \mathrm{cf}(i)$ (so $\kappa \le i < \mathfrak{p}$). Pick an increasing sequence $\langle j_\varepsilon : \varepsilon < \kappa\rangle$ with limit i. Then $\langle A_{j_\varepsilon} : \varepsilon < \kappa\rangle$ is as required. ∎

Remark 1.4. Concerning Proposition 1.3, let us note that Todorčević and Veličković used this idea in [10, Thm 1.5] to exhibit a σ-linked poset of size $\mathfrak{p}$ that is not σ-centered.

Lemma 1.5 *Assume that*

(i) $\overline{A} = \langle A_i : i < \delta\rangle$ *is a sequence of members of* $[\omega]^{\aleph_0}$, $\delta < \mathfrak{t}$,
(ii) $\overline{B} = \langle B_n : n < \omega\rangle \subseteq [\omega]^{\aleph_0}$ *is* $\subseteq^*$*-decreasing,*
(iii) *for each* $i < \delta$ *and* $n < \omega$ *the intersection* $A_i \cap B_n$ *is infinite, and*
(iv) $(\forall i < j < \delta)(\exists n < \omega)(A_j \cap B_n \subseteq^* A_i \cap B_n)$.

Then for some $A \in [\omega]^{\aleph_0}$ *we have*

$$(\forall i < \delta)(A \subseteq^* A_i) \textit{ and } (\forall n < \omega)(A \subseteq^* B_n).$$

Proof Without loss of generality $B_{n+1} \subseteq B_n$ and $\varnothing = \bigcap\{B_n : n < \omega\}$ (as we may use $B'_n = \bigcap_{\ell \le n} B_\ell \setminus \{0, \dots, n\}$). For each $i < \delta$, let $f_i \in {}^\omega\omega$ be defined by

$$f_i(n) = \min\{k \in B_n \cap A_i : k > f_i(m) \text{ for every } m < n\} + 1.$$

Since $\mathfrak{t} \le \mathfrak{b}$, there is $f \in {}^\omega\omega$ such that $(\forall i < \kappa)(f_i <^* f)$ and $n < f(n) < f(n+1)$ for $n < \omega$. Let

$$B^* = \bigcup\{(B_{n+1} \cap [n, f(n+1)) : n < \omega\}.$$

Then $B^* \in [\omega]^{\aleph_0}$ as for n large enough,

$$\min[A_0 \cap B_{n+1} \setminus [0, n)] \le f_0(n+1) < f(n+1).$$

Clearly for each $n < \omega$ we have $B^* \setminus [0, f(n)) \subseteq B_n$, and hence $B^* \subseteq^* B_n$. Moreover, $(\forall i < \kappa)(A_i \cap B^* \in [\omega]^{\aleph_0})$ (as above) and $(\forall i < j < \kappa)(A_j \cap B^* \subseteq^* A_i \cap B^*)$ (remember assumption (iv)). Now applying $\mathfrak{t} > \delta$ to $\langle A_i \cap B^* : i < \delta\rangle$ we get a pseudo-intersection A, which is as required. ∎

Definition 1.6 (1) Let $\mathbf{S}$ be the family of all sequences $\overline{\eta} = \langle \eta_n : n \in B\rangle$ such that $B \in [\omega]^{\aleph_0}$, and for $n \in B$, $\eta_n \in {}^{[n,k)}2$ for some $k \in (n, \omega)$. We let $\mathrm{dom}(\overline{\eta}) = B$ and let $\mathrm{set}(\overline{\eta}) = \bigcup\{\mathrm{set}(\eta_n) : n \in \mathrm{dom}(\overline{\eta})\}$, where $\mathrm{set}(\eta_n) = \{\ell : \eta_n(\ell) = 1\}$.
(2) For $\overline{A} = \langle A_i : i < \alpha\rangle \subseteq [\omega]^{\aleph_0}$, let

$$\mathbf{S}_{\overline{A}} = \left\{\overline{\eta} \in \mathbf{S} : \left(\forall i < \alpha\right)\left(\mathrm{set}(\overline{\eta}) \subseteq^* A_i\right) \text{ and } \left(\forall n \in \mathrm{dom}(\overline{\eta})\right)\left(\mathrm{set}(\eta_n) \ne \varnothing\right)\right\}.$$

(3) For $\overline{\eta}, \overline{\nu} \in \mathbf{S}$, let $\overline{\eta} \le^* \overline{\nu}$ mean that for every n large enough,

$$n \in \mathrm{dom}(\overline{\nu}) \Rightarrow n \in \mathrm{dom}(\overline{\eta}) \wedge \eta_n \trianglelefteq \nu_n$$

(where $\eta_n \trianglelefteq \nu_n$ means "η_n is an initial segment of ν_n").

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(4) For $\overline{\eta}, \overline{\nu} \in \mathbf{S}$, let $\overline{\eta} \leq^{**} \overline{\nu}$ mean that for every $n \in \operatorname{dom}(\overline{\nu})$ large enough, for some $m \in \operatorname{dom}(\overline{\eta})$ we have $\eta_m \subseteq \nu_n$ (as functions).
(5) For $\overline{\eta} \in \mathbf{S}$, let $C_{\overline{\eta}} = \{\nu \in {}^{\omega}2 : (\exists^{\infty} n)(\eta_n \subseteq \nu)\}$.

Observation 1.7 *(1) If $\overline{\eta} \leq^{*} \overline{\nu}$, then $\overline{\eta} \leq^{**} \overline{\nu}$, which implies $C_{\overline{\nu}} \subseteq C_{\overline{\eta}}$.*
(2) For every $\overline{\eta} \in \mathbf{S}$ and a meagre set $B \subseteq {}^{\omega}2$, there is $\overline{\nu} \in \mathbf{S}$ such that $\overline{\eta} \leq^{*} \overline{\nu}$ and $C_{\overline{\nu}} \cap B = \varnothing$.

