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COINTEGRATION AND REPRESENTATION OF COINTEGRATED AUTOREGRESSIVE PROCESSES IN BANACH SPACES

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We extend the notion of cointegration for time series taking values in a potentially infinite dimensional Banach space. Examples of such time series include stochastic processes in $C[0,1]$ equipped with the supremum distance and those in a finite dimensional vector space equipped with a non-Euclidean distance. We then develop versions of the Granger–Johansen representation theorems for I(1) and I(2) autoregressive (AR) processes taking values in such a space. To achieve this goal, we first note that an AR(p) law of motion can be characterized by a linear operator pencil (an operator-valued map with certain properties) via the companion form representation, and then study the spectral properties of a linear operator pencil to obtain a necessary and sufficient condition for a given AR(p) law of motion to admit I(1) or I(2) solutions. These operator-theoretic results form a fundamental basis for our representation theorems. Furthermore, it is shown that our operator-theoretic approach is in fact a closely related extension of the conventional approach taken in a Euclidean space setting. Our theoretical results may be especially relevant in a recently growing literature on functional time series analysis in Banach spaces.

1. INTRODUCTION

Conventionally, the subject of time series analysis concerns time series taking values in finite dimensional Euclidean space. On the other hand, a recent literature on functional time series analysis deals with time series taking values in a possibly infinite dimensional Banach or Hilbert space, for instance, those in $C[0,1]$ equipped with the supremum norm. Examples of such time series are not restricted to function-valued stochastic processes: those in a finite dimensional vector space equipped with a non-Euclidean metric, such as Chebyshev distance or taxicab distance, are also included.

The property of cointegration, which was introduced by Granger (1981) and has been studied in Euclidean space, was recently extended to a more general setting. A recent paper by Chang, Kim, and Park (2016b) appears to be the first to consider the

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possibility of cointegration in an infinite dimensional Hilbert space. More recently, Beare, Seo, and Seo (2017) adopted the notion of cointegration from Chang et al. (2016b) and provided a rigorous treatment of cointegrated linear processes taking values in Hilbert spaces.

The Granger–Johansen representation theorem is a result on the existence and representation of I(1) (and I(2)) solutions to a given vector autoregressive (AR) law of motion that accommodates the possibility of cointegration. Due to crucial contributions by e.g., Engle and Granger (1987), Johansen (1991, 1992, 1995, 2008), Schumacher (1991), Faliva and Zoia (2002, 2010, 2011, 2021), Hansen (2005), and Franchi and Paruolo (2016, 2019), much on this subject is already well known in a Euclidean space setting. For a brief historical overview of this topic, see the introduction of Beare and Seo (2020). More recently, Chang, Hu, and Park (2016a), Hu and Park (2016), and Beare et al. (2017) extended the Granger–Johansen representation theorem for the I(1) case to a general Hilbert space setting. Moreover, Beare and Seo (2020) provided representation theorems for I(1) and I(2) AR processes in such a setting based on analytic operator-valued function theory, and Franchi and Paruolo (2020) developed a more general result for I(d) AR processes for $d \geq 1$.

This paper provides a suitable notion of cointegration and extends the Granger–Johansen representation theorem for Banach-valued, not necessarily Hilbert-valued, AR processes that are I(1) or I(2); that is, our theory can be applied to more general AR processes, for instance, those taking values in $C[0,1]$, $L^q[0,1]$, for $1 \leq q < \infty$, or any finite dimensional vector space equipped with an arbitrary norm. Viewed in the light of our purpose, our representation theorems need to be developed without relying on the following two preconditions commonly required in the literature: (i) a Hilbert space structure and (ii) a special restriction on the AR polynomial. To see this in detail, we briefly review the relevant literature. For a given AR(p) law of motion in a Hilbert space, which is characterized by the AR polynomial $\Phi(z) = I - z\phi_1 - \cdots - z^p\phi_p$, Beare et al. (2017) assume that $\phi_1, \ldots, \phi_p$ are compact operators when $p > 1$ (this compactness assumption is not required if $p = 1$), and provide a sufficient condition for the existence of I(1) solutions and a characterization of such solutions. In their representation theory, compactness of $\phi_1, \ldots, \phi_p$ makes $\Phi(z)$ belong to a special subclass of linear operators, called Fredholm operators, and the mathematical properties of such operators play an important role. More representation theorems in a Hilbert space setting are provided by Hu and Park (2016), Beare and Seo (2020), and Franchi and Paruolo (2020), among which the latter paper more generally deals with I(d) AR(p) processes for $d \geq 1$ and $p \geq 1$. The representation theorems in those papers are closely related to that provided by Beare et al. (2017) in the sense that Fredholmness of $\Phi(z)$ has a crucial role in their developments. The Fredholm assumption explicitly or implicitly employed in the foregoing papers turns out to place nontrivial restrictions on solutions to the AR(p) law of motion: nonstationarity of such a solution is driven by a necessarily finite dimensional unit root process even in an infinite dimensional setting. From another standpoint, Chang et al. (2016a) employ a different assumption that $\Phi(1)$ is a compact operator and provide an I(1) representation result. As opposed to the results under Fredholmness of $\Phi(z)$, it turns out that their compactness assumption always leads to I(1) solutions associated with an infinite dimensional unit root process unless the considered Hilbert space is finite dimensional. To briefly sum up, all of these existing versions are developed in a Hilbert space setting, and each of those relies on a special requirement about $\Phi(z)$, which restricts solutions to the AR(p) law of motion in a specific way. We thus need a novel approach to overcome these limitations in our more general setting.

