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Spectral Properties of a Family of Minimal Tori of Revolution in the Five-dimensional Sphere

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Abstract. The normalized eigenvalues $\Lambda_i(M,g)$ of the Laplace–Beltrami operator can be considered as functionals on the space of all Riemannian metrics g on a fixed surface M. In recent papers several explicit examples of extremal metrics were provided. These metrics are induced by minimal immersions of surfaces in $\mathbb{S}^3$ or $\mathbb{S}^4$. In this paper a family of extremal metrics induced by minimal immersions in $\mathbb{S}^5$ is investigated.

1 Introduction

Let M be a closed surface and let g be a Riemannian metric on M. Then the Laplace–Beltrami operator $\Delta\colon C^\infty(M) \to C^\infty(M)$ is given by the formula

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\Big(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\Big).$$

The spectrum of Δ consists only of eigenvalues. Let us denote them by

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \lambda_3(M,g) \leqslant \cdots,$$

where the eigenvalues are written with their multiplicities.

In this paper the family of functionals

$$\Lambda_i(M,g) = \lambda_i(M,g)\,\mathrm{Area}(M,g)$$

is investigated. Let us fix M. We are interested in investigating $\sup \Lambda_i(M,g)$, where the supremum is taken over the space of all Riemannian metrics on M.

An upper bound for $\Lambda_1(M,g)$ in terms of the genus of M was provided in [28] and later the existence of an upper bound for $\Lambda_i(M,g)$ was shown in [17]. Several recent papers [5–7, 11, 12, 19, 22, 23] deal with finding the exact values of this supremum in the space of all Riemannian metrics on several particular surfaces. We refer the reader to the introduction of [25] for more details.

In an attempt to solve this problem, the following definition was introduced in several papers; see *e.g.,* [6, 22].

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Definition 1.1 A Riemannian metric g on a closed surface M is called an *extremal metric* for a functional $\Lambda_i(M,g)$ if for any analytic deformation g_t such that $g_0 = g$ the following inequality holds:

$$\frac{d}{dt}\Lambda_i(M,g_t)\Big|_{t=0+} \leqslant 0 \leqslant \frac{d}{dt}\Lambda_i(M,g_t)\Big|_{t=0-}.$$

For the correctness of this definition we refer the reader to [1, 2, 7].

A real breakthrough in finding explicit examples of (smooth) extremal metrics became possible due to connection with the theory of minimal surfaces in spheres discovered in [7]. Let $\psi\colon M \looparrowright \mathbb{S}^n$ be a minimal immersion in the unit sphere. We denote by Δ the Laplace–Beltrami operator on M associated with the metric induced by the immersion ψ. Let us introduce Weyl's counting function

$$N(\lambda) = \#\{i \mid \lambda_i(M,g) < \lambda\}.$$

The following theorem provides a general approach to finding smooth extremal metrics.

Theorem 1.2 (El Soufi, Ilias [7]) *Let $\psi\colon M \looparrowright \mathbb{S}^n$ be a minimal immersion of a surface in the unit sphere $\mathbb{S}^n$ endowed with the canonical metric g_{can}. Then the metric $\psi^* g_{can}$ on M is extremal for the functional $\Lambda_{N(2)}(M,g)$.*

In the recent papers [15, 18, 24, 25] this connection was used to provide several examples of extremal metrics on the torus and the Klein bottle. These metrics were induced by minimal immersions of the corresponding surfaces in $\mathbb{S}^3$ and $\mathbb{S}^4$. In this paper a family of minimally immersed surfaces in $\mathbb{S}^5$ is investigated. For any pair of positive integers m,n such that $m \geqslant n$ and $(m,n) = 1$, we consider a doubly 2π-periodic immersion $\phi_{m,n}\colon \mathbb{R}^2 \to \mathbb{S}^5$, given by the formula

(1.1) $$\phi_{m,n}(x,y) = \left(\sqrt{\frac{m+n}{2m+n}}e^{imy}\sin x, \sqrt{\frac{m+n}{m+2n}}e^{iny}\cos x, \sqrt{\frac{n\cos^2 x}{m+2n} + \frac{m\sin^2 x}{2m+n}}e^{-i(m+n)y}\right),$$

where $\mathbb{S}^5$ is considered as the set of unit length vectors in $\mathbb{C}^3$. We denote the image of $\phi_{m,n}$ by $M_{m,n}$. To the best of author's knowledge, the explicit formula (1.1) first appeared in the introduction of [20]. This immersion can be obtained due to a general construction by Mironov (see [21]). We should mention that $M_{m,n}$ were described in conformal coordinates in [9, 13]. The main result of this paper is the following theorem.

