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# THE LIMITING BEHAVIOUR OF CERTAIN SEQUENCES OF CONTINUED FRACTIONS

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We investigate the set of limit points of the continued fractions

$$\frac{1}{x_k+}\frac{1}{x_{k-1}+}\cdots\frac{1}{x_1}, \qquad k=1,2,3,\dots,$$

where $x_1, x_2, \dots$ is a given sequence of positive integers. We show that this set is closed, and that it may include any given countable subset of $[0,1]$ if the integers $x_k$ are chosen appropriately. Our main result, which has applications in transcendence theory, is that the sequence of continued fractions has no rational limit point when the sequence $\{x_k\}$ of partial quotients is bounded.

## 1. INTRODUCTION

Let $\mathbf{X} = \{x_k\}_{k\geqslant 1}$ be a sequence of positive integers. We wish to investigate the nature of $\Lambda(\mathbf{X})$, the set of limit points of the sequence $\{Q_k\}_{k\geqslant 1}$ defined by

$$Q_k = \frac{1}{x_k+}\frac{1}{x_{k-1}+}\cdots\frac{1}{x_1}. \tag{1}$$

Let $\xi = \frac{1}{x_1+}\frac{1}{x_2+}\cdots \in \mathbf{R}-\mathbf{Q}$. Then $Q_k$ is the ratio $\frac{q_k}{q_{k+1}}$ of the denominators of successive convergents to $\xi$, and we shall sometimes write $\Lambda(\xi)$ for $\Lambda(\mathbf{X})$. Conversely, *any irrational number* $\xi$, $0<\xi<1$, *uniquely determines the sequence* $\mathbf{X}$ *and the set* $\Lambda(\mathbf{X})$.

The source of this problem lies in papers by Loxton and van der Poorten [**2**] and Angell [**1**] on functional equation methods in transcendence theory. Given a real irrational

$$\omega = \frac{1}{a_1+}\frac{1}{a_2+}\cdots$$

with bounded partial quotients, it was found necessary to show that the ratio $\frac{q_k}{q_{k+1}}$ approaches an irrational limit as $k$ tends to infinity through some suitable subsequence $\mathbf{K}$ of $\mathbf{N}$—that is, in our present notation, that $\Lambda(\omega)$ contains an irrational. In this paper, we shall first see what can be said about $\Lambda(\mathbf{X})$ without imposing the condition of boundedness on $\mathbf{X}$; among our results are that $\Lambda(\mathbf{X})$ is a closed set, and that $\mathbf{X}$ may be chosen in such a way that $\Lambda(\mathbf{X})$ contains a given countable subset of $[0,1]$. We shall then return to the bounded case. We can show what the original problem requires, and even more:

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**Theorem.** *If* $\mathbf{X}$ *is bounded then* $\Lambda(\mathbf{X})$ *contains no rationals.*

In what follows, lower case Greek letters will denote real numbers in the interval $[0,1]$; the partial quotients in the (finite or infinite) continued fraction expansions of such numbers will be denoted by the corresponding roman letters:

$$\alpha = \frac{1}{a_1+}\frac{1}{a_2+}\cdots$$

The $j$-th complete quotient of $\alpha$ is written

$$\alpha_j = \frac{1}{a_{j+1}+}\frac{1}{a_{j+2}+}\cdots$$

This is valid for $j = 0,1,2,\dots$ if $\alpha$ is irrational, and for $j = 0,1,\dots,n-1$ if $\alpha = \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_n}$ is rational. In the latter case we set $\alpha_n = 0$ and we leave $\alpha_j$ undefined for $j > n$.

## 2. Examples

**1.** If $x_k$ is a constant $x$ for all large $k$ then $\Lambda(\mathbf{X}) = \{\lambda\}$, where

$$\lambda = \frac{1}{x+}\frac{1}{x+}\frac{1}{x+}\cdots = \frac{1}{2}\left(-x+\sqrt{x^2+4}\right)$$

which is irrational.