To accomplish our goal, we first introduce a suitable notion of cointegration in Banach spaces by defining a cointegrating functional that properly generalizes the conventional notion of a cointegrating vector. We then characterize the cointegrating space (to be defined as the collection of cointegrating functionals) based on the Phillips–Solo device (Phillips and Solo, 1992) applied to operator-valued functions. After doing that, representation theorems for I(1) and I(2) AR processes taking values in a Banach space are provided. Our representation theory is derived under more primitive and weaker mathematical conditions in a general Banach space setting where we do not even have the notion of an angle (inner product) between two vectors. From Johansen (1991, 1992) to the foregoing recent papers, geometrical properties induced by an inner product, such as orthogonality, have been employed for the representation theory. However, it will be clarified in this paper that such a richer geometry is not necessarily required: in our representation theory, geometrical properties induced by an inner product have no essential role.

To obtain our representation theorems, we first note that an AR(p) law of motion in a Banach space allows the companion form AR(1) representation in a properly defined product Banach space, and it is thus characterized by a linear operator pencil (to be introduced in detail later) denoted by $\widetilde{\Phi}(z)$. By studying the spectral properties of a linear operator pencil, we find necessary and sufficient conditions for $\widetilde{\Phi}(z)^{-1}$ to have a pole of order 1 and 2, and also obtain a local characterization of $\widetilde{\Phi}(z)^{-1}$ near $z = 1$. These operator-theoretic results not only determine the integration order of solutions to the AR(p) law of motion, but also lead us to a representation of such solutions in terms of the behavior of $\widetilde{\Phi}(z)$ around $z = 1$; that is, our versions of the Granger–Johansen representation theorems for I(1) and I(2) AR processes are obtained. The fact that solutions to the AR(p) law of motion are characterized in terms of a local behavior of $\widetilde{\Phi}(z)$, rather than the original AR polynomial $\Phi(z)$, makes it difficult to compare our results to those developed in a Hilbert/Euclidean space setting. We thus provide further representation results so that important characteristics of I(1) or I(2) solutions are expressed in terms of linear operators associated with $\Phi(z)$. To this end, we first show that our necessary and sufficient condition for $\widetilde{\Phi}(z)^{-1}$ to have a pole of order 1 (resp. 2) is equivalent to a natural generalization of the well-known Johansen I(1) (resp. I(2)) condition, and then we recharacterize solutions to the AR(p) law of motion in a desired way using these equivalent conditions. By this further effort, not only can we obtain a more detailed characterization of such solutions, but also we can better clarify the connection between our representation results and those in the existing literature.

We structure the remainder of the paper as follows. In Section 2, we develop a suitable notion of cointegration in Banach spaces and provide some related results. Our representation theory for I(1) and I(2) AR processes is contained in Section 3, and concluding remarks follow in Section 4. Appendix A reviews background material for our study, and Appendix B collects the proofs of our main results.

2. COINTEGRATION IN BANACH SPACES

Let $\{X_t\}_{t\geq 0}$ be an I(1) time series taking values in Euclidean space of dimension n, denoted by $\mathbb{R}^n$. If there exists a nonzero vector $\beta \in \mathbb{R}^n$ such that $\{\beta^\intercal X_t\}_{t\geq 0}$ is stationary under a suitable choice of X_0, we then say that $\{X_t\}_{t\geq 0}$ is cointegrated with respect to β, and call β a cointegrating vector; see e.g. Johansen (1995, Def. 3.4). In this conventional definition of cointegration, β itself acts as a scalar-valued map defined on $\mathbb{R}^n$, and what makes β a cointegrating vector is *stationarity* of scalar-valued time series $\{\beta^\intercal X_t\}_{t\geq 0}$. We thus may understand cointegration as a property of scalar-valued maps defined on $\mathbb{R}^n$, which leads to the following alternative definition.

Definition 2.1. For an I(1) time series $\{X_t\}_{t\geq 0}$ and any scalar-valued linear map f defined on $\mathbb{R}^n$ (i.e., functional on $\mathbb{R}^n$), if $\{f(X_t)\}_{t\geq 0}$ can be stationary under a suitable choice of X_0, then we say that $\{X_t\}_{t\geq 0}$ is cointegrated with respect to f, and call f a cointegrating functional.

In fact, the above definition is equivalent to the conventional one due to the Riesz representation theorem (see, e.g., Conway, 1994, p. 13), implying that any functional f on $\mathbb{R}^n$ is uniquely identified as a vector β in the following sense: $f(x) = \beta^\intercal x$ for all $x \in \mathbb{R}^n$. Nevertheless, defining cointegration as in Definition 2.1 is advantageous especially when we consider a more general vector space; as will be shown, we may replace $\mathbb{R}^n$ with a separable complex Banach space $\mathcal{B}$ without a serious theoretical complication, and then may obtain a suitable notion of cointegration in $\mathcal{B}$.