Lemma 1.8 *(1) If $\overline{A} = \langle A_i : i < i^* \rangle \subseteq [\omega]^{\aleph_0}$ has finite intersection property and $i^* < \mathfrak{p}$, then $\mathbf{S}_{\overline{A}} \neq \varnothing$.*
(2) Every $\leq^*$-increasing sequence of members of $\mathbf{S}$ of length $< \mathfrak{t}$ has an $\leq^*$-upper bound.
(3) If $\overline{A} = \langle A_i : i < i^* \rangle \subseteq [\omega]^{\aleph_0}$ is $\subseteq^*$-decreasing and $i^* < \mathfrak{p}$, then every $\leq^*$-increasing sequence of members of $\mathbf{S}_{\overline{A}}$ of length $< \mathfrak{p}$ has an $\leq^*$-upper bound in $\mathbf{S}_{\overline{A}}$.

Proof (1) Let $A \in [\omega]^{\aleph_0}$ be such that $(\forall i < i^*)(A \subseteq^* A_i)$ (exists as $i^* < \mathfrak{p}$). Let $k_n = \min(A \setminus (n+1))$, and let $\eta_n \in {}^{[n,k_n+1)}2$ be defined by

$$\eta_n(\ell) = \begin{cases} 0 & \text{if } \ell \in [n, k_n), \\ 1 & \text{if } \ell = k_n. \end{cases}$$

Then $\langle \eta_n : n < \omega \rangle \in \mathbf{S}_{\overline{A}}$.

(2) Let $\langle \overline{\eta}^{\alpha} : \alpha < \delta \rangle$ be a $\leq^*$-increasing sequence and $\delta < \mathfrak{t}$. Let $A^*_{\alpha} =: \operatorname{dom}(\overline{\eta}^{\alpha})$ for $\alpha < \delta$. Then $\langle A^*_{\alpha} : \alpha < \delta \rangle$ is a $\subseteq^*$-decreasing sequence of members of $[\omega]^{\aleph_0}$. As $\delta < \mathfrak{t}$ there is $A^* \in [\omega]^{\aleph_0}$ such that $\alpha < \delta \Rightarrow A^* \subseteq^* A^*_{\alpha}$. Now for $n < \omega$ we define

$$B_n = \bigcup \{ {}^{[m,k)}2 : m \in A^* \text{ and } n \leq m < k < \omega \},$$

and for $\alpha < \delta$ we define

$$A_{\alpha} = \{\eta : \text{ for some } n \in \operatorname{dom}(\overline{\eta}^{\alpha}) \text{ we have } \eta^{\alpha}_n \trianglelefteq \eta\}.$$

One easily verifies that the assumptions of Lemma 1.5 are satisfied upon replacing ω by B_0. Let $A \subseteq B_0$ be given by the conclusion of Lemma 1.5, and put

$$A' = \left\{ n : \text{ for some } \eta \in A \text{ we have } \eta \in \bigcup \{ {}^{[n,k)}2 : k \in (n, \omega) \} \right\}.$$

Plainly, the set A' is infinite. We let $\overline{\eta}^* = \langle \eta_n : n \in A' \rangle$ where η_n is any member of $A \cap B_n \setminus B_{n+1}$.

(3) Assume that $\overline{A} = \langle A_i : i < i^* \rangle \subseteq [\omega]^{\aleph_0}$ is $\subseteq^*$-decreasing, $i^* < \mathfrak{p}$, and $\langle \overline{\eta}^{\alpha} : \alpha < \delta \rangle \subseteq \mathbf{S}_{\overline{A}}$ is $\leq^*$-increasing, and $\delta < \mathfrak{p}$. Let us consider the following forcing notion $\mathbb{P}$.

A condition in $\mathbb{P}$ is a quadruple $p = (\overline{\nu}, u, w, a) = (\overline{\nu}^p, u^p, w^p, a^p)$ such that
(a) $u \in [\omega]^{<\aleph_0}$, $\overline{\nu} = \langle \nu_n : n \in u \rangle$, and for $n \in u$ we have:
- $\nu_n \in {}^{[n,k_n)}2$ for some $k_n \in (n, \omega)$, and

- $\mathrm{set}(\nu_n) \neq \varnothing$,

(b) $w \subseteq \delta$ is finite, and
(c) $a \subseteq i^*$ is finite.

The order $\leq_{\mathbb{P}} = \leq$ *of* $\mathbb{P}$ is given by $p \leq q$ if and only if ($p, q \in \mathbb{P}$ and)

(i) $u^p \subseteq u^q$, $w^p \subseteq w^q$, $a^p \subseteq a^q$, and $\overline{\nu}^q \restriction u^p = \overline{\nu}^p$,
(ii) If $p \neq q$, then $\max(u^p) < \min(u^q \setminus u^p)$ and for $n \in u^q \setminus u^p$, we have
  (a) $(\forall \alpha \in w^p)(n \in \mathrm{dom}(\overline{\eta}^\alpha) \ \wedge \ \eta^\alpha_n \lhd \nu^q_n)$,
  (b) $(\forall i \in a^p)(\mathrm{set}(\nu^q_n) \subseteq A_i)$.

Plainly, $\mathbb{P}$ is a σ-centered forcing notion, and the sets

$$\mathcal{I}^{\alpha,i}_m = \left\{ p \in \mathbb{P} : \alpha \in w^p \ \wedge \ i \in a^p \ \wedge \ |u^p| > m \right\}$$

(for $\alpha < \delta$, $i < i^*$ and $m < \omega$) are open and dense in $\mathbb{P}$. Since $|\delta| + |i^*| + \aleph_0 < \mathfrak{p}$, we may choose a directed set $G \subseteq \mathbb{P}$ meeting all the sets $\mathcal{I}^{\alpha,i}_n$. Putting $\overline{\nu} = \bigcup\{\overline{\nu}^p : p \in G\}$, we will get an upper bound to $\langle \overline{\eta}^\alpha : \alpha < \delta \rangle$ in $\mathbf{S}_{\overline{A}}$. ■

Lemma 1.9 *Assume the following.*

(i) $\mathfrak{p} < \mathfrak{t}$ *and* $\mathcal{B} = \{B_\alpha : \alpha < \mathfrak{p}\}$ *exemplifies* $\mathfrak{p}$ *(see Definition* 1.2*).*
(ii) $\overline{A} = \langle A_i : i < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ *is* $\subseteq^*$*-decreasing,* $\kappa < \mathfrak{p}$*, and conditions* (a) *and* (b) *of Proposition* 1.3 *hold.*
(iii) $\mathrm{pr}\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ *is a bijection satisfying* $\mathrm{pr}(\alpha_1, \alpha_2) \geq \alpha_1, \alpha_2$.