**2.** If $\mathbf{X}$ is eventually periodic of period $p$ then

$$\Lambda(\mathbf{X}) = \{[\overline{a_p,a_{p-1},\dots,a_1}],[\overline{a_{p-1},\dots,a_1,a_p}],[\overline{a_1,a_p,\dots,a_2}]\},$$

a finite set of quadratic irrationals, and has precisely $p$ elements. Here $[\overline{a_p,\dots,a_1}]$ denotes the periodic continued fraction

$$\frac{1}{a_p+}\cdots\frac{1}{a_1+}\frac{1}{a_p+}\cdots\frac{1}{a_1+}\cdots.$$

**3.** $0 \in \Lambda(\mathbf{X})$ if and only if $\mathbf{X}$ is unbounded.

**Proof:** For any $k$,

$$\frac{1}{x_k+1} \leqslant Q_k \leqslant \frac{1}{x_k}.$$

(In fact both inequalities are strict when $k \geqslant 3$.) Hence $Q_k$ is bounded away from zero if $\mathbf{X}$ is bounded; conversely, if some subsequence of $\mathbf{X}$ increases without limit, then the corresponding subsequence of $\{Q_k\}_{k\geqslant 1}$ tends to zero. ∎

**4.** Similarly, $\Lambda(\mathbf{X}) = \{0\}$ (that is, $\lim_{k\to\infty} x_k Q_k = 0$) if and only if $\lim_{k\to\infty} = \infty$.

**5.** In fact $\lim_{k\to\infty} Q_k$ exists (that is, $\Lambda(\mathbf{X})$ is a singleton) if and only if either

(i) $\mathbf{X}$ is eventually constant; or

(ii) $\lim_{k\to\infty} x_k = \infty$

(that is, in the cases covered by examples 1 and 4).

PROOF: We have

$$x_k = \frac{1}{Q_k} - Q_{k-1}.$$

If $Q_k \to 0$ as $k \to \infty$ then $x_k \to \infty$; if, on the other hand, $Q_k$ tends to a non-zero limit then $x_k$ tends to a (finite) limit. In the latter case, since each $x_k$ is an integer, $\mathbf{X}$ must be eventually constant. This establishes one half of the result; the converse is given by examples 1 and 4 above. ▮

## 3. Some general results.

It is clear by counting arguments that $\Lambda(\mathbf{X})$ cannot be an arbitary subset of $[0,1]$; in this section we state and prove a few properties of $\Lambda(\mathbf{X})$.

**Definition.** Two real numbers $\xi$, $\eta$ (not necessarily in $[0,1]$) are said to be *equivalent* if

$$\xi = \frac{a\eta + b}{c\eta + d}$$

for some integers, $a$, $b$, $c$, $d$ with $ad - bc = \pm 1$.

LEMMA. *Let* $\mathbf{X}$ *be as above and* $\mathbf{Y} = \{y_k\}_{k\geqslant 1}$, *where* $y_k = x_{k+1}$. *Then* $\Lambda(\mathbf{X}) = \Lambda(\mathbf{Y})$.

PROOF: Define $Q_k$ as in (1) and

$$\begin{aligned} Q'_k &= \frac{1}{y_k+}\,\frac{1}{y_{k-1}+}\cdots\frac{1}{y_1} \\ &= \frac{1}{x_{k+1}+}\,\frac{1}{x_k+}\cdots\frac{1}{x_2}. \end{aligned}$$

Then

$$Q_{k+1} = \frac{x_1 p + p'}{x_1 q + q'} \tag{2}$$

where $\frac{p}{q} = Q'_k$, and $\frac{p'}{q'}$ is the second last convergent in the continued fraction of $Q'_k$. Therefore

$$\begin{aligned} |Q_{k+1} - Q'_k| = \left|Q_{k+1} - \frac{p}{q}\right| &= \frac{1}{q(x_1 q + q')} \qquad \text{from (2)} \\ &\to 0 \end{aligned}$$

since $q \to \infty$ as $k \to \infty$. Hence any limit point of $\{Q_k\}$ is a limit point of $\{Q'_k\}$, and conversely. ▮

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**Corollary.** *If $\xi$ and $\eta$ are equivalent irrational numbers between 0 and 1, then* $\Lambda(\xi) = \Lambda(\eta)$.

**Proof:** If $\xi$ and $\eta$ are equivalent, their continued fractions have the forms

$$\xi = \frac{1}{x_1+}\cdots\frac{1}{x_m+}\frac{1}{z_1+}\frac{1}{z_2+}\cdots$$
$$\eta = \frac{1}{y_1+}\cdots\frac{1}{y_n+}\frac{1}{z_1+}\frac{1}{z_2+}\cdots;$$

by $m+n$ applications of the lemma we have

$$\Lambda(\xi) = \Lambda(\mathbf{Z}) = \Lambda(\eta).$$

Examples 1 and 2 above follow easily from this result. The converse is false; for we can clearly construct inequivalent $\xi = \frac{1}{x_1+}\frac{1}{x_2+}\cdots$ and $\eta = \frac{1}{y_1+}\frac{1}{y_2+}\cdots$ with the property $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = \infty$; and then we have, from example 4, $\Lambda(\xi) = \Lambda(\eta) = \{0\}$.

