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A CENTRAL LIMIT THEOREM WITH CONDITIONING ON THE DISTANT PAST

BY

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Serfling (1968) has considered a central limit theorem in which assumptions are made concerning the expectation of variables conditioned on their distant predecessors. Dvoretsky (1972, theorem 5.3) has continued this investigation. Serfling showed that both martingales and φ-mixing sequences satisfied his conditions, and Dvoretsky extended this to Strong mixing sequences of random variables.

In this paper we provide a simple proof of the main result of both of these papers. In doing so, we drop the assumption that moments higher than the second exist for the random variables in question and replace this by a weaker uniform integrability assumption. In addition, we do not require, as both Dvoretsky and Serfling do, rates of convergence for the quantities in (a) and (b) of theorem 1.

Let $X_1, X_2, \ldots$ be a sequence of random variables, put $S_n = \sum_{i=1}^n X_i$, and $E_k U = E(U \mid X_1, X_2, \ldots, X_k)$. Denote the L_p norms, $(E^{1/p} |U|^p)$ by $\|U\|_p$ and convergence in law or in distribution by "$\xrightarrow{w}$".

1. Theorem. *Suppose for some positive constant σ^2, and functions: $\psi(n)$ and $\varphi(n)$ for which* $\lim_{n\to\infty} \psi(n) = \lim_{n\to\infty} \varphi(n) = 0$, *we have;*

(a) $\left\| E_k \dfrac{S_{k+n}-S_k}{n^{1/2}} \right\|_1 \leq \psi(n)$ and

(b) $\left\| E_k \dfrac{(S_{k+n}-S_k)^2}{n} - \sigma^2 \right\|_1 \leq \phi(n)$ for all k and n, and

(c) The set $\left\{ \dfrac{(S_{k+n}-S_k)^2}{n};\, k = 1, 2, \ldots, n = 1, 2, \ldots \right\}$ is uniformly integrable.

Then $S_n/n^{1/2}\sigma \xrightarrow{w} N(0, 1)$.

Proof. Clearly we may assume without loss of generality that $\sigma^2 = 1$, and that the functions ψ and φ are strictly positive and non-increasing. Then for any fixed integer k, we have, by assumption, $k^{1/2}\psi((n/k)-1) \to 0$ as $n \to \infty$. Therefore, by lemma 1, p. 188 of Chung (1968) there exists a non-decreasing sequence of integers $k_n \to \infty$ such that $k_n^{1/2}\psi((n/k_n)-1) \to 0$ as $n \to \infty$. This implies, by the nature of ψ, that $(n/k_n) \to \infty$.

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Now for each n, let $m(i)=[in/k_n]$ for all $i\leq k_n$, where the square brackets indicate the "integer contained in". Define an array $X_{n,i}=n^{-1/2}(S_{m(i)}-S_{m(i-1)})$ for $1\leq i\leq k_n$.

Now observe that the following three conditions are sufficient for corollary 2.1 of Dvoretsky (1972).

(2) The set $\{k_nX_{n,i}^2; 1\leq i\leq k_n, n=1,2,\ldots\}$ is uniformly integrable.

(3) $\max_{i\leq k_n} k_n \|E_{m(i-1)}X_{n,i}\|_1\to 0$ as $n\to\infty$, and

(4) $\max_{i\leq k_n} \|k_nE_{m(i-1)}X_{n,i}^2-1\|_1\to 0$ as $n\to\infty$.

Generally, 2 implies the Lindeberg condition, 3 implies Dvoretsky's 2.8 and 4 implies 2.9. We will verify these three conditions for the array $\{X_{n,i}\}$. Note that

$$(5)\qquad \frac{n}{k_n}-1\leq m(i)-m(i-1)\leq\frac{n}{k_n}+1\quad\text{for all}\quad i\leq k_n.$$

Condition 2. Clearly

$$k_nX_{n,i}^2\leq(k_n/n)(n/k_n+1)\frac{(S_{m(i)}-S_{m(i-1)})^2}{m(i)-m(i-1)}$$

by (5), and this is uniformly integrable by c, since the product of the first two terms $\to 1$ as $n\to\infty$.

Condition 3.

$$\begin{aligned}k_n\|E_{m(i-1)}X_{n,i}\|_1&=k_nn^{-1/2}\|E_{m(i-1)}(S_{m(i)}-S_{m(i-1)})\|_1\\&\leq k_nn^{-1/2}((n/k_n+1)^{1/2}\psi((n/k_n-1)\end{aligned}$$

by (a) and (5). But this converges to 0 as $n\to\infty$, since $k_n/n\to 0$ and $k_n^{1/2}\psi((n/k_n)-1)\to 0$.

Condition 4. Similarly,

$$\begin{aligned}&\|k_nE_{m(i-1)}X_{n,i}^2-1\|_1\\&\quad=(k_n/n)\|E_{m(i-1)}(S_{m(i)}-S_{m(i-1)})^2-(n/k_n)\|_1\\&\quad\leq\left(\frac{k_n}{n}\right)\|E_{m(i-1)}(S_{m(i)}-S_{m(i-1)})^2-(m(i)-m(i-1))\|_1+\left(\frac{k_n}{n}\right)\text{ by (5),}\\&\quad\leq\left(1+\frac{k_n}{n}\right)\phi\left(\frac{n}{k_n}-1\right)+\frac{k_n}{n}\end{aligned}$$

by (b) and (5), and this converges to 0 as $n\to\infty$. Q.E.D.

We defer further discussion of this theorem to McLeish (3), in which it is shown that a sufficient condition for both (a) and (c) to hold for a doubly infinite stationary sequence with $EX_0^2<\infty$, and $ES_n^2/n\to\sigma^2$ is the condition $\sum_{m=1}^{\infty}\|E_{-m}X_0\|_2^{\theta}<\infty$, for some $\theta<2$. This is applied to proving invariance principles under φ-mixing (e.g. Billingsley, theorem 20.1), strong mixing, and various considerably more general types of mixing conditions.

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