Throughout this section, we formally introduce cointegrated I(1) and I(2) processes taking values in a Banach space and characterize the collection of cointegrating functionals. Prior to a detailed mathematical treatment of those, it may be helpful to see an example of functional time series of economic or statistical interest that can motivate our more general setting.

2.1. Example: Banach-Valued Time Series and Cointegration

Our more general setting is of central relevance for applications involving functional time series. As mentioned and analyzed in Hörmann, Horváth, and Reeder (2013) and Horváth, Kokoszka, and Rice (2014), possibly one of the most natural functional time series is a sequence of intraday price curves of a financial asset. Let

$X_t(s)$ be the price of a financial asset at time $s \in [s_{\min}, s_{\max}]$ on day $t \in \{1,2,\ldots\}$. By reparametrizing s into $u = (s - s_{\min})/(s_{\max} - s_{\min})$, $X_t := \{X_t(u), u \in [0,1]\}$ may be viewed as a random element in $C[0,1]$, the Banach space of continuous functions on $[0,1]$ equipped with the usual sup norm. Linear functionals defined on $C[0,1]$ reveal various characteristics of X_t. For example, consider f_1, f_2, and f_3 defined by

$$f_1(x) = x(1), \qquad f_2(x) = \int_0^1 x(u)du, \qquad f_3(x) = x(1) - \int_0^1 x(u)du, \tag{2.1}$$

where $x \in C[0,1]$. Then, $f_1(X_t)$ (resp. $f_2(X_t)$) computes the closing (resp. average) price on day t, and $f_3(X_t)$ gives their difference.

We may assume that the price curves observed on two adjacent days, say, X_{t-1} and X_t, are tightly connected in the sense that $X_t(0) = X_{t-1}(1)$ or loosely connected in the sense that an overnight jump $\varepsilon_t(0) := X_t(0) - X_{t-1}(1)$ is allowed. In either case, $\{X_t(u) - X_{t-1}(1)\}_{u\in[0,1]}$ denotes the cumulative intraday returns on day t. For illustrative purposes, we may model such a sequence of returns as follows: for a stationary process $\{\varepsilon_t\}_{t\geq 0}$,[1]

$$X_t(u) - X_{t-1}(1) = \varepsilon_t(u), \quad u \in [0,1], \tag{2.2}$$

where $\{\varepsilon_t(1)\}_{t\geq 0}$ is assumed to have a positive variance for reasons to become apparent. By introducing a linear operator ϕ_1 defined by $\phi_1(x)(u) = x(1)$ for $u \in [0,1]$ and then suppressing dependence on u for convenience, (2.2) can be written as a curve-valued AR(1) process as follows:

$$X_t = \phi_1 X_{t-1} + \varepsilon_t. \tag{2.3}$$

If we take the functional f_1 to both sides of (2.3), we find that $X_t(1) = X_{t-1}(1) + \varepsilon_t(1)$; that is, the time series of closing prices $\{f_1(X_t)\}_{t\geq 0}$ is a random walk with stationary increments. This implies that $\{X_t\}_{t\geq 0}$ is a nonstationary curve-valued process; if it were stationary, $\{f_1(X_t)\}_{t\geq 0}$ would be stationary since f_1 is a continuous linear transformation. On the other hand, note that $f_3(X_t) = f_3(\varepsilon_t)$ holds and thus $\{f_3(X_t)\}_{t\geq 0}$ is stationary, from which we find that f_3 transforms the curve-valued nonstationary process $\{X_t\}_{t\geq 0}$ into a scalar-valued stationary process. In view of Definition 2.1, f_3 may be called a cointegrating functional, which is, of course, informal at this point since we have not yet provided our formal definition of a cointegrating functional.

From the definition of ϕ_1 and (2.3), we also observe that the following holds:

$$Y_t(u) := X_t(u) - X_0(1) = \sum_{s=1}^{t} \varepsilon_s(1) + (\varepsilon_t(u) - \varepsilon_t(1)), \quad t \geq 1. \tag{2.4}$$

We know from (2.4) that, under a suitable initialization, X_t can be decomposed into the sum of two different components: the first is the random constant function X_t^N

[1] Empirical evidence about stationarity of cumulative stock return curves of a financial asset was provided in Horváth et al. (2014); however, we need to be careful in interpreting such evidence in our context since their results are obtained by viewing intraday price curves as random elements of the usual Hilbert space $L^2[0,1]$ rather than $C[0,1]$.