Clearly $\Lambda(\mathbf{X}) \subseteq [0,1]$; conversely, we can show by an example that any $\alpha \in [0,1]$ is in $\Lambda(\mathbf{X})$ for some $\mathbf{X}$. If $\alpha = 0$, see examples 3 and 4 above; otherwise, suppose first that $\alpha$ is rational and write

$$\alpha = \frac{p}{q} = \frac{1}{a_1+}\cdots\frac{1}{a_n}.$$

If $\eta$ is any (finite or infinite) continued fraction of the form

$$\eta = \frac{1}{a_1+}\cdots\frac{1}{a_n+}\frac{1}{m+}\frac{1}{b_1+}\frac{1}{b_2+}\cdots$$

then $\frac{p}{q}$ is a convergent to $\eta$ and we have

$$\left|\frac{p}{q} - \eta\right| < \frac{1}{mq^2} = \frac{C}{m}$$

since $q$ is fixed. The sequence

(3) $$\mathbf{X} = \{1, a_n, a_{n-1}, \ldots, a_1, 2, a_n, a_{n-1}, \ldots, a_1, 3, \ldots\}$$

then does what is required; for if $m > 0$ we have

$$Q_{m(n+1)} = \frac{1}{a_1+}\cdots\frac{1}{a_n+}\frac{1}{m+}\cdots\frac{1}{a_n+}\frac{1}{1}$$

and by the previous result

$$\left|\frac{p}{q} - Q_{m(n+1)}\right| < \frac{C}{m}.$$

Hence $Q_{m(n+1)} \to \alpha$ as $m \to \infty$, and $\alpha \in \Lambda(\mathbf{X})$. If on the other hand $\alpha = \frac{1}{a_1+}\frac{1}{a_2+}\cdots \notin \mathbf{Q}$, consider the sequence

$$\mathbf{X} = \{1, a_1, 2, a_2, a_1, 3, a_3, a_2, a_1, 4, \dots\}. \tag{4}$$

We have

$$Q_{\frac{1}{2}(m^2+3m)} = \frac{1}{a_1+}\cdots\frac{1}{a_m+}\frac{1}{m+}\cdots\frac{1}{a_1+}\frac{1}{1}$$

and so

$$\left|Q_{\frac{1}{2}(m^2+3m)} - \frac{p_m}{q_m}\right| < \frac{1}{mq_m^2}$$

where $\frac{p_m}{q_m} = \frac{1}{a_1+}\cdots\frac{1}{a_m}$. But $\left|\alpha - \frac{p_m}{q_m}\right| < \frac{1}{q_m^2}$ since $\frac{p_m}{q_m}$ is a convergent to $\alpha$; hence

$$\left|\alpha - Q_{\frac{1}{2}(m^2+3m)}\right| < \frac{m+1}{mq_m^2}.$$

Thus $Q_{\frac{1}{2}(m^2+3m)} \to \alpha$ as $m \to \infty$, and $\alpha \in \Lambda(\mathbf{X})$. ∎

It is amusing to note that in these constructions we have reversed classical procedure by using an irrational, or a "complicated" rational, as an approximation to one of its convergents.

**Remarks.**

**1.** By similar means we can construct $\mathbf{X}$ so that $\Lambda(\mathbf{X})$ contains any two given numbers in $[0,1]$. For example, if $\alpha = \frac{1}{a_1+}\cdots\frac{1}{a_n}$, $\beta = \frac{1}{b_1+}\frac{1}{b_2+}\cdots$, choose

$$\mathbf{X} = \{1, b_1, 1, a_n, \dots, a_1, 2, b_2, b_1, 2, a_n, \dots, a_1, 3, b_3, b_2, b_1, \dots\}.$$

Then $\alpha, \beta \in \Lambda(\mathbf{X})$. We can even make $\Lambda(\mathbf{X})$ contain a given countably infinite subset of $[0,1]$; in particular, $\Lambda(\mathbf{X})$ may include all rationals between 0 and 1.

