On Self-Adjoint Partial Differential Equations of the Second Order.

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§ 1. Let $F(u) = \sum_{i,k} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu$ be a linear differential expression involving n independent variables x_i, the coefficients A_{ik}, B_i and C being functions of the independent variables but not involving the dependent variable u. Associated with $F(u)$ is the *adjoint* expression

$$G(v) = \sum_{i,k} \frac{\partial^2}{\partial x_i \partial x_k}(A_{ik} v) - \sum_i \frac{\partial}{\partial x_i}(B_i v) + Cv.*$$

If the expressions $F(u)$ and $G(u)$ are identical, $F(u)$ is said to be *self-adjoint*, and the equation $F(u) = 0$ is a self-adjoint linear partial differential equation of the second order.

If a second order linear partial differential equation is obtained by annulling the variation of an integral according to the methods of the calculus of variations, this equation must be self-adjoint, and conversely. Also by applying the theory of continuous groups of transformations to such an integral, certain conservation theorems † satisfied by the solution of the partial differential equation will be obtained.

Consequently the investigation of the conditions which the coefficients of a general linear second order partial differential equation must satisfy to be self-adjoint appears to be of interest.

* See Courant u. Hilbert: *Methoden der Mathematischen Physik.* Band I., Kap. IV., § 8.

† *E.g.* in Dynamics, the theorems of conservation of energy and momentum. See a paper by EMMY NOETHER: Gött. Nach. (1918), p. 238. Also the present author's paper *Proc. Edin. Math. Soc.*, 42, p 61.

The principal result here obtained is that *in general a linear partial differential equation of the second order with constant coefficients can be made self-adjoint by multiplication by a factor* $e^{\sum\limits_i \lambda_i x_i}$ where the coefficients λ_i are certain constants. The exceptional case is when the equation is of parabolic type.

§ 2. Let us denote derivatives by suffixes. Then we may write $F(u) = \sum\limits_{i,k} a^{ik} u_{ik} + 2\sum\limits_i b^i u_i + cu$, where the coefficients a^{ik}, b^i, c are functions of the variables x_i, and where $a^{ik} = a^{ki}$. We easily see that

$$G(v) = \sum_{i,k} a^{ik} v_{ik} + 2\sum_i v_i\left(\sum_k a_k^{ik} - b^i\right) + cv + v\sum_i\left(\sum_k a_{ik}^{ik} - 2b_i^i\right).$$

If the coefficients satisfy the n first order partial differential equations $\sum a_k^{ik} = 2b^i \quad (i = 1, 2, \ldots\ldots n)$ then $F(u)$ is identical with $G(u)$ and is consequently self-adjoint. If the coefficients do not satisfy these equations, it may be possible to find a function $\phi(x_1, x_2, \ldots\ldots x_n)$ which is such that $\phi F(u)$ is self-adjoint. For the purpose of finding conservation theorems, this would be just as useful. The function ϕ must satisfy the n equations

$$\sum_k \phi a_k^{ik} + \sum_k \phi_k a^{ik} = 2\phi b^i \qquad (i = 1, 2, \ldots\ldots n).$$

The case of particular interest is the equation with constant coefficients. Such an equation can only be self-adjoint if the coefficients b^i are all zero. But if $\phi F(u)$ is self-adjoint and the expression $F(u)$ has constant coefficients, ϕ must satisfy the n equations $\sum\limits_k a^{ik}\phi_k = 2b^i\phi \quad (i = 1, 2, \ldots\ldots n)$.

This system of equations has the solution $\phi = e^{\sum \lambda_i x_i}$ where

$$2b^r = a^{r1}\lambda_1 + a^{r2}\lambda_2 + \ldots\ldots + a^{rn}\lambda_n (r = 1, 2, \ldots n).$$

If we exclude the case of an equation of parabolic type which is such that the determinant of the coefficients a^{ik} vanishes, the constants λ_i can be uniquely determined; hence any second order linear non-parabolic partial differential equation with constant coefficients can be made self-adjoint by multiplication by a factor $e^{\sum \lambda_i x_i}$ and so can be derived from a calculus of variations problem.

§ 3. To discuss the case when the determinant $| a^{ik} |$ vanishes, it is convenient to make use of a linear change of the independent variables x_i to $\underline{x}_i$, where $\underline{x}_i = \sum_k l_{ki} x_k$. Denoting $\frac{\partial u}{\partial \underline{x}_i}$ by $\underline{u}_i$ and so on, we have

$$F(u) = \sum_{i,k} \underline{u}_{ik} \left(\sum_{r,s} a^{rs} l_{ri} l_{sk}\right) + 2\sum_i \underline{u}_i \left(\sum_r b^r l_{ri}\right) + cu.$$

Since $| a^{ik} | = 0$, we can choose $l_{11}, l_{21}, \dots l_{n1}$ so as to satisfy the n equations $a^{1s} l_{11} + a^{2s} l_{21} + \dots + a^{ns} l_{n1} = 0 \; (s = 1, 2, \dots n)$. When this is done, the expression $F(u)$ becomes

$$\sum_{i,k}{}' \underline{a}^{ik} \underline{u}_{ik} + 2\sum_i{}' \underline{b}^i \underline{u}_i + cu + 2\underline{b}^1 \underline{u}_1$$

where $\sum'$ means that the summation is made over the values $2, 3, \dots n$. instead of $1, 2, \dots n$, and where the coefficients $\underline{a}^{ik}$, $\underline{b}^i$ are constants.

Now it can easily be shewn that $\phi F(u)$ will be self-adjoint after such a change only if it was before. The conditions then are that

$$2\underline{b}^i = \underline{a}^{i2} \underline{\lambda}_2 + \underline{a}^{i3} \underline{\lambda}_3 + \dots + \underline{a}^{in} \underline{\lambda}_n \quad (i = 2, 3, \dots n)$$

$$2\underline{b}^1 = 0$$

where $\phi = e^{\sum_i \underline{\lambda}_i \underline{x}_i}$.

Hence *if the expression* $F(u)$ *has constant coefficients which are such that* $| a^{ik} | = 0$, *and if it becomes self-adjoint on multiplication by a factor* $e^{\sum_i \lambda_i x_i}$, *it must be reducible to an expression in* $(n-1)$ *independent variables.*

Ex. 1. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\gamma}{c^2}\frac{\partial u}{\partial t} - \lambda u = 0$

is not self-adjoint. But considered in the form

$$e^{\gamma t}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\gamma}{c^2}\frac{\partial u}{\partial t} - \lambda u\right) = 0$$

it is self-adjoint, and can be derived from the Calculus of Variations Problem

$$\delta \iiint e^{\gamma t}\left\{\left(\frac{\partial u}{dx}\right)^2 + \left(\frac{\partial u}{dy}\right)^2 - \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \lambda u^2\right\} dx\,dy\,dt = 0.$$

Ex. 2. The equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \alpha\dfrac{\partial u}{\partial x} + \beta\dfrac{\partial u}{\partial y} + \gamma\dfrac{\partial u}{\partial z} + \lambda u = 0$

is not self-adjoint, and can only be made self-adjoint by multiplication by an exponential factor when $\gamma = 0$, that is, when the equation reduces to one in two independent variables.