Main Theorem *For any pair of positive integers m,n such that $m \geqslant n$ and $(m,n) = 1$, the immersion $\phi_{m,n}$ is minimal. The corresponding surface $M_{m,n}$ is a torus. If $mn \equiv 0$* mod 2*, then the metric induced on $M_{m,n}$ by the immersion is extremal for the functional $\Lambda_{4(m+n)-3}(\mathbb{T}^2,g)$. If $mn \equiv 1$* mod 2*, then the metric induced on $M_{m,n}$ by the immersion is extremal for the functional $\Lambda_{2(m+n)-3}(\mathbb{T}^2,g)$.*

The proof of this theorem is similar to the proof of the main theorem in [24] by Penskoi. However, we should mention that the exposition here is much simplified;

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e.g., we do not use the theory of the Magnus–Winkler–Ince equation. We also fill a gap by giving a rigorous proof of [24, Proposition 20].

We provide the exact value of the corresponding functional in terms of elliptic integrals of the first and the second kind given respectively by the formulae

$$K(k) = \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx, \qquad E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx.$$

Following the paper [15] we also prove the non-maximality of the metric on $M_{m,n}$.

Proposition 1.3 *If* $mn \equiv 0 \mod 2$, *then*

$$\Lambda_{4(m+n)-3}(M_{m,n}) = 16\pi\left(\sqrt{m^2+2mn}\,E\left(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\right) - \frac{mn}{\sqrt{m^2+2mn}}K\left(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\right)\right).$$

If $mn \equiv 1 \mod 2$, *then*

$$\Lambda_{2(m+n)-3}(M_{m,n}) = 8\pi\left(\sqrt{m^2+2mn}\,E\left(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\right) - \frac{mn}{\sqrt{m^2+2mn}}K\left(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\right)\right).$$

For every pair $\{m,n\} \neq \{1,1\}$ *the metric on* $M_{m,n}$ *is not maximal for the corresponding functional.*

Remark 1.4 It is easy to check that $\phi_{1,1}$ is an immersion of the flat equilateral torus in $\mathbb{S}^5$ by first eigenfuctions, and as it was shown in [22] that this metric is maximal for the functional $\Lambda_1(\mathbb{T}^2, g)$.

The paper is organized in the following way. In Section 2.1 we describe $M_{m,n}$ as a part of a general construction from [21] by Mironov. Then in Section 2.3 we reduce the problem of finding $N(2)$ for Δ to the similar problem for a family of periodic Sturm–Liouville operators. Finally, Section 3 contains the proof of Main Theorem, and Section 4 is dedicated to the proof of Proposition 1.3.

2 Preliminaries

2.1 Construction of Minimal Lagrangian Submanifolds in $\mathbb{C}^n$ by Mironov

Let M be a k-dimensional submanifold of $\mathbb{R}^n$ given by equations

$$e_{1j}u_1^2 + \cdots + e_{nj}u_n^2 = d_j, \qquad j = 1,\ldots,n-k,$$

where $d_j \in \mathbb{R}$ and $e_{ij} \in \mathbb{Z}$. Since $\dim M = k$, the vectors $e_j = (e_{j1},\ldots,e_{j(n-k)}) \in \mathbb{Z}^{n-k}$, $j = 1,\ldots,n$ form a lattice Λ of maximal rank in $\mathbb{R}^{n-k}$. Let us denote by Λ^* the dual lattice to Λ,

$$\Lambda^* = \left\{ y \in \mathbb{R}^{n-k} \mid (e_i, y) \in \mathbb{Z}, i = 1,\ldots,n \right\},$$

where $(x,y) = x_1y_1 + \cdots + x_{n-k}y_{n-k}$.