Then we can find a sequence $\langle \overline{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ *such that*

(a) $\overline{\eta}^\alpha \in \mathbf{S}_{\overline{A}}$ *for* $\alpha < \mathfrak{p}$ *and* $\overline{\eta}^{\mathfrak{p}} \in \mathbf{S}$,
(b) $\langle \overline{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ *is* $\leq^*$*-increasing,*
(c) *if* $\alpha < \mathfrak{p}$ *and* $n \in \mathrm{dom}(\overline{\eta}^{\alpha+1})$ *is large enough, then* $\mathrm{set}(\eta^{\alpha+1}_n) \cap B_\alpha \neq \varnothing$ *(hence* $(\forall^\infty n \in \mathrm{dom}(\overline{\eta}^\beta))(\mathrm{set}(\eta^\beta_n) \cap B_\alpha \neq \varnothing)$ *holds for every* $\beta \in [\alpha + 1, \mathfrak{p}]$*),*
(d) *if* $\alpha = \mathrm{pr}(\beta, \gamma)$*, then* $\mathrm{set}(\eta^{\alpha+1}_n) \cap B_\beta \neq \varnothing$ *and* $\mathrm{set}(\eta^{\alpha+1}_n) \cap B_\gamma \neq \varnothing$ *for* $n \in \mathrm{dom}(\overline{\eta}^{\alpha+1})$*, and the truth values of*

$$\min(\mathrm{set}(\eta^{\alpha+1}_n) \cap B_\beta) < \min(\mathrm{set}(\eta^{\alpha+1}_n) \cap B_\gamma)$$

are the same for all $n \in \mathrm{dom}(\overline{\eta}^{\alpha+1})$*,*
(e) *in* (d)*, if* $\beta < \kappa$ *we can replace* B_β *by* A_β*; similarly with* γ*; and if* $\beta, \gamma < \kappa$ *then we can replace both.*

Proof We choose $\overline{\eta}^\alpha$ by induction on α. For $\alpha = 0$, it is trivial; for α limit $< \mathfrak{p}$, we use Lemma 1.8(3) (and $|\alpha| < \mathfrak{p}$). At a successor stage $\alpha + 1$, we let β, γ be such that $\mathrm{pr}(\beta, \gamma) = \alpha$ and we choose $B'_\alpha \in [\omega]^{\aleph_0}$ such that $B'_\alpha \subseteq B_\alpha \cap B_\beta \cap B_\gamma$ and $(\forall i < \kappa)(B'_\alpha \subseteq^* A_i)$. Next, for $n \in \mathrm{dom}(\overline{\eta}^\alpha)$, we choose η'_n such that $\eta^\alpha_n \lhd \eta'_n$ and

$$\varnothing \neq \{\ell : \eta'_n(\ell) = 1 \text{ and } \ell g(\eta^\alpha_n) \leq \ell < \ell g(\eta'_n)\} \subseteq B'_\alpha.$$

Then we let $\overline{\eta}^{\alpha+1} = \langle \eta'_n : n \in \mathrm{dom}(\overline{\eta}^\alpha) \rangle$. By shrinking the domain of $\overline{\eta}^{\alpha+1}$ there is no problem to take care of clause (d). It should also be clear that we may ensure clause (e) as well.

For $\alpha = \mathfrak{p}$, use Lemma 1.8(2). ■

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Definition 1.10 Let κ_1, κ_2 be infinite regular cardinals. *A* (κ_1, κ_2)*-peculiar cut in* ${}^{\omega}\omega$ is a pair $\big(\langle f_i : i < \kappa_1\rangle, \langle f^\alpha : \alpha < \kappa_2\rangle\big)$ of sequences of functions in ${}^{\omega}\omega$ such that the following hold:

(a) $(\forall i < j < \kappa_1)(f_j <^* f_i)$;
(b) $(\forall \alpha < \beta < \kappa_2)(f^\alpha <^* f^\beta)$;
(c) $(\forall i < \kappa_1)(\forall \alpha < \kappa_2)(f^\alpha <^* f_i)$;
(d) if $f : \omega \to \omega$ is such that $(\forall i < \kappa_1)(f \le^* f_i)$, then $f \le^* f^\alpha$ for some $\alpha < \kappa_2$;
(e) if $f : \omega \to \omega$ is such that $(\forall \alpha < \kappa_2)(f^\alpha \le^* f)$, then $f_i \le^* f$ for some $i < \kappa_1$.

Proposition 1.11 *If* $\kappa_2 < \mathfrak{b}$*, then there is no* $(\aleph_0, \kappa_2)$*-peculiar cut.*

Proof Assume towards contradiction that $\mathfrak{b} > \kappa_2$, but there is an $(\aleph_0, \kappa_2)$-peculiar cut, say $\big(\langle f_i : i < \omega\rangle, \langle f^\alpha : \alpha < \kappa_2\rangle\big)$ is such a cut. Let S be the family of all increasing sequences $\overline{n} = \langle n_i : i < \omega\rangle$ with $n_0 = 0$. For $\overline{n} \in S$ and $g \in {}^{\omega}\omega$, we say that $\overline{n}$ *obeys* g if $(\forall i < \omega)(g(n_i) < n_{i+1})$. Also for $\overline{n} \in S$, define $h_{\overline{n}} \in {}^{\omega}\omega$ by

$$h_{\overline{n}} \restriction [n_i, n_{i+1}) = f_i \restriction [n_i, n_{i+1}) \qquad \text{for } i < \omega.$$

Now, let $g^* \in {}^{\omega}\omega$ be an increasing function such that for every $n < \omega$ and $m \ge g^*(n)$ we have

$$f_{n+1}(m) < f_n(m) < \cdots < f_1(m) < f_0(m).$$

Note that

(1) if $\overline{n} \in S$ obeys g^*, then $(\forall i < \omega)(h_{\overline{n}} <^* f_i)$.