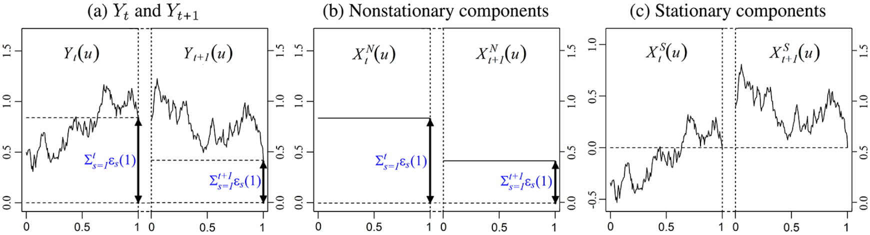

FIGURE 1. Decomposition of $Y_t(u) = X_t(u) - X_0(1)$ into the sum of X_t^N and X_t^S. *Notes:* For a clear graphical illustration, we let (i) Y_t (resp. ε_{t+1}) be a sample path of Brownian motion initialized at 0.5 (resp. 0) and (ii) let Y_t and Y_{t+1} be horizontally separated in the first plot (even if Y_t and Y_{t+1} are tightly connected under (i)). The horizontal axis in each plot represents values of $u \in [0,1]$.

defined by $X_t^N(u) = Y_t(1) = \sum_{s=1}^t \varepsilon_s(1)$, and the second is the random continuous function X_t^S which is given by $X_t^S(u) = \varepsilon_t(u) - \varepsilon_t(1)$ and hence satisfies that $X_t^S(1) = 0$ (Figure 1 graphically illustrates this decomposition). Since $\{X_t^S\}_{t\geq 1}$ is stationary, nonstationarity of $\{Y_t\}_{t\geq 1}$ results from the fact that the value of X_t^N follows a random walk. Thus, the decomposition given by (2.4) distinguishes the nonstationary and the stationary components of Y_t from each other, and, as will be discussed in detail later (Remark 2.2), this is a natural extension of the projection-based decomposition of a cointegrated vector-valued time series; that is, the transformations $Y_t \mapsto X_t^N$ and $Y_t \mapsto X_t^S$ can be viewed as done by a certain projection acting on $C[0,1]$ (Remark 2.3).

One may be interested in describing the above characteristics of the model (2.3) using the existing theory of cointegration, assuming that the intraday price curves are random elements of the usual Hilbert space $L^2[0,1]$, the space of square integrable functions on $[0,1]$ equipped with inner product $\langle f,g\rangle = \int_0^1 f(u)g(u)du$. In this case, however, the AR(1) operator ϕ_1 given in (2.3) and the functionals f_1 and f_3 given in (2.1) lack an essential continuity property required in the existing theory, and dealing with those linear maps in this context is far beyond what has been covered in the literature.[2] This example, therefore, shows that our general Banach space setting is useful to accommodate more various functional time series as subjects of the theory of cointegration.

Many economic/statistical time series can be understood as Banach-valued (especially $C[0,1]$-valued) stochastic processes. Some existing examples in the literature include daily electricity demand curves (Petris, 2013) and annual temperature profiles (Dette, Kokot, and Aue, 2020).[3] Moreover, Banach space methodology is sometimes naturally in demand when researchers want to adopt a

[2] ϕ_1, f_1, and f_3 are not continuous with respect to the topology of $L^2[0,1]$. Such a linear map is equivalently said to be unbounded. As far as this author knows, unbounded linear operators or functionals have not been considered in the existing theory of cointegration and the Granger–Johansen representation.

[3] Nielsen, Seo, and Seong (2019) recently considered similar empirical examples (Ontario monthly electricity demand curves and Australian annual temperature curves) in the $L^2[0,1]$ Hilbert space setting and found empirical evidence that those are cointegrated time series.

different notion of distance for functional time series analysis. For example, Dette et al. (2020) noted that two curves with rather different visual shapes may still have a small L^2-distance and thus be identified as similar in the usual $L^2[0,1]$ setting. They therefore employed the sup-distance, which is expected to better reflect the visualization of curve-valued observations in statistical analysis, by assuming that such observations are random elements of $C[0,1]$. In this regard, nonstationary, and possibly cointegrated, time series considered in the $L^2[0,1]$ setting can be potentially reconsidered in a Banach space setting; as an example of such time series, curves of age-specific employment rates (Nielsen et al., 2019; Seo, 2020), population counts (Shang et al., 2016), mortality rates (Gao and Shang, 2017), or fertility rates (Hyndman and Ullah, 2007) can be mentioned.

2.2. Notation

We review our notation for the subsequent discussions. The setting for our analysis is a separable complex Banach space $\mathcal{B}$ equipped with norm $\|\cdot\|_{\mathcal{B}}$. To conveniently introduce our notation, we let $\widetilde{\mathcal{B}}$ denote another such space, equipped with norm $\|\cdot\|_{\widetilde{\mathcal{B}}}$; for example, $\widetilde{\mathcal{B}}$ may be set to $\mathcal{B}$ or the complex plane $\mathbb{C}$.