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Consider the map $\phi: M \times (\mathbb{R}^{n-k}/\Lambda^*) \to \mathbb{C}^n$ given by the explicit formula

$$\phi(u_1, \dots, u_n, y) = (u_1 e^{2\pi i (e_1, y)}, \dots, u_n e^{2\pi i (e_n, y)}).$$

We endow $\mathbb{C}^n$ with the standard symplectic form

$$\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n.$$

Recall that an immersion $\psi: N \looparrowright \mathbb{C}^n$ is called Lagrangian if $\psi^* \omega = 0$.

Theorem 2.1 (Mironov [21]) *Suppose that $e_1 + \cdots + e_n = 0$. Then the immersion ϕ is a minimal Lagrangian immersion.*

Let us now consider a particular case

$$M = \left\{ (x_1, x_2, x_3) \mid m x_1^2 + n x_2^2 - (m+n) x_3^2 = 0 \right\} \subset \mathbb{R}^3.$$

Then by Theorem 2.1, the immersion ϕ is a minimal Lagrangian immersion. It is easy to see that in this case $\operatorname{Im} \phi$ is a cone $C(M_{m,n})$ over $M_{m,n}$. It is a standard fact that $C(M_{m,n})$ is minimal in $\mathbb{C}^3$ if and only if $M_{m,n}$ is minimal in $\mathbb{S}^5 \subset \mathbb{C}^3$; see *e.g.*, [26].

2.2 Symmetries of $\phi_{m,n}$

The goal of this section is to prove the following proposition.

Proposition 2.2 *Suppose $m \neq n$. If $mn \equiv 1 \bmod 2$, then one has $\phi_{m,n}(x, y) = \phi_{m,n}(x+\pi, y+\pi)$ and $\phi_{m,n}|_{[0,2\pi)\times[0,2\pi)}$ is a double cover almost everywhere. If $mn \equiv 0 \bmod 2$, then $\phi_{m,n}|_{[0,2\pi)\times[0,2\pi)}$ is one-to-one almost everywhere. Thus $M_{m,n}$ is a torus for each $m, n > 0$, $(m, n) = 1$.*

Remark 2.3 In fact, according to the paper [21], one can omit the words "almost everywhere" in the previous proposition.

Proof Since $(m, n) = 1$, there are no symmetries of the form $(x, y) \mapsto (x, y + \alpha)$. Examining the third coordinate of $\phi_{m,n}$, we see that the only possible symmetry has the form

$$(x, y) \mapsto \left((-1)^{\varepsilon_1} x + (-1)^{\varepsilon_2} \pi, y + \frac{2\pi}{m+n} \right),$$

where $\varepsilon_i = 0, 1$. Substituting this into the first two coordinates of $\phi_{m,n}$ we obtain the statement of the proposition. ∎

2.3 Associated Periodic Sturm–Liouville Problem

In this section we reduce the problem of finding $N(2)$ for the Laplace–Beltrami operator on $M_{m,n}$ to a similar problem for the associated Sturm–Liouville operator.

Let us introduce the notations

$$\sigma(x) = \sqrt{m^2 + 4mn + n^2 - (m^2 - n^2)\cos 2x},$$
$$\rho(x) = (m+n)(m+n-(m-n)\cos 2x).$$

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Direct calculations show that the metric on $M_{m,n}$ is given by

$$\rho(x)\Big(\sigma(x)^{-2}dx^2 + \frac{1}{2}dy^2\Big).$$

Then a straightforward calculation shows that the following formula holds for the Laplace–Beltrami operator,

$$\Delta f = -\frac{1}{\rho(x)}\Big(\sigma(x)\frac{\partial}{\partial x}\Big(\sigma(x)\frac{\partial f}{\partial x}\Big) + 2\frac{\partial^2 f}{\partial y^2}\Big). \tag{2.1}$$

Proposition 2.4 *Assume* $mn \equiv 0 \bmod 2$. *The number* λ *is the eigenvalue of Laplace–Beltrami operator* (2.1) *if and only if there exists* $l \in \mathbb{Z}_{\geqslant 0}$ *such that there is a solution of the following associated periodic Sturm–Liouville problem:*

$$-\sigma(x)\frac{d}{dx}\Big(\sigma(x)\frac{dg(x)}{dx}\Big) + 2l^2 g(x) = \lambda\rho(x)g(x), \qquad g(x+2\pi) \equiv g(x). \tag{2.2}$$