Now, for $\alpha < \kappa_2$ define $g^\alpha \in {}^{\omega}\omega$ by

(2) $g^\alpha(n) = \min\big\{k < \omega : k > n+1 \wedge \big(\forall i \le n\big)\big(\exists \ell \in [n,k)\big)\big(f^\alpha(\ell) < f_i(\ell)\big)\big\}$.

Since $\kappa_2 < \mathfrak{b}$, we may choose $g \in {}^{\omega}\omega$ such that

$$g^* < g \quad \text{and} \quad (\forall \alpha < \kappa_2)(g^\alpha <^* g).$$

Pick $\overline{n} \in S$ which obeys g and consider the function $h_{\overline{n}}$. It follows from (1) that $h_{\overline{n}} <^* f_i$ for all $i < \omega$, so by the properties of an $(\aleph_0, \kappa_2)$-peculiar cut there is $\alpha < \kappa_2$ such that $h_{\overline{n}} \le^* f^\alpha$. Then, for sufficiently large $i < \omega$, we have

- $f_i \restriction [n_i, n_{i+1}) = h_{\overline{n}} \restriction [n_i, n_{i+1}) \le f^\alpha \restriction [n_i, n_{i+1})$, and
- $n_i < g^\alpha(n_i) < g(n_i) < n_{i+1}$.

The latter implies that for some $\ell \in [n_i, n_{i+1})$ we have $f^\alpha(\ell) < f_i(\ell)$, contradicting the former. ■

Theorem 1.12 *Assume* $\mathfrak{p} < \mathfrak{t}$*. Then for some regular cardinal* κ*, there exists a* $(\kappa, \mathfrak{p})$*-peculiar cut in* ${}^{\omega}\omega$ *and* $\aleph_1 \le \kappa < \mathfrak{p}$.

Proof Use Proposition 1.3 and Lemma 1.9 to choose $\mathcal{B}, \kappa, \overline{A}$, pr and $\langle \overline{\eta}^\alpha : \alpha \le \mathfrak{p}\rangle$ so that:

(i) $\mathcal{B} = \{B_\alpha : \alpha < \mathfrak{p}\}$ exemplifies $\mathfrak{p}$,
(ii) $\overline{A} = \langle A_i : i < \kappa\rangle \subseteq [\omega]^{\aleph_0}$ is $\subseteq^*$-decreasing, $\kappa = \mathrm{cf}(\kappa) < \mathfrak{p}$ and conditions (a) and (b) of Proposition 1.3 hold,

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(iii) $\mathrm{pr}: \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is a bijection satisfying $\mathrm{pr}(\alpha_1, \alpha_2) \geq \alpha_1, \alpha_2$,
(iv) the sequence $\langle \overline{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ satisfies conditions (a)–(e) of Lemma 1.9.

It is enough to find a suitable cut $\langle f_i : i < \kappa \rangle, \langle f^\alpha : \alpha < \mathfrak{p} \rangle \subseteq {}^{A^*}\omega$ for some infinite $A^* \subseteq \omega$ (as by renaming, A^* is ω). Let

(v) $A^* = \mathrm{dom}(\overline{\eta}^{\mathfrak{p}})$,
(vi) for $i < \kappa$, we let $f_i \colon A^* \to \omega$ be defined by

$$f_i(n) = \min \left\{ \ell : [\eta_n^{\mathfrak{p}}(n+\ell) = 1 \ \wedge n + \ell \notin A_i] \text{ or } \mathrm{dom}(\eta_n^{\mathfrak{p}}) = [n, n+\ell) \right\},$$

(vii) for $\alpha < \mathfrak{p}$, we let $f^\alpha \colon A^* \to \omega$ be defined by

$$f^\alpha(n) = \min \left\{ \ell + 1 : [\eta_n^{\mathfrak{p}}(n+\ell) = 1 \ \wedge \ n + \ell \in B_\alpha] \text{ or } \mathrm{dom}(\eta_n^{\mathfrak{p}}) = [n, n+\ell) \right\}.$$

Note that (by the choice of f_i, *i.e.,* clause (vi)):

(viii) $\bigcup \{ [n, n + f_i(n)) \cap \mathrm{set}(\eta_n^{\mathfrak{p}}) : n \in A^* \} \subseteq^* A_i$ for every $i < \kappa$.

Also,

$(\circledast)_1^{\mathrm{a}}$ $f_j \leq^* f_i$ for $i < j < \kappa$.

[Because, if $i < j < \kappa$, then $A_j \subseteq^* A_i$, and hence for some n^* we have that $A_j \setminus n^* \subseteq A_i$. Therefore, for every $n \in A^* \setminus n^*$ in the definition of f_i, f_j in clause (vi), if ℓ can serve as a candidate for $f_i(n)$ then it can serve for $f_j(n)$, so (as we use the minimum there) $f_j(n) \leq f_i(n)$. Consequently $f_j \leq^* f_i$.]

Now, we want to argue that we may find a subsequence of $\langle f_i : i < \kappa \rangle$ which is $<^*$-decreasing. For this it is enough to show that

$(\circledast)_1^{\mathrm{b}}$ for every $i < \kappa$, for some $j \in (i, \kappa)$ we have $f_j <^* f_i$.

So assume towards contradiction that for some $i(*) < \kappa$, we have

$$(\forall j)(i(*) < j < \kappa \ \Rightarrow \ \neg(f_j <^* f_{i(*)})).$$

For $j < \kappa$ put $B_j^* =: \{ n \in A^* : f_j(n) \geq f_{i(*)}(n) \}$. Then $B_j^* \in [A^*]^{\aleph_0}$ is $\subseteq^*$-decreasing, so there is a pseudo-intersection B^* of $\langle B_j^* : j < \kappa \rangle$ (so $B^* \in [A^*]^{\aleph_0}$ and $(\forall j < \kappa)(B^* \subseteq^* B_j^*)$). Now, let $A' = \bigcup \{ \mathrm{set}(\eta_n^{\mathfrak{p}}) \cap [n, n + f_{i(*)}(n)) : n \in B^* \}$.

$(*)$ A' is an infinite subset of ω.