A linear operator $A:\mathcal{B}\mapsto\widetilde{\mathcal{B}}$ is said to be bounded if $\|Ax\|_{\widetilde{\mathcal{B}}}\leq M\|x\|_{\mathcal{B}}$ for some $M<\infty$ and any $x\in\mathcal{B}$. Such an operator is obviously continuous on $\mathcal{B}$. Unless otherwise noted, every linear operator considered in this paper is bounded. Let $\mathcal{L}(\mathcal{B},\widetilde{\mathcal{B}})$ denote the space of bounded linear operators from $\mathcal{B}$ to $\widetilde{\mathcal{B}}$ equipped with the operator norm $\|A\|_{\text{op}}=\sup_{\|x\|\leq 1}\|Ax\|_{\widetilde{\mathcal{B}}}$. We are mostly concerned with the case $\mathcal{B}=\widetilde{\mathcal{B}}$, so let $\mathcal{L}(\mathcal{B})$ denote $\mathcal{L}(\mathcal{B},\mathcal{B})$, and let $I\in\mathcal{L}(\mathcal{B})$ denote the identity operator acting on $\mathcal{B}$. For any $A\in\mathcal{L}(\mathcal{B},\widetilde{\mathcal{B}})$, we let $\operatorname{ran}A$ (resp. $\ker A$) denote the set $\{Ax:x\in\mathcal{B}\}$ (resp. $\{x\in\mathcal{B}:Ax=0\}$). Commonly, $\operatorname{ran}A$ (resp. $\ker A$) is called the range (resp. the kernel) of A, and it is well known that $\ker A$ is necessarily closed, whereas $\operatorname{ran}A$ may not be so. If $\dim(\operatorname{ran}A)<\infty$, then A is said to be a finite rank operator. For any subspace $V\subset\mathcal{B}$, let V' denote the space of bounded linear functionals from V to $\mathbb{C}$ equipped with the operator norm, i.e., $V'=\mathcal{L}(V,\mathbb{C})$, which is commonly called the topological dual of V.

For any subset V of $\mathcal{B}$, let $\operatorname{cl}V$ denote the closure of V, i.e., the union of V and its limit points. For subspaces V_1 and V_2 of $\mathcal{B}$, we let V_1+V_2 denote the set $\{v_1+v_2: v_1\in V_1, v_2\in V_2\}$, which is called the algebraic sum of V_1 and V_2. If $V_1+V_2=\mathcal{B}$ and $V_1\cap V_2=\{0\}$ hold for closed subspaces V_1 and V_2, we then say that $\mathcal{B}$ is the direct sum of V_1 and V_2, and write $\mathcal{B}=V_1\oplus V_2$. In this case, V_1 (resp. V_2) is said to be complemented by V_2 (resp. V_1), and V_2 (resp. V_1) is called a complementary subspace of V_1 (resp. V_2). These definitions can be extended for a finite collection of subspaces $V_1,V_2,\ldots,V_k$ in an obvious way. For any set $V\subset\mathcal{B}$, we let $\operatorname{Ann}(V)$ denote the annihilator of V, defined by the set $\{f\in\mathcal{B}':f(x)=0,\forall x\in V\}$, which turns out to be a closed subspace of $\mathcal{B}'$(Fabian et al., 2010, p. 56). For any closed subspace V of $\mathcal{B}$, we let $\mathcal{B}/V$ denote the quotient space equipped with the quotient norm $\|\cdot\|_{\mathcal{B}/V}$, which is briefly reviewed in Appendix A.1.

The definitions of a $\mathcal{B}$-random variable X, its expectation $\mathbb{E}X$, covariance C_X, and cross-covariance $C_{X,Y}$ with another $\mathcal{B}$-random variable Y are given in Appendix A.2. We are mostly concerned with the collection of $\mathcal{B}$-valued random variables X satisfying $\mathbb{E}X = 0$ and $\mathbb{E}\|X\|_{\mathcal{B}}^2 < \infty$, which is denoted by $\mathfrak{L}^2(\mathcal{B})$. For $X \in \mathfrak{L}^2(\mathcal{B})$, we say that X has a positive definite covariance if $fC_X(f) = 0$ implies that $f = 0$.

2.3. Cointegrated I(d) Processes in Banach Spaces

Throughout this paper, we will need to consider I(d) processes in $\mathcal{B}$ and in $\mathbb{C}$ for $d \in \{1,2\}$, with innovations in $\mathcal{B}$, so it is convenient to define the I(d) property with another separable complex Banach space $\widetilde{\mathcal{B}}$ as in Section 2.2. Our definition of the I(d) property is adapted from Beare and Seo (2020) and Franchi and Paruolo (2020) for our more general setting. As a key building block for the I(d) property, we first define the I(0) property.

Definition 2.2. A sequence $X = \{X_t\}_{t \geq t_0}$ in $\mathfrak{L}^2(\widetilde{\mathcal{B}})$ is said to be I(0) if

$$X_t - \mathbb{E}(X_t) = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}, \quad t \geq t_0, \tag{2.5}$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is an i.i.d. sequence in $\mathfrak{L}^2(\mathcal{B})$ with positive definite covariance C_{ε_0} and $\{\theta_j\}_{j\geq 0}$ is a sequence in $\mathcal{L}(\mathcal{B},\widetilde{\mathcal{B}})$ satisfying $\sum_{j=0}^{\infty} \|\theta_j\|_{\text{op}} < \infty$ and $\sum_{j=0}^{\infty} \theta_j \neq 0$.

Remark 2.1. $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ in Definition 2.2 is an i.i.d. sequence in $\mathfrak{L}^2(\mathcal{B})$, which is called a strong $\mathcal{B}$-white noise (Bosq, 2000, p. 148). The i.i.d. condition is imposed for simplicity, and the results to be developed remain valid under a weaker condition that C_{ε_t} does not depend on t and $C_{\varepsilon_t,\varepsilon_s} = 0$ for all t and $s \neq t$. Such a sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is called a weak $\mathcal{B}$-white noise (Bosq, 2000, p. 161).

