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CENTRALIZING MAPPINGS OF PRIME RINGS

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ABSTRACT. Let R be a prime ring and U be a nonzero ideal or quadratic Jordan ideal of R. If L is a nontrivial automorphism or derivation of R such that $uL(u)-L(u)u$ is in the center of R for every u in U, then the ring R is commutative.

If R is a ring, a mapping L of R to itself is called *centralizing* on a subset S of R if $xL(x)-L(x)x$ is in the center of R for every x in S. Posner [5] has shown that the existence of a nontrivial centralizing derivation on a prime ring forces the ring to be commutative. In [2] the author obtained the same result for a centralizing automorphism. Then in [3] these two results were generalized by showing that the ring is commutative if the automorphism or derivation centralizes and leaves invariant a nonzero ideal in the ring. In this paper the ideal invariance assumption is shown to be unnecessary. Thus the existence of a nontrivial automorphism or derivation which is centralizing on a nonzero ideal in a prime ring implies that the ring is commutative.

Then using the fact that every nonzero quadratic Jordan ideal contains a nonzero (associative) ideal [4], we find that the mapping need only be centralizing on a nonzero quadratic Jordan ideal. In the derivation case this extends a theorem of Awtar [1, Theorem 3] to prime rings of arbitrary characteristic. Awtar proved that if a prime ring of characteristic not equal to two has a nontrivial derivation which is centralizing on a nonzero Jordan ideal, then the ideal is contained in the center of the ring.

Recall that a ring R is prime if $aRb=0$ implies that $a=0$ or $b=0$. Furthermore, if I is a nonzero ideal in a prime ring with $aIb=0$, then $a=0$ or $b=0$. Let $[x, y]=xy-yx$ and note the important identity $[x, yz]=y[x, z]+[x, y]z$. This identity shows that the mapping $I_x(y)=[x, y]$ is a derivation, the *inner derivation* determined by x. I_x is zero if and only if x is in the *center* $Z=\{z \in R \mid [z, R]=0\}$.

LEMMA 1. [5, Lemma 1]. *If D is a nonzero derivation of a prime ring R, then the left and right annihilators of $D(R)$ are zero. In particular, $a[b, R]=0$ or $[b, R]a=0$ implies that $I_b=0$ (b is in Z) or $a=0$.*

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LEMMA 2. *Let I be a nonzero right ideal in a prime ring R.*

(a) *If R has a derivation D which is zero on I, then D is zero on R.*

(b) *If R has a homomorphism T which is the identity on I, then T is the identity on R.*

Proof. (a) If $D(I)=0$, then $0=D(IR)=D(I)R+ID(R)=ID(R)$. By Lemma 1 D must be zero since I is nonzero. (b) Let x be in I and a, b be in R. Then $xab=T(xab)=T(xa)T(b)=xaT(b)$. Thus $xa(b-T(b))=0$ and either $x=0$ or $b-T(b)=0$. But I is nonzero and so contains an $x\neq 0$. This forces $T(b)=b$ for all b in R.

LEMMA 3. *If the prime ring R contains a commutative nonzero right ideal I, then R is commutative.*

Proof. If x is in I, then $I_x(I)=[x,I]=0$ since I is commutative. By Lemma 2 $I_x=0$ on R and x is in the center. Thus $[x,R]=0$ for every x in I. Hence $I_a(I)=0$ for all a in R and again by Lemma 2, $I_a=0$ and a is the center for all a in R. Therefore R is commutative.

LEMMA 4. *Let b and ab be in the center of a prime ring R. If b is not zero, then a is in Z, the center of R.*

Proof. $0=[ab,r]=a[b,r]+[a,r]b=[a,r]b$ for all r in R. By Lemma 1 $b=0$ or a is in Z. Hence a must be in Z.

Now if L is a linear mapping on R and S is a subset of R closed under addition such that L is centralizing on S, then by linearization

(1) $$[x,L(y)]+[y,L(x)] \text{ is in } Z \text{ for all } x \text{ and } y \text{ in } S.$$

In particular, $[x,[x,L(y)]+[y,L(x)]]=0$. Using the Jacobi identity on this last equation gives

(2) $$[x,[L(y),x]]+[L(x),[x,y]]=0 \quad \text{for all } x \text{ and } y \text{ in } S.$$

If the characteristic of a prime ring is not equal to two and L is either an automorphism or derivation such that $[x,L(x)]$ is in Z for all x in some ideal I, then it can easily be shown that $[x,L(x)]=0$ for all x in I. In fact, this holds under somewhat weaker hypotheses.