[Because, by Lemma 1.9(a) we have $\overline{\eta}^0 \in \mathbf{S}_{\overline{A}}$ and hence $\mathrm{set}(\overline{\eta}^0) \subseteq^* A_{i(*)}$ and $(\forall n \in \mathrm{dom}(\overline{\eta}^0)(\mathrm{set}(\eta_n^0) \neq \varnothing)$ (see Definition 1.6(2)). By Lemma 1.9(b) we know that for every large enough $n \in \mathrm{dom}(\overline{\eta}^{\mathfrak{p}})$, we have $n \in \mathrm{dom}(\overline{\eta}^0)$ and $\eta_n^0 \trianglelefteq \eta_n^{\mathfrak{p}}$. For every large enough $n \in \mathrm{dom}(\overline{\eta}^0)$, we have $\mathrm{set}(\overline{\eta}^0) \setminus \{0, \dots, n-1\} \subseteq A_{i(*)}$, and hence for every large enough $n \in \mathrm{dom}(\overline{\eta}^{\mathfrak{p}})$, we have $\eta_n^0 \trianglelefteq \eta_n^{\mathfrak{p}}$ and $\varnothing \neq \mathrm{set}(\eta_n^0) \subseteq A_{i(*)}$. Consequently, for large enough $n \in B^*$, $[n, n + f_{i(*)}(n)) \cap \mathrm{set}(\eta_n^{\mathfrak{p}}) \neq \varnothing$ and we are done.]

$(**)$ $A' \subseteq^* A_j$ for $j \in (i(*), \kappa)$ (and hence for all $j < \kappa$).

[Because $f_j \restriction B^* =^* f_{i(*)} \restriction B^*$ for $j \in (i(*), \kappa)$.]

$(***)$ $A' \cap B_\alpha$ is infinite for $\alpha < \mathfrak{p}$.

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[Because, by clauses (c) and (a) of Lemma 1.9, for every large enough $n \in \mathrm{dom}(\overline{\eta}^{\alpha+1})$, we have $\mathrm{set}(\eta_n^{\alpha+1}) \cap B_\alpha \neq \varnothing$ and $\mathrm{set}(\eta_n^{\alpha+1}) \subseteq A_{i(*)}$.]

Properties $(*)$–$(***)$ contradict Proposition 1.3(b), finishing the proof of $(\circledast)_1^{\mathrm{b}}$.

Thus passing to a subsequence if necessary, we may assume that

$(\circledast)_1^{\mathrm{c}}$ the demand in (a) of Definition 1.10 is satisfied, *i.e.*, $f_j <^* f_i$ for $i < j < \kappa$.

Now,

$(\circledast)_2$ $(\forall i < \kappa)(\forall \alpha < \mathfrak{p})(f^\alpha <^* f_i)$.

[Because if $i < \kappa$, $\alpha < \mathfrak{p}$, then for large enough $n \in A^*$ we have that $\mathrm{set}(\eta_n^{\alpha+1}) \subseteq A_i$, $\mathrm{set}(\eta_n^{\alpha+1}) \cap B_\alpha \neq \varnothing$, and $\eta_n^{\alpha+1} \trianglelefteq \eta_n^{\mathfrak{p}}$. Then for those n we have $f^\alpha(n) \leq f_i(n)$. Now we may conclude that actually $f^\alpha <^* f_i$.]

$(\circledast)_3^{\mathrm{a}}$ The set (of functions) $\{f_i : i < \kappa\} \cup \{f^\alpha : \alpha < \mathfrak{p}\}$ is linearly ordered by $\leq^*$.

$(\circledast)_3^{\mathrm{b}}$ In fact, if f', f'' are in the family then either $f' =^* f''$ or $f' <^* f''$ or $f'' <^* f'$.

[This follows from $(\circledast)_1$, $(\circledast)_2$, and clauses (d) and (e) of Lemma 1.9.]

Choose inductively a sequence $\overline{\alpha} = \langle \alpha(\varepsilon) : \varepsilon < \varepsilon^* \rangle \subseteq \mathfrak{p}$ such that:

- $\alpha(\varepsilon)$ is the minimal $\alpha \in \mathfrak{p} \setminus \{\alpha(\zeta) : \zeta < \varepsilon\}$ satisfying $(\forall \zeta < \varepsilon)(f^{\alpha(\zeta)} <^* f^\alpha)$, and
- we cannot choose $\alpha(\varepsilon^*)$.

We ignore (until $(\circledast_7)$) the question of the value of ε^*. Now,

$(\circledast)_4$ $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^* \rangle$ satisfy clauses (a)–(c) of Definition 1.10.

[This follows from $(\circledast)_1$–$(\circledast)_3$ and the choice of $\alpha(\varepsilon)$'s above.]

$(\circledast)_5$ $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^* \rangle$ satisfy clause (e) of Definition 1.10.

[To see this, assume towards contradiction that $f : A^* \to \omega$ and

$$\big(\forall i < \kappa\big)\big(f \leq^* f_i\big) \ \text{ but } \ \big(\forall \varepsilon < \varepsilon^*\big)\big(\neg(f \leq^* f^{\alpha(\varepsilon)})\big).$$

Clearly, without loss of generality, we may assume that $[n, n + f(n)) \subseteq \mathrm{dom}(\eta_n^{\mathfrak{p}})$ for $n \in A^*$. Let $A' = \bigcup \big\{ [n, n + f(n)) \cap \mathrm{set}(\eta_n^{\mathfrak{p}}) : n \in A^* \big\}$. Now for every $i < \kappa$, $A' \subseteq^* A_i$ because $f \leq^* f_i$ and by the definition of f_i. Also, for every $\alpha < \mathfrak{p}$, the intersection $A' \cap B_\alpha$ is infinite. For it follows from the choice of the sequence $\overline{\alpha}$ that for some $\varepsilon < \varepsilon^*$ we have $\neg(f^{\alpha(\varepsilon)} <^* f^\alpha)$, and thus $f^\alpha \leq^* f^{\alpha(\varepsilon)}$ (remembering $(\circledast)_3$). Hence, if $n \in A^*$ is large enough, then $f^\alpha(n) \leq f^{\alpha(\varepsilon)}(n)$ and for infinitely many $n \in A^*$ we have $f^\alpha(n) \leq f^{\alpha(\varepsilon)}(n) < f(n) \leq f_0(n) \leq |\mathrm{dom}(\eta_n^{\mathfrak{p}})|$. For every such n we have $n + f^\alpha(n) - 1 \in A' \cap B_\alpha$. Together, A' contradicts clause (ii) of the choice of $\langle A_i : i < \kappa \rangle, \langle B_\alpha : \alpha < \mathfrak{p} \rangle$, specifically the property stated in Proposition 1.3(b).]