As in a Euclidean space setting, (2.5) may be conveniently expressed as

$$X_t - \mathbb{E}(X_t) = \Theta(L)\varepsilon_t,$$

where $\Theta(z) = \sum_{j=0}^{\infty} \theta_j z^j$ and L denotes the lag operator. Note that $\Theta(\cdot)$ is an operator-valued function defined on $\mathbb{C}$, which is called an operator pencil (see Appendix A.4); if $\Theta(\cdot)$ is matrix-valued, it is called a matrix pencil. Based on the I(0) property given by Definition 2.2, we define the I(1) and I(2) properties as follows.

Definition 2.3. For $d \in \{1,2\}$, a sequence in $\mathfrak{L}^2(\widetilde{\mathcal{B}})$ is said to be I(d) if its dth difference is an I(0) process admitting a representation (2.5) with $\{\theta_j\}_{j\geq 0}$ satisfying $\sum_{j=1}^{\infty} j^d \|\theta_j\|_{\text{op}} < \infty$.

Note that we require some summability conditions for I(1) and I(2) sequences, which are introduced for mathematical convenience in order to facilitate the use of the Phillips–Solo device in Section 2.4.

Cointegration of an I(d) process in $\mathcal{B}$ may be defined by extending Definition 2.1 in an obvious way. Let X be an I(d) process in $\mathcal{B}$, f be an element of $\mathcal{B}'$, and $\Delta := I - L$ be the difference operator. If the scalar-valued time series $\{f(\Delta^{d-1}X_t)\}_{t\geq 0}$ is stationary (Δ^0 is understood as the identity operator) in $\mathbb{C}$ for a suitable choice of X_0, we then say that X is cointegrated and call f a cointegrating functional. Obviously, the collection of cointegrating functionals constitutes a subspace of $\mathcal{B}'$, so we call it the cointegrating space.

2.4. Characterization of the Cointegrating Space

In this section, we characterize the cointegrating space associated with I(1) or I(2) processes in $\mathcal{B}$. A key input to our results is the Phillips–Solo device (Phillips and Solo, 1992, Lem. 2.1 and Sect. 4) for obtaining an algebraic decomposition of a linear filter into long-run and transitory components. Even if the Phillips–Solo device was presented in Phillips and Solo (1992) as a way to decompose matrix pencils when the usual matrix norm is considered, it can be directly extended to our Banach space setting by just replacing matrix pencils (resp. the usual matrix norm) with operator pencils (resp. the operator norm); no further changes are required from their proofs.

For $d \in \{1,2\}$, let $X = \{X_t\}_{t\geq -d+1}$ be an I(d) sequence in $\mathcal{B}$, admitting the following representation:

$$\Delta^d X_t = \Theta(L)\varepsilon_t, \quad t \geq 1. \tag{2.6}$$

Under the summability conditions given in Definition 2.3, we may apply the Phillips–Solo device to obtain

$$\Theta(L) = \Theta(1) + \Delta\Theta^{\star}(L), \tag{2.7}$$

where $\Theta^{\star}(L) = -\sum_{j=0}^{\infty}\theta_j^{\star}L^j$ and $\theta_j^{\star} = \sum_{k=j+1}^{\infty}\theta_k$. In (2.7), $\Theta(1)$ (resp. $\Delta\Theta^{\star}(L)$) is called the long-run (resp. transitory) component of $\Theta(L)$; see Phillips and Solo (1992). We may deduce from (2.7) that (2.6) allows the following representation, called the Beveridge–Nelson decomposition: for $d \in \{1,2\}$ and for some τ_0,

$$\Delta^{d-1}X_t = \tau_0 + \Theta(1)\sum_{s=1}^{t}\varepsilon_s + \nu_t, \quad t \geq 0, \tag{2.8}$$

where $\{\nu_t\}_{t\geq 0}$ is stationary and $\nu_t = \Theta^{\star}(L)\varepsilon_t$ for each t. Given (2.8), the cointegrating space $\mathfrak{C}(X)$ associated with X is formally defined as follows: for $d \in \{1,2\}$,

$$\mathfrak{C}(X) = \{f \in \mathcal{B}' : \{f(\Delta^{d-1}X_t)\}_{t\geq 0} \text{ is stationary for some } \tau_0 \in \mathfrak{L}^2(\mathcal{B})\}.$$

We also define

$$\mathfrak{A}(X) = \operatorname{ran}\Theta(1),$$

which is called the attractor space of X. We then provide useful results to characterize $\mathfrak{C}(X)$ when (i) there is no restriction on $\mathfrak{A}(X)$ and (ii) $\operatorname{cl}\mathfrak{A}(X)$ is

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complemented in $\mathcal{B}$. In case (ii), $\mathrm{cl}\,\mathfrak{A}(X)$ allows a complementary subspace, but which is not uniquely determined in general; for example, if $\mathcal{B} = \mathbb{R}^2$ and $\mathrm{cl}\,\mathfrak{A}(X) = \mathrm{span}\{(1,0)\}$, then the span of any arbitrary vector that is not included in $\mathrm{span}\{(1,0)\}$ can be a complementary subspace. We will also see later, in Remark 2.3, that various subspaces can complement $\mathrm{cl}\,\mathfrak{A}(X)$ of the cointegrated time series considered in Section 2.1.