LEMMA 5. *Let R be a prime ring of characteristic not equal to two and U be a Jordan subring of R. If L is a Jordan homorphism or Jordan derivation of U such that $[x,L(x)]$ is in the center of R for all x in U, then $[x,L(x)]=0$ for all x in U.*

Proof. Let T be a Jordan homomorphism of U and replace y by x^2 in (1). Then $[x,T(x^2)]+[x^2,T(x)]$ is in Z for all x in U. Thus $T(x)[x,T(x)]+[x,T(x)]T(x)+x[x,T(x)]+[x,T(x)]x=2(x+T(x)[x,T(x)]$ is in Z. By Lemma 4, either $[x,T(x)]=0$ or $x+T(x)$ is in Z. But if $x+T(x)$ is in Z, then $[x,x+T(x)]=[x,T(x)]=0$. So $[x,T(x)]=0$ for all x in U.

If D is a Jordan derivation on U, again replace y by x^2 in (1) to obtain $[x, D(x^2)]+[x^2, D(x)]=4x[x, D(x)]$ is in Z. By Lemma 4, $[x, D(x)]=0$ for all x in U.

It would be nice to have $[x, L(x)]=0$ for arbitrary characteristic.

LEMMA 6. *Let I be a right ideal in a prime ring R. If L is a derivation or homomorphism of R such that $[x, L(x)]$ is in Z for all x in I, then $[x, L(x)]=0$ for all x in I.*

Proof. If the characteristic of R is not two, Lemma 5 implies that $[x, L(x)]=0$ on I. So suppose R has characteristic equal to two. Let x and y be in I and L be a linear mapping, then $[[x, y], L(x)]+[x^2, L(y)]=x([y, L(x)]+[x, L(y)])+([y, L(x)]+[x, L(y)])x=2x([y, L(x)]+[x, L(y)])=0$ by (1) and the fact that R has characteristic two. Letting $z=L(x)$, we obtain

(3) $$[[x, y], z]+[x^2, L(y)]=0 \quad \text{for } x \text{ and } y \text{ in } I,\ z=L(x).$$

As a special case of (3) when $x=y$,

(4) $$[x^2, z]=0 \quad \text{for all } x \text{ in } I,\ z=L(x).$$

Since I is a right ideal, let $y=xz$ in (3) to obtain $0=[[x, xz], z]+[x^2, L(xz)]=[x[x, z], z]+[x^2, L(xz)]=[x, z]^2+[x^2, L(xz)]$. So we have

(5) $$[x, z]^2=[x^2, L(xz)] \quad \text{for all } x \text{ in } I,\ z=L(x).$$

If $L=D$ is a derivation, then $[x^2, D(xz)]=[x^2, z^2+xD(z)]=x[x^2, D(z)]=x(D([x^2, z])-[D(x^2), z])=0$ by (4) and the fact that $D(x^2)=[x, D(x)]$ is central. So by (5), $[x, z]^2=0$ and hence $[x, z]=[x, D(x)]=0$ since R is prime. If $L=T$ is a homomorphism, then using (4) in (5) gives $[x, z]^2=[x^2, zT(z)]=z[x^2, T(z)]$. Let $y=xzx$ in (3) so that $0=[x, z][x^2, z]+[x^2, zT(z)z]=z[x^2, T(z)]z$ by (4). Hence $[x, z]^2z=0$ and thus $[x, z]=[x, T(x)]=0$ since R is prime.

Now if a linear mapping L is such that $[x, L(x)]=0$ for all x in some subset S closed under addition in R, this can be linearized to

(6) $$[x, L(y)]+[y, L(x)]=0 \quad \text{for all } x \text{ and } y \text{ in } S.$$

We now have enough information to prove the main theorem of this paper.