The corresponding eigenspace is spanned by the functions of the form $g(l,x)\sin lx$ *and* $g(l,x)\cos lx$*, where* l *is any positive integer number such that a solution of equation* (2.2) *exists and* $g(l,x)$ *is the corresponding solution.*

If $mn \equiv 1 \bmod 2$*, then the statement remains the same with the boundary conditions*

$$g(x+\pi) \equiv (-1)^l g(x). \tag{2.3}$$

Proof Let us remark that Δ commutes with $\frac{\partial^2}{\partial y^2}$. Thus, these operators have a common basis of eigenfunctions of the form $g(l,x)\cos lx$ and $g(l,x)\sin lx$. By substituting these eigenfunctions into formula (2.1) we obtain equation (2.2). Since any function on $M_{m,n}$ should be doubly 2π-periodic, we have $l \in \mathbb{Z}_{\geqslant 0}$ and boundary conditions in (2.2).

In the case $mn \equiv 1 \bmod 2$, any function $f \in C^\infty(M_{m,n})$ should satisfy the condition $f(x+\pi, y+\pi) = f(x,y)$. This condition implies immediately boundary conditions (2.3). ■

For a general Sturm–Liouville problem the following classic proposition holds; see *e.g.*, [4].

Proposition 2.5 *Consider a periodic Sturm–Liouville problem in the form*

$$-\frac{d}{dt}\Big(p(t)\frac{d}{dt}g(t)\Big) + q(t)g(t) = \lambda r(t)g(t), \qquad g(t+t_0) \equiv g(t), \tag{2.4}$$

where $p(t), r(t) > 0$ *and* $p(t+t_0) \equiv p(t), q(t+t_0) \equiv q(t), r(t+t_0) \equiv r(t)$. *Let us denote by* λ_i *and* $g_i(t)$ $(i = 0,1,2,\dots)$ *the eigenvalues and eigenfunctions of problem* (2.4). *Then the following inequalities hold:*

$$\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \lambda_5 \leqslant \lambda_6 \leqslant \cdots$$

For $\lambda = \lambda_0$ *there exists a one-dimensional eigenspace spanned by* $g_0(t)$. *For* $i \geqslant 0$ *if* $\lambda_{2i+1} < \lambda_{2i+2}$*, then there is a one-dimensional* λ_{2i+1}*-eigenspace spanned by* $g_{2i+1}(t)$ *and there is a one-dimensional* λ_{2i+2}*-eigenspace spanned by* $g_{2i+2}(t)$. *If* $\lambda_{2i+1} = \lambda_{2i+2}$*, then*

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there is a two-dimensional eigenspace spanned by $g_{2i+1}(t)$ *and* $g_{2i+2}(t)$ *with eigenvalue* $\lambda = \lambda_{2i+1} = \lambda_{2i+2}$.

The eigenfunction $g_0(t)$ *has no zeros on* $[0, t_0)$. *The eigenfunctions* $g_{2i+1}(t)$ *and* $g_{2i+2}(t)$ *each have exactly* $2i + 2$ *zeros on* $[0, t_0)$.

Proposition 2.6 *For* $l \geqslant 0$ *the eigenvalues* $\lambda_i(l)$ *of problem* (2.2) *are strictly increasing functions of the parameter* l.

Proof The Raleigh quotient for equation (2.2) is defined by the formula

$$R_l[f] = \frac{\int_0^{2\pi} \big(\sigma(x)(f')^2 + \frac{2l^2}{\sigma(x)} f^2\big)\, dx}{\int_0^{2\pi} \frac{\rho(x)}{\sigma(x)} f^2\, dx}.$$

By the variational characterization of the eigenvalues (see *e.g.*, [10]), one has

$$\lambda_k(l) = \inf_{E_k} \sup_{f \in E_k} R_l[f],$$

where the infimum is taken over all $(k+1)$-dimensional subspaces E_k in the space of all 2π-periodic functions of the Sobolev space $H^1[0, 2\pi]$. Moreover, the infimum is reached on the space $V_k(l)$ formed by the first $(k+1)$ eigenfunctions. Let us remark that $R_{l_1}[f] < R_{l_2}[f]$ if $0 \leqslant l_1 < l_2$.