We thus hereafter let $\mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$ denote the collection of the complementary subspaces of $\mathrm{cl}\,\mathfrak{A}(X)$, i.e., any element $V \in \mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$ satisfies

$$\mathcal{B} = \mathrm{cl}\,\mathfrak{A}(X) \oplus V. \tag{2.9}$$

If $\mathcal{B}$ is a Hilbert space, $\mathfrak{A}(X)^{\perp}$ is always a complementary subspace of $\mathrm{cl}\,\mathfrak{A}(X)$ (Conway, 1994, pp. 35–36), hence $\mathfrak{A}(X)^{\perp} \in \mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$; however, there may not exist a subspace V satisfying (2.9) if $\mathcal{B}$ is an infinite dimensional Banach space (see Remark 2.4). If the direct sum (2.9) holds for any $V \in \mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$, we may define the unique projection $\mathrm{P}_V \in \mathcal{L}(\mathcal{B})$ onto V along $\mathrm{cl}\,\mathfrak{A}(X)$, i.e., P_V is the unique linear operator satisfying the following properties:

$$\mathrm{P}_V = \mathrm{P}_V^2, \qquad \mathrm{ran}\,\mathrm{P}_V = V, \qquad \ker \mathrm{P}_V = \mathrm{cl}\,\mathfrak{A}(X); \tag{2.10}$$

see Megginson (2012, Thm. 3.2.11). If $\mathcal{B}$ is a Hilbert space and $V = \mathfrak{A}(X)^{\perp}$ as mentioned above, P_V becomes the orthogonal projection onto $\mathfrak{A}(X)^{\perp}$, but for any other $V \in \mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$ which is not equal to $\mathfrak{A}(X)^{\perp}$, P_V becomes a nonorthogonal projection. Our first result in this section characterizes the cointegrating space $\mathfrak{C}(X)$ in terms of $\mathfrak{A}(X)$ and P_V defined above.

PROPOSITION 2.1. *If $X = \{X_t\}_{t \geq -d+1}$ is I(d) for $d \in \{1,2\}$, the following hold.*

(i) $\mathfrak{C}(X) = \mathrm{Ann}(\mathfrak{A}(X))$.

(ii) *If $V \in \mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$, then $\mathfrak{C}(X) = \{f \circ \mathrm{P}_V : f \in \mathcal{B}'\}$, where P_V satisfies (2.10).*

Proposition 2.1(i) shows that $\mathfrak{C}(X)$ is given by the annihilator of $\mathfrak{A}(X)$, and thus a closed subspace of $\mathcal{B}'$ regardless of whether $\mathfrak{A}(X)$ is closed or not. This characterization is obtained without any additional condition on $\mathfrak{A}(X)$, but it is instead less informative than that given in Proposition 2.1(ii). If $\mathrm{cl}\,\mathfrak{A}(X)$ allows a complementary subspace $V \in \mathsf{C}(\mathrm{cl}\,\mathfrak{A}(X))$, we then know from the result given by Proposition 2.1(ii) that $\mathfrak{C}(X)$, which is a subspace of $\mathcal{B}'$, is in fact fully characterized by the projection $\mathrm{P}_V \in \mathcal{L}(\mathcal{B})$ satisfying (2.10). This result in turn leads us to have a natural decomposition of $\{\Delta^{d-1}X_t\}_{t\geq 0}$ into two components with different kinds of cointegrating behaviors as in a Hilbert/Euclidean space setting; see Remark 2.2. Some more remarks on Proposition 2.1 and the direct sum condition (2.9) are in order.

Remark 2.2. In our Banach space setting, the cointegrating space $\mathfrak{C}(X)$ is, by definition, a subspace of $\mathcal{B}'$ which is in general different from $\mathcal{B}$. However, the result given in Proposition 2.1(ii) makes it possible to understand

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$\mathfrak{C}(X)$ as a subspace of $\mathcal{B}$. Consider the Beveridge–Nelson decomposition (2.8). Using P_V defined under the direct sum (2.9), we may decompose $\Delta^{d-1}X_t$ into $(I-\mathrm{P}_V)\Delta^{d-1}X_t$ and $\mathrm{P}_V\Delta^{d-1}X_t$. The former is the unit root component in the sense that $\{f(I-\mathrm{P}_V)\Delta^{d-1}X_t\}_{t\geq 0}$ cannot be stationary for all $f \in \mathcal{B}'$ as long as $f(I-\mathrm{P}_V) \neq 0$, whereas the latter is the stationary component in the sense that $\{f\mathrm{P}_V\Delta^{d-1}X_t\}_{t\geq 0}$ can be made stationary under a suitable choice of τ_0 for all $f \in \mathcal{B}'$. This projection-based decomposition of a cointegrated time series is what has been done in a Euclidean space setting (see, e.g., Johansen, 1995, pp. 40–41), and as discussed, it is also possible in our setting without a richer geometry of a Hilbert space.