**2.** If $\mathbf{X}$ is the sequence defined by (3) we can calculate $\Lambda(\mathbf{X})$ precisely. For $j = 0, 1, 2, \dots, n-1$ we have

$$\left|Q_{m(n+1)-j} - \alpha_j\right| < \frac{C}{m},$$

where $\alpha_j$ is as defined in Section 1, and therefore $\alpha_j \in \Lambda(\mathbf{X})$. For $j = n$, moreover,

$$\left|Q_{m(n+1)-j}\right| < \frac{1}{m},$$

so $0 \in \Lambda(\mathbf{X})$. Now suppose $\lambda \in \Lambda(\mathbf{X})$; then some sequence $\{Q_k\}_{k \in \mathbf{K}}$ converges to $\lambda$. Such a sequence contains, for some fixed $j = 0, 1, \dots, n$, infinitely many terms

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$Q_{m(n+1)-j}$: the sequence of such terms tends to the limit $\alpha_j$ (recall that $\alpha_n = 0$ by definition). Hence $\lambda = \alpha_j$; therefore $\Lambda(\mathbf{X})$ is precisely the set

$$\{0, \frac{1}{a_n}, \frac{1}{a_{n-1}+}\frac{1}{a_n}, \dots, \frac{1}{a_1+}\cdots\frac{1}{a_n}\}.$$

Observe that $\Lambda(\mathbf{X}) \subseteq \mathbf{Q}$.

**3.** If $\mathbf{X}$ is the sequence (4), then the above reasoning shows that

$$\alpha_j = \lim_{m\to\infty} Q_{\frac{1}{2}(m^2+3m)-j}$$

and hence $\Lambda(\mathbf{X}) \supseteq \{0, \alpha_0, \alpha_1, \dots\}$. Furthermore, if any finite sequence $b_1, \dots, b_n$ ($n \geqslant 1$) occurs infinitely often in $\{a_k\}_{k\geqslant 1}$ then the sequence $\{Q_k\}_{k\geqslant 1}$ contains terms of the form

$$\frac{1}{b_1+}\cdots\frac{1}{b_n+}\frac{1}{m+}\cdots\frac{1}{a_1+}\frac{1}{1}$$

for arbitarily large $m$; hence $\frac{1}{b_1+}\cdots\frac{1}{b_n} \in \Lambda(\mathbf{X})$. It would seem that $\Lambda(\mathbf{X})$ may contain a wide variety of numbers; we close this remark with the observation that it must contain any point of accumulation of all the numbers mentioned so far, for we have:

THEOREM. *For any sequence* $\mathbf{X}$, $\Lambda(\mathbf{X})$ *is closed.*

PROOF: Let $\lambda$ be an accumulation point of $\Lambda(\mathbf{X})$; write $\lambda = \lim_{i\to\infty} \xi^{(i)}$, $\xi^{(i)} \in \Lambda(\mathbf{X})$: without loss of generality $\xi^{(1)} \neq Q_1$. Define $k_j$ inductively by setting $k_1 = 1$ and choosing $k_{j+1}$ to be the least integer $s > k_j$ such that

$$0 < \left|Q_s - \xi^{(j+1)}\right| < \frac{1}{2}\left|Q_{k_j} - \xi^{(j)}\right|;$$

this is always possible since, for a given $j$, $\{Q_k\}_{k\geqslant 1}$ contains elements arbitrarily close to $\xi^{(j+1)}$. (Note that the condition $Q_s \neq \xi^{(j+1)}$ is necessary in order that the process may continue; for the same reason we specified $Q_{k_1} \neq \xi^{(1)}$ above.) Then $\{Q_{k_j}\}_{j\geqslant 1}$ is a subsequence of $\{Q_k\}_{k\geqslant 1}$, and

$$\begin{aligned}\left|\lambda - Q_{k_j}\right| &\leqslant \left|\lambda - \xi^{(j)}\right| + \left(\frac{1}{2}\right)^{j-1}\left|Q_1 - \xi^{(1)}\right| \\ &\to 0 \text{ as } j \to \infty.\end{aligned}$$

Hence $\lambda = \lim_{j\to\infty} Q_{k_j} \in \Lambda(\mathbf{X})$. ∎

COROLLARY. *For some* $\mathbf{X}$, $\Lambda(\mathbf{X}) = [0,1]$.

PROOF: As in Remark 1 above, $\Lambda(\mathbf{X})$ may contain every rational in $[0,1]$; but $\Lambda(\mathbf{X})$ is closed. ∎

## 4. Bounded sequences

We have seen (example 2, p.71–72) that without the boundedness condition on X our original problem (to show that $\Lambda(\mathbf{X})$ contains an irrational number) may have no solution. However, in the bounded case the situation is different: we can prove a far stronger result than we actually require.