$(\circledast)_6$ $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^* \rangle$ satisfy clause (e) of Definition 1.10.

[Assume towards contradiction that $f : A^* \to \omega$, and

$$\big(\forall \varepsilon < \varepsilon^*\big)\big(f^{\alpha(\varepsilon)} \leq^* f\big) \ \text{ but } \ \big(\forall i < \kappa\big)\big(\neg(f_i \leq^* f)\big).$$

It follows from $(\circledast)_1$ (and the assumption above) that we may choose an infinite set $A^{**} \subseteq A^*$ such that $\big(\forall i < \kappa\big)\big((f \restriction A^{**}) <^* (f_i \restriction A^{**})\big)$. Let

$$A'' = \bigcup \{[n, n + f(n)) \cap \mathrm{set}(\eta_n^{\mathfrak{p}}) : n \in A^{**}\} \subseteq \omega.$$

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Since $(f \restriction A^{**}) <^* (f_i \restriction A^{**})$, we easily see that $A'' \subseteq^* A_i$ for all $i < \kappa$ (remember (viii)). As in the justification for $(\circledast)_5$ above, if $\alpha < \mathfrak{p}$, then for some $\varepsilon < \varepsilon^*$ we have $f^\alpha \leq^* f^{\alpha(\varepsilon)}$ and we may conclude from our assumption towards contradiction that $f^\alpha \leq^* f$ for all $\alpha < \mathfrak{p}$. As in $(\circledast)_5$ we conclude that for every $\alpha < \mathfrak{p}$ the intersection $A'' \cap B_\alpha$ is infinite, contradicting the choice of $\langle A_i : i < \kappa \rangle$, $\langle B_\alpha : \alpha < \mathfrak{p} \rangle$.]

$(\circledast)_7$ $\varepsilon^* = \mathfrak{p}$.

[Because the sequence $\langle \alpha(\varepsilon) : \varepsilon < \mathfrak{p} \rangle$ is an increasing sequence of ordinals $< \mathfrak{p}$, hence $\varepsilon^* \leq \mathfrak{p}$. If $\varepsilon^* < \mathfrak{p}$, then by the Bell theorem we get a contradiction to $(\circledast)_4$–$(\circledast)_6$ above; cf. Proposition 2.1 below.]

So $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \mathfrak{p} \rangle$ are as required: clauses (a)–(c) of Definition 1.10 hold by $(\circledast)_4$, clause (d) by $\circledast_5$, and clause (e) by $(\circledast)_6$. Finally, since $\mathfrak{t} \leq \mathfrak{b}$, we may use Proposition 1.11 to conclude that (under our assumption $\mathfrak{p} < \mathfrak{t}$) there is no $(\aleph_0, \mathfrak{p})$-peculiar cut and hence $\kappa \geq \aleph_1$. ■

Remark 1.13. The existence of $(\kappa, \mathfrak{p})$-peculiar cuts for $\kappa < \mathfrak{p}$ is independent from "ZFC+$\mathfrak{p} = \mathfrak{t}$". We will address this issue in a forthcoming paper [9].

2 Peculiar Cuts and MA

Proposition 2.1 *Assume that $\kappa_1 \leq \kappa_2$ are infinite regular cardinals and there exists a (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$. Then for some σ-centered forcing notion $\mathbb{Q}$ of cardinality κ_1 and a sequence $\langle \mathcal{I}_\alpha : \alpha < \kappa_2 \rangle$ of open dense subsets of $\mathbb{Q}$, there is no directed $G \subseteq \mathbb{Q}$ such that $(\forall \alpha < \kappa_2)(G \cap \mathcal{I}_\alpha \neq \varnothing)$. Hence $\mathbf{MA}_{\kappa_2}(\sigma$-centered) fails and thus $\mathfrak{p} \leq \kappa_2$.*

Proof Let $\big(\langle f_i : i < \kappa_1 \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle\big)$ be a (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$. Define a forcing notion $\mathbb{Q}$ as follows.

A condition in $\mathbb{Q}$ is a pair $p = (\rho, u)$ such that $\rho \in {}^{\omega>}\omega$ and $u \subseteq \kappa_1$ is finite.

The order $\leq_{\mathbb{Q}} = \leq$ *of* $\mathbb{Q}$ is given by $(\rho_1, u_1) \leq (\rho_2, u_2)$ if and only if (both are in $\mathbb{Q}$ and) the following hold:

(a) $\rho_1 \trianglelefteq \rho_2$,
(b) $u_1 \subseteq u_2$,
(c) if $n \in [\ell g(\rho_1), \ell g(\rho_2))$ and $i \in u_1$, then $f_i(n) \geq \rho_2(n)$.

Plainly, $\mathbb{Q}$ is a forcing notion of cardinality κ_1. It is σ-centered, since for each $\rho \in {}^{\omega>}\omega$, the set $\{(\eta, u) \in \mathbb{Q} : \eta = \rho\}$ is directed.

For $j < \kappa_1$, let $\mathcal{I}_j = \{(\rho, u) \in \mathbb{Q} : j \in u\}$, and for $\alpha = \omega\beta + n < \kappa_2$, let

$$\mathcal{I}^\alpha = \big\{ (\rho, u) \in \mathbb{Q} : \big(\exists m < \ell g(\rho)\big) \big(m \geq n \ \wedge \rho(m) > f^\beta(m) \big) \big\}.$$

Clearly $\mathcal{I}_j, \mathcal{I}^\alpha$ are dense open subsets of $\mathbb{Q}$. Suppose towards contradiction that there is a directed $G \subseteq \mathbb{Q}$ intersecting all $\mathcal{I}^\alpha, \mathcal{I}_j$ for $j < \kappa_1$, $\alpha < \kappa_2$. Put $g = \bigcup\{\rho : (\exists u)((\rho, u) \in G)\}$. Then

- g is a function; its domain is ω (as $G \cap \mathcal{I}^n \neq \varnothing$ for $n < \omega$), and
- $g \leq^* f_i$ (as $G \cap \mathcal{I}_i \neq \varnothing$), and
- $\{n < \omega : f^\alpha(n) < g(n)\}$ is infinite (as $G \cap \mathcal{I}^{\omega\alpha+n} \neq \varnothing$ for every n).