Then $\lambda_k(l_1) \leqslant \sup_{f \in V_k(l_2)} R_{l_1}[f]$. The latter supremum is reached on some function $g \in V_k(l_2)$. Thus one has

$$\lambda_k(l_1) \leqslant R_{l_1}[g] < R_{l_2}[g] \leqslant \sup_{f \in V_k(l_2)} = \lambda_k(l_2),$$

which completes the proof. ∎

3 Proof of Main Theorem

We need the following classic theorem (see *e.g.*, [16]).

Theorem 3.1 *Let* $M \looparrowright \mathbb{S}^n$ *be a minimally immersed surface of the unit sphere* $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. *Then the restrictions* $x^1|_M, \dots, x^{n+1}|_M$ *on* M *of the standard coordinate functions of* $\mathbb{R}^{n+1}$ *are eigenfunctions of the Laplace–Beltrami operator on* M *with eigenvalue* 2.

According to Theorem 3.1, the components of $\phi_{m,n}$ are eigenfunctions of the Laplace–Beltrami operator on $M_{m,n}$. Since the function

$$\sqrt{\frac{n\cos^2 x}{m+2n} + \frac{m\sin^2 x}{2m+n}}$$

does not have zeroes on $[0, 2\pi)$, we have by Proposition 2.5 that

$$g_0(m+n, x) = \sqrt{\frac{n\cos^2 x}{m+2n} + \frac{m\sin^2 x}{2m+n}}$$

and $\lambda_0(m+n) = 2$. By Proposition 2.6 one has $\lambda_0(l) < 2$ for $l < m+n$. The function $\cos ny \cos x$ corresponds to $l = n$, whereas the function $\cos my \sin x$ corresponds to $l = m$. At the same time both $\sin x$ and $\cos x$ have 2 zeroes on $[0, 2\pi)$. Thus, again by

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Proposition 2.5, either $\lambda_1(m) = 2$ and $\lambda_2(n) = 2$ or $\lambda_1(n) = 2$ and $\lambda_2(m) = 2$. In the latter case we have a contradiction, since $m > n$ and by Proposition 2.6 $2 = \lambda_1(n) < \lambda_1(m) \leqslant \lambda_2(m) = 2$. Thus, $\lambda_1(l) < 2$ for $l < m$ and $\lambda_2(l) < 2$ for $l < n$. The last part of the proof of the Main Theorem is based on the following proposition, which we prove later in this section.

Proposition 3.2 *The eigenvalue* $\lambda_3(l)$ *of problem* (2.2) *satisfies the inequality* $\lambda_3(0) > 2$.

Recall that for every $\lambda_i(l)$ with $l > 0$ there are two eigenfuctions of the Laplace–Beltrami operator on $M_{m,n}$. This observation completes the proof in the case $mn \equiv 0 \bmod 2$.

If $mn \equiv 1 \bmod 2$, then one has to take into account the symmetry $(x, y) \mapsto (x + \pi, y + \pi)$; *i.e.*, if l is even, then we need to count only π-periodic solutions of equation (2.2), and if l is odd, then we need to count only π-antiperiodic solutions of (2.2). Application of Proposition 2.5 with $t_0 = \pi, 2\pi$ yields the fact that g_{2i+1} and g_{2i+2} are π-antiperiodic if and only if i is odd and π-periodic otherwise. Obvious calculations now complete the proof of the Main Theorem. ∎

The rest of this section is dedicated to the proof of Proposition 3.2.

3.1 Lamé Equation

In this section we recall several facts concerning the Lamé equation, usually written as

$$\frac{d^2\phi}{dz^2} + \left(h - \widehat{n}(\widehat{n}+1)k^2 \operatorname{sn}^z\right)\phi = 0. \tag{3.1}$$

We write $\widehat{n}$, since n is already used as a parameter in the family $M_{m,n}$.

We use a trigonometric form of the Lamé equation

$$\left[1 - (k\cos y)^2\right]\frac{d^2\phi}{dy^2} + k^2 \sin y \cos y \frac{d\phi}{dy} + \left[h - \widehat{n}(\widehat{n}+1)(k\cos y)^2\right]\phi = 0. \tag{3.2}$$

Equation (3.2) can be obtained from equation (3.1) using the following change of variables

$$\operatorname{sn} z = \cos y \quad \Longleftrightarrow \quad y = \frac{\pi}{2} - \operatorname{am} z,$$

where am is the Jacobi amplitude function; see *e.g.*, [8].