THEOREM 1. *Let R be a prime ring and I be a nonzero ideal in R. If L is a nontrivial automorphism or derivation of R such that $xL(x)-L(x)x$ is in the center of R for every x in I, then the ring R is commutative.*

Proof. Let T be an automorphism of R satisfying the hypotheses of the theorem. By Lemma 6, $[x, T(x)]=0$ for all x in I. Replacing y by xy in (6) results in $0=[x, T(x)T(y)]+[xy, T(x)]=T(x)[x, T(y)]+x[y, T(x)]$. Using (6) on the commutator in the last term gives $(T(x)-x)[x, T(y)]=0$ for all x and y in I. Since I is an ideal, we may replace y in this last equation by ya where a is any element in R. Then $0=(T(x)-x)[x, T(y)T(a)]=(T(x)-x)([x, T(y)]T(a)+T(y)[x, T(a)])=(T(x)-x)T(y)[x, T(a)]$ and so $(T(x)-x)T(I)\times[x, T(a)]=0$ for all x in I and a in R. Now T is an automorphism and I is a nonzero ideal so $T(I)$ is also a nonzero ideal. Since R is prime, either $T(x)-x=0$ or $[x, T(a)]=0$ for all a in R. Hence for any element x in I, T fixes x or x is in the center of R.

T is not the identity on R and so by Lemma 2, T is not the identity on I. Thus there is an element $x\neq 0$ in I such that $T(x)\neq x$ and x is in Z. Let y be any other element in I. If y is not in the center, then neither is $x+y$ and T fixes both y and $x+y$. But then, $T(x+y)=T(x)+T(y)=T(x)+y=x+y$ and so $T(x)=x$, a contradiction. Hence y is in Z for every y in I. This means that I is commutative and by Lemma 3, R is also commutative.

Now let D be a nonzero derivation of R which centralizes I. By Lemma 6, $[x, D(x)]=0$ for all x in I. As in the automorphism case, replace y by xy in (6) to obtain $0=[x, D(xy)]+[xy, D(x)]=[x, D(x)y]+[x, xD(y)]+[xy, D(x)]$. Thus $0=D(x)[x, y]+x([x, D(y)]+[y, D(x)])$ and by (6) this last term is zero. Therefore $D(x)[x, y]=0$ for all x in y in I. I is an ideal so y may be replaced by ya where a is any element in R. Then $0=D(x)[x, ya]=D(x)y[x, a]+D(x)[x, y]a=D(x)y[x, a]$. Thus $D(x)I[x, a]=0$ for all x in I and a in R. R prime implies that $D(x)=0$ or $[x, a]=0$ for all a in R. So for any element x in I, $D(x)=0$ or x is in Z.

D is not zero on R so by Lemma 2, D is not zero on I. Hence there exists an element $x\neq 0$ in I such that $D(x)\neq 0$ and x is in Z. Let y by any other element in I. Then the same kind of argument used in the automorphism case shows that y is in Z and thus I is commutative. Again by Lemma 3, R is commutative.

It is easy to extend this theorem to the case where the centralized ideal is quadratic Jordan. This generalizes Awtar's theorem for centralizing derivations.

THEOREM 2. *Let R be a prime ring and U be a nonzero quadratic Jordan ideal of R. If L is a nontrivial automorphism or derivation of R which is centralizing on U, then R is commutative.*

Proof. McCrimmon [4] has shown that every nonzero quadratic Jordan ideal contains a nonzero associative ideal I. Apply Theorem 1 to the ideal I to conclude that R is commutative.

The following example due to McCrimmon shows that in the automorphism case the results cannot be extended to semi-prime rings. Let R be the direct sum of two copies of a simple ring S which is not commutative. R is then semi-prime. Let T be the exchange automorphism defined on R by $T(x_1, x_2) = (x_2, x_1)$. The ideal $S \oplus 0$ satisfies the hypotheses of both theorems but R is not commutative.

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References

1. R. Awtar, *Lie and Jordan structures in prime rings with derivations*, Proc. Amer. Math. Soc., **41** (1973), 67–74.

2. J. Mayne, *Centralizing automorphisms of prime rings*, Canad, Math. Bull., **19(1)** (1976), 113–115.

3. J. Mayne, *Ideals and centralizing mappings in prime rings*, Proc. Amer. Math. Soc., **86** (1982), 211–212. Erratum **89** (1983), 187.

4. K. McCrimmon, *On Herstein's theorems relating Jordan and associative algebras*, J. Algebra, **13** (1969), 382–392.

5. E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093–1100.

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