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The properties of the function g clearly contradict our assumptions on $\langle f_i : i < \kappa_1 \rangle$, $\langle f^\alpha : \alpha < \kappa_2 \rangle$. ■

Corollary 2.2 *If there exists an* $(\aleph_0, \kappa_2)$*-peculiar cut, then* $\mathrm{cov}(\mathcal{M}) \leq \kappa_2$.

Theorem 2.3 *Let* $\mathrm{cf}(\kappa_2) = \kappa_2 > \aleph_1$. *Assume* $\mathbf{MA}_{\aleph_1}$ *holds. Then there is no* $(\aleph_1, \kappa_2)$*-peculiar cut in* ${}^\omega\omega$.

Proof Suppose towards contradiction that $\mathrm{cf}(\kappa_2) = \kappa_2 > \aleph_1$, $(\langle f_i : i < \omega_1 \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle)$ is an $(\aleph_1, \kappa_2)$-peculiar cut and $\mathbf{MA}_{\aleph_1}$ holds true. We define a forcing notion $\mathbb{Q}$ as follows.

A condition in $\mathbb{Q}$ is a pair $p = (u, \bar{\rho}) = (u^p, \bar{\rho}^p)$ such that

(a) $u \subseteq \omega_1$ is finite, $\bar{\rho} = \langle \rho_i : i \in u \rangle = \langle \rho_i^p : i \in u \rangle$,
(b) for some $n = n^p$, for all $i \in u$ we have $\rho_i \in {}^n\omega$,
(c) for each $i \in u$ and $m < n^p$ we have $\rho_i(m) \leq f_i(m)$,
(d) if $i_0 = \max(u)$ and $m \geq n^p$, then $f_{i_0}(m) > 2 \cdot |u^p| + 885$.
(e) $\langle f_i \restriction [n^p, \omega) : i \in u \rangle$ is $<$-decreasing.

The order $\leq_{\mathbb{Q}} = \leq$ *of* $\mathbb{Q}$ is given by $p \leq q$ if and only if ($p, q \in \mathbb{Q}$ and)

(f) $u^p \subseteq u^q$,
(g) $\rho_i^p \trianglelefteq \rho_i^q$ for every $i \in u^p$,
(h) if $i < j$ are from u^p, then $\rho_i^q \restriction [n^p, n^q) < \rho_j^q \restriction [n^p, n^q)$,
(i) if $i < j$, $i \in u^q \setminus u^p$ and $j \in u^p$, then for some $m \in [n^p, n^q)$ we have $f_j(m) < \rho_i^q(m)$.

Claim 2.3.1 $\mathbb{Q}$ *is a ccc forcing notion of size* $\aleph_1$.

Proof of the Claim Plainly, the relation $\leq_{\mathbb{Q}}$ is transitive and $|\mathbb{Q}| = \aleph_1$. Let us argue that the forcing notion $\mathbb{Q}$ satisfies the ccc.

Let $p_\varepsilon \in \mathbb{Q}$ for $\varepsilon < \omega_1$. Without loss of generality $\langle p_\varepsilon : \varepsilon < \omega_1 \rangle$ is without repetition. Applying the Δ-Lemma we can find an unbounded set $\mathcal{U} \subseteq \omega_1$ and $m(*) < n(*) < \omega$ and $n' < \omega$ such that for each $\varepsilon \in \mathcal{U}$ we have the following:

(i) $|u^{p_\varepsilon}| = n(*)$ and $n^{p_\varepsilon} = n'$; let $u^{p_\varepsilon} = \{\alpha_{\varepsilon,\ell} : \ell < n(*)\}$ and $\alpha_{\varepsilon,\ell}$ increases with ℓ;
(ii) $\alpha_{\varepsilon,\ell} = \alpha_\ell$ for $\ell < m(*)$ and $\rho_{\varepsilon,\ell} = \rho_\ell^*$ for $\ell < n(*)$;
(iii) if $\varepsilon < \zeta$ are from $\mathcal{U}$ and $k, \ell \in [m(*), n(*))$, then $\alpha_{\varepsilon,\ell} < \alpha_{\zeta,k}$.

Let $\varepsilon < \zeta$ be elements of $\mathcal{U}$ such that $[\varepsilon, \zeta) \cap \mathcal{U}$ is infinite. Pick $k^* > n'$ such that for each $k \geq k^*$ we have

- the sequence $\langle f_\alpha(k) : \alpha \in \{\alpha_{\varepsilon,\ell} : \ell < n(*)\} \cup \{\alpha_{\zeta,\ell} : \ell < n(*)\} \rangle$ is strictly decreasing,
- $f_{\alpha_{\zeta,n(*)-1}}(k) > 885 \cdot (n(*) + 1)$,
- $f_{\alpha_{\zeta,m(*)}}(k) + n(*) + 885 < f_{\alpha_{\varepsilon,n(*)-1}}(k)$.

(The choice is possible because $\langle f_i : i < \omega_1 \rangle$ is $<^*$-decreasing and by the selection of ε, ζ we also have $\lim\limits_{k \to \infty} \left(f_{\alpha_{\varepsilon,n(*)-1}}(k) - f_{\alpha_{\zeta,m(*)}}(k) \right) = \infty$.)