In order to prove Proposition 3.2 we need the following proposition.

Proposition 3.3 *Assume* $\widehat{n} = 1$. *Then the eigenvalue* $h_3(k)$ *is greater than* 2 *for every* $0 < k < 1$.

Proof According to [27] the number $h_3(k)$ can be characterized as the first eigenvalue of problem (3.2) with boundary conditions

$$\phi(y + \pi) \equiv \phi(y) \qquad \phi(y) \equiv -\phi(\pi - y). \tag{3.3}$$

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First let us rewrite equation (3.2) in the form

$$\frac{d}{dx}\Big(\sqrt{1-(k\cos x)^2}\frac{d\phi}{dx}\Big)+\frac{h-2(k\cos x)^2}{\sqrt{1-(k\cos x)^2}}\phi=0. \tag{3.4}$$

Let us denote $p(x)=\sqrt{1-(k\cos x)^2}$. We introduce an auxiliary Sturm–Liouville problem of the form

$$-\frac{d}{dx}\Big(p(x)\frac{d\phi}{dx}\Big)+p(x)\phi=\lambda p(x)\phi. \tag{3.5}$$

It easy to see that a function $\phi(x)$ is a solution of equation (3.4) with $h(k)=2$ if and only if $\phi(x)$ is a solution of equation (3.5) with $\lambda(k)=3$.

Therefore $h_3(k)\neq 2$ if and only if the Rayleigh quotient

$$R_k[f]=\frac{\int_0^\pi p(k,x)((f')^2+f^2)\,dx}{\int_0^\pi p(k,x)f^2\,dx} \tag{3.6}$$

is greater than 3 for any function f satisfying condition (3.3). Indeed, by the variational characterization of the eigenvalues the first eigenvalue $\widehat{\lambda}_0(k)$ of the problem (3.5) with boundary conditions (3.3) is equal to $\inf R[f]$, where the infimum is taken over the subspace $\mathcal{L}$ of functions $f\in H^1[0,\pi]$ satisfying conditions (3.3).

Then let us remark that the Rayleigh quotient (3.6) is a decreasing function of k. Indeed, if $k_1>k_2$, then $p(k_1,x)<p(k_2,x)$, and we have

$$\int_0^\pi p(k_1,x)(f')^2\,dx<\int_0^\pi p(k_2,x)(f')^2\,dx.$$

By adding $\int_0^\pi p(k_1,x)f^2\,dx\int_0^\pi p(k_2,x)f^2\,dx$ to both sides, we obtain

$$\int_0^\pi p(k_1,x)\big((f')^2+f^2\big)\int_0^\pi p(k_2,x)f^2\,dx<\int_0^\pi p(k_2,x)\big((f')^2+f^2\big)\,dx\int_0^\pi p(k_1,x)f^2\,dx.$$

This inequality implies $R_{k_1}[f]<R_{k_2}[f]$.

Therefore, since $p(1,x)=\sin x$ on $[0,\pi]$, one has the inequality

$$\widehat{\lambda}_0(k)>\inf_{f\in\mathcal{L}}\frac{\int_0^\pi((f')^2+f^2)\sin x\,dx}{\int_0^\pi f^2\sin x\,dx}. \tag{3.7}$$

Any function $f\in\mathcal{L}$ can be expressed in the form $g(\cos x)$, where g is a function in the segment $[-1,1]$ such that

$$\int_{-1}^1\frac{g^2(t)}{\sqrt{1-t^2}}\,dt<\infty,\quad \int_{-1}^1(g')^2(t)\sqrt{1-t^2}\,dt<\infty,\quad g(t)\equiv -g(-t).$$

Consequently, g lies in a wider space $\mathcal{H}$ given by

$$\mathcal{H}=\big\{g(t)\in L^2[-1,1]\mid g'(t)\sqrt{1-t^2}\in L^2[-1,1],\ g(t)\equiv -g(-t)\big\}.$$