THEOREM. *If* $\mathbf{X}$ *is bounded then* $\Lambda(\mathbf{X})$ *contains no rationals.*

PROOF: This is an immediate consequence of the following theorem. ∎

**Definition.** For any positive integer $M$, let $\mathbf{B}_M$ be the set of all real numbers in $[0,1]$ which can be expanded in a continued fraction with partial quotients at most $M$. For $0 \leqslant \xi \leqslant 1$ write

$$\mu(\xi, M) = \inf_{\beta \in B_M,\, \beta \neq \xi} |\xi - \beta|.$$

The following result shows that a number can well be approximated by numbers with bounded partial quotients (if and) only if it is an irrational whose partial quotients satisfy the same bound.

THEOREM. *Let* $M$ *be a positive integer. Then* $\mu(\xi, M) = 0$ *if and only if* $\xi$ *is an irrational element of* $\mathbf{B}_M$.

PROOF: The converse statement is quickly proved: if $\xi \in \mathbf{B}_m - \mathbf{Q}$ then the convergents $\beta_m = \frac{p_m}{q_m}$ to $\xi$ satisfy

$$\beta_m \in \mathbf{B}_M, \qquad \beta_m \neq \xi, \qquad \lim_{m\to\infty} \beta_m = \xi$$

so $\mu(\xi, M) = 0$. To prove the forward half of the theorem we first note that if $\xi$ and $\beta$ are (finite or infinite) continued fractions

$$\xi = \frac{1}{x_1+}\frac{1}{x_2+}\cdots, \qquad \beta = \frac{1}{b_1+}\frac{1}{b_2+}\cdots,$$

and if $x_1 = b_1, \ldots, x_m = b_m$, then

$$\begin{aligned} |\xi - \beta| = |\xi_0 - \beta_0| &= \xi_0\beta_0 \left| \left(\frac{1}{\xi_0} - x_1\right) - \left(\frac{1}{\beta_0} - b_1\right) \right| \\ &= \xi_0\beta_0 \,|\xi_1 - \beta_1| = \ldots \\ &= \xi_0 \cdots \xi_{m-1}\beta_0 \cdots \beta_{m-1}\,|\xi_m - \beta_m|. \end{aligned}$$

Supppose first that $\xi = \frac{1}{x_1+} \cdots \frac{1}{x_n} \in \mathbf{Q}$. Let

(5) $$\beta = \frac{1}{b_1+} \cdots \frac{1}{b_{n+2}+} \cdots$$

be a finite or infinite continued fraction in $\mathbf{B}_M$. Since $\beta$ has at least $n+2$ partial quotients, $\beta \neq \xi$. Define $m$ to be the greatest integer $s$ such that $x_1 = b_1, \dots, x_s = b_s$. Then $m \leqslant n$ and we have

$$
\begin{aligned}
|\xi - \beta| &\geqslant \xi_0 \dots \xi_{m-1}\beta_0 \dots \beta_{m-1} |\xi_m - \beta_m| \\
&\geqslant \xi_0 \dots \xi_{n-1}(M+1)^{-n} |\xi_m - \beta_m|. \qquad (6)
\end{aligned}
$$

(The last step is necessary since $m$ depends on $\beta$.) We now have

(i) if $m = n$ then $\xi_m = 0$ and

$$|\xi_m - \beta_m| = \beta_m \geqslant \frac{1}{M+1};$$

(ii) if $m < n$ and $x_{m+1} > b_{m+1}$ then

$$|\xi_m - \beta_m| = \beta_m - \xi_m \geqslant \frac{1}{b_{m+1}+}\,\frac{1}{1+}\,\frac{1}{M+1} - \frac{1}{x_{m+1}}$$

using the fact (from (5)) that $\beta$ has at least $n+2$ partial quotients. Hence

$$
\begin{aligned}
|\xi_m - \beta_m| &\geqslant \frac{(M+2)(x_{m+1} - b_{m+1}) - (M+1)}{x_{m+1}((M+2)b_{m+1} + (M+1))} \\
&\geqslant \frac{1}{N(M^2+3M+1)}
\end{aligned}
$$

where $N = \max_{1 \leqslant j \leqslant n} x_j$; and

(iii) if $m < n$ and $x_{m+1} < b_{m+1}$ then $|\xi_m - \beta_m| \geqslant \dfrac{1}{(N+1)(M^2+M+1)}$ as in (ii).

