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STABILITY OF STRONGLY REGULAR GRAPHS

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In this note we characterize strongly regular graphs which are stable.

1. Introduction

Transposition in the automorphism group of a graph is a necessary condition for a graph to be stable, but sufficiency conditions for stability of graphs have not yet been found. However, Holton [2] has shown that a tree is stable if and only if it contains a transposition in its automorphism group. Here we prove a similar result for strongly regular graphs.

We refer to [1] for the definitions and results not mentioned here. For a vertex u of a graph G, by $N_G(u)$ we denote the set of all vertices adjacent to u. By $\overline{N_G(u)}$, we denote the set $N_G(u) \cup \{u\}$.

If $W \subseteq V(G)$, then by G_W we mean the induced subgraph $\langle V(G)-W \rangle$ of G. By $\Gamma(G)_W$, we denote the maximal subgroup of $\Gamma(G)$ each element of which fixes each vertex in W; here we consider $\Gamma(G)_W$ as acting only on $V(G) - W$. A graph G is said to be *semi-stable* (at $v \in V(G)$) if $\Gamma(G_v) = \Gamma(G)_v$. If there exists a sequence $v_1, v_2, \ldots, v_n$ of all the vertices of G such that $G_{\{v_1,\ldots,v_k\}}$ is semi-stable at v_{k+1} for $1 \le k \le n-1$, we say that G is stable.

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An r-regular graph G of order n is said to be *strongly regular* if

(i) the number of vertices adjacent to both end vertices of an edge is constant and is equal to λ,

(ii) the number of vertices adjacent to two non-adjacent vertices is constant and is equal to μ.

n, r, λ and μ are called the parameters of G.

Now we list the following results which we require to prove the main result.

LEMMA 1 [3]. *If a graph G is stable, then either G is K_1 or $\Gamma(G)$ contains a transposition.*

LEMMA 2 [3]. *If a graph G is stable then $\overline{G}$ is stable.*

LEMMA 3 [4]. *If G is a strongly regular graph with parameters n, r, λ and μ, then $\overline{G}$ is a strongly regular graph with parameters $\overline{n}$, $\overline{r}$, $\overline{\lambda}$, and $\overline{\mu}$ where $\overline{n} = n$, $\overline{r} = n - r - 1$, $\overline{\lambda} = n - 2r + \mu - 2$ and $\overline{\mu} = n - 2r + \lambda$.*

2. Main result

THEOREM 1. *A nontrivial strongly-regular graph G with parameters n, r, λ and μ has a transposition in the automorphism group if and only if $G \cong mK_r$ or $\overline{mK_{n-r-1}}$ for some $m \geq 1$.*

Proof. If $G \cong mK_r$ or $\overline{mK_{n-r-1}}$ and is not equal to K_1, then certainly it has a transposition.

Suppose G is strongly-regular and $\Gamma(G)$ contains a transposition (uv). If $[u, v] \in E(G)$, then we prove that $G \cong mK_r$. Otherwise we prove that $G \cong \overline{mK_{n-r-1}}$.

Let $[u, v] \in E(G)$. Since $(uv) \in \Gamma(G)$,

$$\overline{N_G(u)} = \overline{N_G(v)}$$

or

$$\overline{N_G(u)} - \{u, v\} = \overline{N_G(v)} - \{u, v\} .$$

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But

$$d(u) = d(v) = r ;$$

therefore $\lambda = r - 1$.

Let $w \in \overline{N_G(u)}$. Since $[u, w] \in E(G)$ and $\lambda = r - 1$, $\overline{N_G(u)} = \overline{N_G(w)}$. Thus $\overline{N_G(u)}$ induces a complete graph of order r in G . If $n > r$, then choose any vertex $x \notin \overline{N_G(u)}$. Since G is strongly regular with $\lambda = r - 1$, $\overline{N_G(x)}$ also induces a graph isomorphic to K_r in G . If still there is any vertex $y \notin \overline{N_G(u)} \cup \overline{N_G(x)}$, then $\overline{N_G(y)}$ will induce a graph isomorphic to K_r . Proceeding in this way, till we exhaust all vertices of G , we find that every component of G is isomorphic to K_r , that is, $G = mK_r$ for some $m \geq 1$.

If $[u, v] \notin E(G)$ then $[u, v] \in E(\overline{G})$. Since $(uv) \in \Gamma(G)$, $(uv) \in \Gamma(\overline{G})$. From Lemma 3, we infer that $\overline{G}$ is strongly regular with parameters $\overline{n}, \overline{r}, \overline{\lambda}$, and $\overline{\mu}$ such that $\overline{n} = n$, $\overline{r} = n - r - 1$, $\overline{\lambda} = n - 2r + \mu - 2$ and $\overline{\mu} = n - 2r + \lambda$. Since $[u, v] \notin E(G)$,

$$N_G(u) = N_G(v) .$$

Hence

$$\mu = |N_G(u)| = r .$$

Therefore

$$\overline{\lambda} = n - r - 2 .$$

Since $\overline{G}$ is a strongly regular graph such that $[u, v] \in E(\overline{G})$, $(uv) \in \Gamma(\overline{G})$ and $\overline{\lambda}$ is equal to $n - r - 2$, it follows from earlier discussions that $\overline{G} \cong mK_{n-r-1}$ for some $m \geq 1$. Therefore $G \cong \overline{mK_{n-r-1}}$. This completes the proof.

COROLLARY 1. *A non-trivial strongly-regular graph is stable if and only if it contains a transposition in the automorphism group.*

Proof. Let G be a strongly-regular graph with parameters n, r, λ and μ . If $\Gamma(G)$ contains a transposition, it follows from Theorem 1 that $G \cong mK_r$ or $\overline{mK_{n-r-1}}$. Since complete graphs are stable, mK_r and

mK_{n-r-1} are also stable. From Lemma 2, we conclude that $\overline{mK_{n-r-1}}$ is stable. Therefore G is stable.

If G is stable, it is clear from Lemma 2 that $\Gamma(G)$ contains a transposition. This completes the proof.

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