Given any function $g\in\mathcal{H}$ consider the orthonormal basis in $L^2[-1,1]$ formed by normalized Legendre polynomials $\sqrt{(2n+1)/2}P_n(t)$. Let us recall that the Legendre

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polynomials satisfy the Legendre equation,

$$\frac{d}{dt}\Big((1-t^2)\frac{dP_n(t)}{dt}\Big) = -n(n+1)P_n(t).$$

Suppose that

$$g(t) = \sum_{i=1}^{\infty} a_n \sqrt{\frac{2n+1}{2}} P_n(t)$$

is a Fourier expansion for $g(t)$ that starts with $i = 1$ due to the oddity of $g(t)$. Then $g'(t)\sqrt{1-t^2} \in L^2[-1,1]$ and the associated Legendre functions $P_n^1(t) = \sqrt{1-t^2}P_n'(t)$ form an orthogonal basis in L^2 and let

$$g'(t)\sqrt{1-t^2} = \sum_{i=1}^{\infty} b_m \sqrt{\frac{2m+1}{2}} P_m^1(t).$$

Recall that for $m, n \geqslant 1$ one has the orthogonality property

$$\int_{-1}^{1} P_n^1(t)P_m^1(t)\,dt = \frac{2n(n+1)}{2n+1}\delta_{m,n}.$$

If by $(\cdot,\cdot)$ we denote the L^2-inner product in $\mathcal{H}$, then

$$\sqrt{\frac{2}{2n+1}}n(n+1)\,b_n = \big(g'(t)\sqrt{1-t^2}, P_n^1(t)\big) = -\Big(g(t), \frac{d}{dt}\Big((1-t^2)\frac{dP_n(t)}{dt}\Big)\Big)$$
$$= (g(t), n(n+1)P_n(t)) = \sqrt{\frac{2}{2n+1}}\,n(n+1)a_n.$$

It follows that $a_n = b_n$. Now the expression under the inf on the right-hand side of inequality (3.7) in terms of $g(t)$ has the form

$$R_1[g] = \frac{\int_{-1}^{1}(1-t^2)g'^2(t) + g^2(t)\,dt}{\int_{-1}^{1} g^2(t)\,dt}.$$

Substituting the series for $g(t)$ and $g'(t)\sqrt{1-t^2}$ into this quotient, we see that the infimum is reached on $g(t) = P_1(t) = t$, and the quotient is equal to 3. Thus, $\widehat{\lambda}_0(k) > 3$ for $0 < k < 1$.

Then it is easy to see that $h_3(0) = 4$ and $h_3(k)$ depend continuously on k. Since $h_3(k) \neq 2$, one has $h_3(k) > 2$ for every $k \in (0,1)$. ∎

3.2 Proof of Proposition 3.2

Let us first remark that equation (2.2) is the Lamé equation with parameters

$$k^2 = \frac{m^2 - n^2}{m^2 + 2mn}, \qquad h = \frac{(m^2+mn)\lambda - l^2}{m^2 + 2mn}, \qquad \widehat{n}(\widehat{n}+1) = \lambda.$$

Suppose the contradiction to the statement, *i.e.*, $\lambda_3(0) < 2$. Then, since $\lambda_3(n) > \lambda_2(n) = 2$, there exists a number l_2 such that $\lambda_3(l_2) = 2$. Then for $l = l_2$, equation (2.2) with $\lambda = 2$ has a solution with 4 zeroes on $[0, 2\pi)$. Therefore, so does the Lamé

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equation with $\widehat{n}(\widehat{n}+1)=\lambda$. But such a solution corresponds to either $h_3(k)$ or $h_4(k)$, and one has

$$h_4(k)\geqslant h_3(k)\geqslant 2 \quad \text{or} \quad \frac{2(m^2+mn)-l_2^2}{m^2+2mn}\geqslant 2,$$

which implies $l_2^2<0$. We obtain a contradiction. ∎

4 Value of the Corresponding Functional

In this section we prove Proposition 1.3. We start with the formula for the area of $M_{m,n}$.