Now define $q = (u^q, \bar{\rho}^q)$ as follows:

- $u^q = u^{p_\varepsilon} \cup u^{p_\zeta}$, $n^q = k^* + 1$;

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- if $n < n'$, $i \in u^{p_\varepsilon}$, then $\rho_i^q(n) = \rho_i^{p_\varepsilon}(n)$;
- if $n < n'$, $i \in u^{p_\zeta}$, then $\rho_i^q(n) = \rho_i^{p_\zeta}(n)$;
- if $i = \alpha_{\varepsilon,\ell}$, $\ell < n(*)$, $n \in [n', k^*)$, then $\rho_i^q(n) = \ell$, and if $j = \alpha_{\zeta,\ell}$, $m(*) \leq \ell < n(*)$, then $\rho_j^q(n) = n(*) + \ell + 1$;
- if $j = \alpha_{\zeta,\ell}$, $\ell < n(*)$, then $\rho_j^q(k^*) = \ell$, and if $i = \alpha_{\varepsilon,\ell}$, $m(*) \leq \ell < n(*)$, then $\rho_i^q(k^*) = f_{\alpha_{\zeta,m(*)}}(k^*) + \ell + 1$.

It is well defined (as $\rho^{p_\varepsilon}_{\alpha_{\varepsilon,\ell}} = \rho^{p_\zeta}_{\alpha_{\zeta,\ell}}$ for $\ell < m(*)$). Also $q \in \mathbb{Q}$. Lastly, one easily verifies that $p_\varepsilon \leq_{\mathbb{Q}} q$ and $p_\zeta \leq_{\mathbb{Q}} q$, so indeed $\mathbb{Q}$ satisfies the ccc. ■

For $i < \omega_1$ and $n < \omega$, let

$$\mathfrak{I}_{i,n} = \left\{ p \in \mathbb{Q} : \left[u^p \nsubseteq i \text{ or for no } q \in \mathbb{Q} \text{ we have } p \leq_{\mathbb{Q}} q \wedge u^q \nsubseteq i \right] \text{ and } n^p \geq n \right\}.$$

Plainly, the sets $\mathfrak{I}_{i,n}$ are open dense in $\mathbb{Q}$. Also, for each $i < \omega_1$ there is $p_i^* \in \mathbb{Q}$ such that $u^{p_i} = \{i\}$. It follows from Claim 2.3.1 that for some $i(*)$, $p^*_{i(*)} \Vdash_{\mathbb{Q}}$ "$\{j < \omega_1 : p_j^* \in \underset{\sim}{G}\}$ is unbounded in ω_1". Note also that if p is compatible with $p^*_{i(*)}$ and $p \in \mathfrak{I}_{i,n}$ then $u_p \nsubseteq i$.

Since we have assumed $\mathbf{MA}_{\aleph_1}$ and $\mathbb{Q}$ satisfies the ccc (by Claim 2.3.1), we may find a directed set $G \subseteq \mathbb{Q}$ such that $p^*_{i(*)} \in G$ and $\mathfrak{I}_{i,n} \cap G \neq \varnothing$ for all $n < \omega$ and $i < \omega_1$. Note that then the set $\mathcal{U} := \bigcup\{u^p : p \in G\}$ is unbounded in ω_1.

For $i \in \mathcal{U}$ let $g_i = \bigcup\{\rho_i^p : p \in G\}$. Clearly each $g_i \in {}^\omega\omega$ (as G is directed, $\mathfrak{I}_{i,n} \cap G \neq \varnothing$ for $i < \omega_1$, $n < \omega$). Also $g_i \leq f_i$ by clause (c) of the definition of $\mathbb{Q}$, and $\langle g_i : i \in \mathcal{U}\rangle$ is $<^*$-increasing by clause (h) of the definition of $\leq_{\mathbb{Q}}$. Hence for each $i < j$ from $\mathcal{U}$ we have $g_i <^* g_j \leq^* f_j <^* f_i$. Thus by property (d) of Definition 1.10 of a peculiar cut, for every $i \in \mathcal{U}$ there is $\gamma(i) < \kappa_2$ such that $g_i <^* f^{\gamma(i)}$. Let $\gamma(*) = \sup\{\gamma(i) : i \in \mathcal{U}\}$. Then $\gamma(*) < \kappa_2$ (as $\kappa_2 = \mathrm{cf}(\kappa_2) > \aleph_1$). Now, for each $i \in \mathcal{U}$ we have $g_i <^* f^{\gamma(*)} <^* f_i$, and thus for $i \in \mathcal{U}$ we may pick $n_i < \omega$ such that

$$n \in [n_i, \omega) \Rightarrow g_i(n) < f^{\gamma(*)}(n) < f_i(n).$$

For some n^* the set $\mathcal{U}_* = \{i \in \mathcal{U} : n_i = n^*\}$ is unbounded in ω_1. Let $j \in \mathcal{U}_*$ be such that $\mathcal{U}_* \cap j$ is infinite. Pick $p \in G$ such that $j \in u^p$ and $n^p > n^*$ (remember $G \cap \mathfrak{I}_{j,n^*+1} \neq \varnothing$ and G is directed). Since u^p is finite, we may choose $i \in \mathcal{U}_* \cap j \setminus u^p$, and then $q \in G$ such that $q \geq p$ and $i \in u^q$. If follows from clause (i) of the definition of the order $\leq$ of $\mathbb{Q}$ that there is $n \in [n^p, n^q)$ such that $f_j(n) < \rho_i^q(n) = g_i(n)$. Since $n > n^* = n_i = n_j$, we get $f_j(n) < g_i(n) < f^{\gamma(*)}(n) < f_j(n)$, a contradiction. ■

Remark 2.4. The proof of Theorem 2.3 actually used Hausdorff gaps on which much is known (see, *e.g.,* Abraham and Shelah [1, 2]). More precisely, the proof could be presented as a two-step argument:

(1) from $\mathbf{MA}_{\aleph_1}$ one gets that every decreasing ω_1-sequence is half of a Hausdorff gap, and
(2) if $\kappa_2 = \mathrm{cf}(\kappa_2) > \aleph_1$, then the ω_1-part of a peculiar (ω_1, κ_2)-cut cannot be half of a Hausdorff gap.

Corollary 2.5 *If* $\mathbf{MA}_{\aleph_1}$*, then* $\mathfrak{p} = \aleph_2 \Leftrightarrow \mathfrak{t} = \aleph_2$*. In other words,*

$$\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_2.$$

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