Hence

$$\inf |\xi - \beta| \geqslant \frac{\xi_0 \dots \xi_{n-1}(M+1)^{-n}}{(N+1)(M^2+3M+1)} = C(\xi, M) > 0,$$

where the infimum here extends over all $\beta \in \mathbf{B}_M$ of the form (5). Since by doing this we have excluded only finitely many elements of $\mathbf{B}_M$, we have

$$\mu(\xi, M) > 0.$$

Suppose, on the other hand, that $\xi$ is irrational and not in $\mathbf{B}_M$. Let $n$ be maximal such that $x_1, x_2, \dots, x_n \leqslant M$; for any $\beta \in \mathbf{B}_M$ of the form

$$\frac{1}{b_1+} \cdots \frac{1}{b_{n+3}+} \cdots$$

let $m$, as before, be the greatest integer $s$ such that $x_1 = b_1, \ldots, x_s = b_s$. Then $m \leqslant n$ and (6) is again valid. We have

(i) if $m = n$ then

$$|\xi_m - \beta_m| = \beta_m - \xi_m \geqslant \frac{1}{M+}\frac{1}{1+}\frac{1}{M+1} - \frac{1}{M+1}$$
$$= \frac{1}{(M+1)(M^2+3M+1)},$$

where we have relied on the form (7) of $\beta$; and

(ii) if $m < n$ then $|\xi_m - \beta_m| \geqslant C'(\xi, M)$, where $C'(\xi, M)$ is the constant of (ii) or (iii) above.

Hence

$$\inf|\xi - \beta| \geqslant \frac{\xi_0 \ldots \xi_{n-1}(M+1)^{-n}}{(N+1)(M+1)(M^2+3M+1)} = C(\xi, M) > 0,$$

where, as for the case $\xi \in \mathbf{Q}$, the infimum excludes only finitely many values of $\beta$. This completes the proof of the theorem. ∎

**Remarks.**

1. In the case where $\mathbf{X}$ is bounded, it is still possible that $\Lambda(\mathbf{X})$ be infinite. For example, write $\alpha^{(j)} = \frac{1}{2+} \cdots \frac{1}{2+}\frac{1}{1}$, $\beta^{(j)} = \frac{1}{2+} \cdots \frac{1}{2+}\frac{1}{2}$, where each continued fraction has just $j+1$ partial quotients. If

$$\xi^{(j)} = \frac{1}{2+} \cdots \frac{1}{2+}\frac{1}{1+} \cdots$$

is a finite or infinite continued fraction whose sequence of partial quotients begins with precisely $j$ twos, then $\xi^{(j)}$ lies in $I^{(j)}$, the closed interval between $\alpha^{(j)}$ and $\beta^{(j)}$ (that is, $I^{(j)} = [\alpha^{(j)}, \beta^{(j)}]$ or $[\beta^{(j)}, \alpha^{(j)}]$ according as $j$ is odd or even). It may be checked that these intervals are pairwise disjoint. Now let $\mathbf{X}$ be the sequence

$$\mathbf{X} = \{1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 2, 2, 1, 2, 2, 1, 2, 1, \ldots\}$$

which contains each finite sequence $1, 2, \ldots, 2$ infinitely often. Then for each $j$, infinitely many $Q_k$ lie in $I^{(j)}$; and hence each $I^{(j)}$ contains at least one limit point of $\{Q_k\}_{k \geqslant 1}$. Hence $\Lambda(\mathbf{X})$ is infinite. (In fact for any $j$ the sequence $\{Q_{k_{jn}}\}_{n \geqslant j}$, where $k_{jn} = 1 + 3 + 6 + \cdots + \frac{1}{2}n(n+1) - \frac{1}{2}j(j-1)$, lies entirely in $I^{(j)}$.)

2. Apart from the limit points already mentioned in the $I^{(j)}$, $\Lambda(\mathbf{X})$ also contains any accumulation point of all these; one such point is easily seen to be

$$\frac{1}{2+}\frac{1}{2+} \cdots = \sqrt{2} - 1$$

which lies in none of the $I^{(j)}$.

**3.** We conclude with a question: if $\mathbf{X}$ is a bounded sequence of positive integers, can $\Lambda(\mathbf{X})$ be uncountable? It may help to recall that $\Lambda(\mathbf{X})$ is a closed set which contains no rationals.

## References

[1] D.D. Angell, *Mahler's Method in Transcencence Theory* (Ph.D. thesis, University of New South Wales, 1987).

[2] J.H. Loxton and A.J. van der Poorten, 'Arithmetic properties of certain functions in several variables III', *Bull. Austral. Math. Soc.* **16** (1977), 15–47.

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