$$\begin{aligned}(4.1)\quad &\operatorname{Area}(M_{m,n})\\ &=\frac{2\pi}{\sqrt{2}}\int_0^{2\pi}\frac{m^2+2mn+n^2-(m^2-n^2)\cos 2x}{\sqrt{m^2+4mn+n^2-(m^2-n^2)\cos 2x}}\,dx\\ &=8\pi\int_0^{\frac{\pi}{2}}\frac{m^2+mn-(m^2-n^2)\sin^2 x}{\sqrt{m^2+2mn-(m^2-n^2)\sin^2 x}}\,dx\\ &=8\pi\Bigg(\sqrt{m^2+2mn}E\Big(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\Big)-\frac{mn}{\sqrt{m^2+2mn}}K\Big(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\Big)\Bigg).\end{aligned}$$

If $mn\equiv 1 \bmod 2$, then one has to take into account the symmetry $(x,y)\mapsto(x+\pi,y+\pi)$, hence this number has to be divided by 2.

Now, following [14], we prove the non-maximality of the metric on $M_{m,n}$. Let us recall two propositions from [14].

Proposition 4.1 *The following inequality holds:* $\sup\Lambda_n(\mathbb{T}^2,g)>8\pi n$.

Proposition 4.2 *For every $k\in[0,1]$, one has*

$$K(k)-\frac{2}{2-k^2}E(k)\geqslant 0.$$

By Proposition 4.1 the following proposition implies non-maximality of the tori $M_{m,n}$.

Proposition 4.3 *If $mn\equiv 1 \bmod 2$ and $m\neq 1$, then the following inequality holds:*

$$8\pi\big(2(m+n)-3\big)\geqslant\Lambda_{2m+2n-3}(M_{m,n}).$$

If $mn\equiv 0 \bmod 2$, then the following inequality holds:

$$8\pi(4(m+n)-3)\geqslant\Lambda_{4m+4n-3}(M_{m,n}).$$

Proof Assume $mn\equiv 1 \bmod 2$. Then by formula (4.1),

$$\begin{aligned}(4.2)\quad &\Lambda_{2m+2n-3}(M_{m,n})\\ &=2\operatorname{Area}(M_{m,n})\\ &=8\pi\Bigg(\sqrt{m^2+2mn}E\Big(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\Big)-\frac{mn}{\sqrt{m^2+2mn}}K\Big(\sqrt{\frac{m^2-n^2}{m^2+2mn}}\Big)\Bigg).\end{aligned}$$

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Let us apply Proposition 4.2 with $k = \sqrt{\frac{m^2-n^2}{m^2+2mn}}$. Then we have

$$-\frac{m^2+4mn+n^2}{2m^2+4mn}K(k) \leqslant E(k).$$

Applying this inequality to formula (4.2), we have

$$\Lambda_{2m+2n-3} \leqslant 8\pi\sqrt{m^2+2mn}\Big(1-\frac{2mn}{m^2+4mn+n^2}\Big)E(k).$$

Therefore, in order to prove the first inequality, it is sufficient to obtain the inequality

$$\sqrt{m^2+2mn}\Big(1-\frac{2mn}{m^2+4mn+n^2}\Big)E(k) \leqslant 2m+2n-3. \tag{4.3}$$

Let us divide both parts of inequality (4.3) by m and denote the ratio $\frac{n}{m}$ by $x \in [0,1]$. Then formula (4.3) transforms into

$$\sqrt{1+2x}\Big(1-\frac{2x}{1+4x+x^2}\Big)E\Big(\sqrt{\frac{1-x^2}{1+2x}}\Big) \leqslant 2(1+x)-\frac{3}{m}.$$

Since $E(\widehat{k}) \leqslant \frac{\pi}{2}$ for each $\widehat{k} \in [0,1]$, this inequality could be obtained from

$$\frac{6}{m} \leqslant 4(1+k)-\pi\sqrt{1+2k}. \tag{4.4}$$

Inequality (4.4) holds for $m \geqslant 7$. Thus we have several exceptional cases:

$$\{m,n\} = \{3,1\},\ \{5,1\}\ \{5,3\}\ \{7,1\},\ \{7,3\},\ \{7,5\}.$$

For these cases, inequality (4.3) can be verified explicitly using the tables of elliptic integrals in [3].

Proof of the second inequality is obtained in the same way. There are also exceptional cases: $\{m,n\} = \{2,1\}, \{3,2\}$. ■

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