Remark 2.3. In the example given in Section 2.1, any solution to the AR(1) law of motion (2.3) in $\mathcal{B}(= C[0,1])$ satisfies that $\Delta X_t = \phi_1\varepsilon_t + \Delta(\varepsilon_t - \phi_1\varepsilon_t)$ (see (2.4)) and, as shown in Section B.4, $\mathrm{cl}\,\mathfrak{A}(X) = \mathrm{ran}\,\phi_1 = \mathrm{C}_0$ holds, where C_0 denotes the collection of constant functions. In this case, there are many different choices of V satisfying (2.9). One possible candidate is what we already considered in Section 2.1. Note that $x \in \mathcal{B}$ is uniquely decomposed into $x = x_1 + x_2$, where $x_1(u) = x(1)$ (and thus $x_1 \in \mathrm{C}_0$) and $x_2(u) = x(u) - x(1)$ for $u \in [0,1]$. Thus, C_0 is obviously complemented by $\mathrm{C}_1 = \{y \in \mathcal{B} : y(1) = 0\}$, and, in this case, P_V is given by $\mathrm{P}_V x(u) = x(u) - x(1)$. How this projection decomposes X_t into the unit root and stationary components is already illustrated in Figure 1; specifically, note that (ignoring τ_0) $(I-\mathrm{P}_V)X_t = X_t^N$ and $\mathrm{P}_V X_t = X_t^S$. It may be similarly deduced that $\{y \in \mathcal{B} : y(a) = 0\}$ for $a \in [0,1]$ (in this case, $\mathrm{P}_V x(u) = x(u) - x(a)$) or $\{y \in \mathcal{B} : \int_0^1 y(u)du = 0\}$ (in this case, $\mathrm{P}_V x(u) = x(u) - \int_0^1 x(u)du$) can be another candidate for V. We note that $(I-\mathrm{P}_V)X_t = X_t^N$ holds regardless of which V is chosen, meaning that the unit root component is always uniquely identified. On the other hand, the stationary component depends on V (e.g., ignoring τ_0, $\mathrm{P}_V X_t(u) = \varepsilon_t(u) - \varepsilon_t(1)$ if $V = \mathrm{C}_1$, whereas $\mathrm{P}_V X_t(u) = \varepsilon_t(u) - \int_0^1 \varepsilon_t(u)du$ if $V = \{y \in \mathcal{B} : \int_0^1 y(u)du = 0\}$). In a Hilbert/Euclidean space setting, this projection-based decomposition has been discussed for the case where $V = \mathfrak{A}(X)^\perp$ and thus P_V is an orthogonal projection. However, it is now clear that orthogonality has no essential role in such a decomposition; V may be set to a subspace that is not orthogonal to the attractor space, and this only leads us to have a different stationary component without affecting the unit root component.

Remark 2.4. If $\mathcal{B}$ is an infinite dimensional Banach space, a closed subspace may not be complemented (Megginson, 2012, pp. 301–302), hence (2.9) is not generally true. It, however, turns out that either of the following is a sufficient (but not necessary) condition for the existence of V satisfying (2.9):

(i) $\dim(\mathfrak{A}(X)) < \infty$, (ii) $\dim(\mathcal{B}/\mathfrak{A}(X)) < \infty$;

see Megginson (2012, Thm. 3.2.18). The two conditions lead to different dimensionalities of the cointegrating space of $\mathcal{B}'$. In case (i), V is necessarily infinite dimensional, and we deduce from Proposition 2.1 that the cointegrating space is

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also infinite dimensional. In case (ii), on the other hand, V is finite dimensional, hence the cointegrating space is finite dimensional as well.

One may be interested in how the general results given by Proposition 2.1 reduce to what we have known about cointegration in a Hilbert/Euclidean space setting. Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. If $\mathcal{B}=\mathcal{H}$, the Riesz representation theorem (see, e.g., Conway, 1994, p. 13) implies that every $f\in\mathcal{H}'$ is given by the map $\langle\cdot,y\rangle:\mathcal{H}\mapsto\mathbb{C}$ for a unique element $y\in\mathcal{H}$. Therefore, we may alternatively define the cointegrating space as follows: for $d\in\{1,2\}$,

$$\mathfrak{C}_{\mathcal{H}}(X)=\{y\in\mathcal{H}:\{\langle\Delta^{d-1}X_t,y\rangle\}_{t\geq 0}\text{ is stationary for some }\tau_0\in\mathfrak{L}^2(\mathcal{H})\}. \tag{2.11}$$

Moreover, in this case, we know that the direct sum (2.9) holds for $V=\mathfrak{A}(X)^{\perp}$, which makes P_V become the orthogonal projection onto $\mathfrak{A}(X)^{\perp}$. Under all these simplifications, Proposition 2.1 reduces to the following characterization, which is identical to the description of $\mathfrak{C}_{\mathcal{H}}(X)$ given by Beare et al. (2017).

COROLLARY 2.1. *If* $X=\{X_t\}_{t\geq -d+1}$ *is I(d) for* $d\in\{1,2\}$ *and* $\mathcal{B}=\mathcal{H}$*, then* $\mathfrak{C}_{\mathcal{H}}(X)=\mathfrak{A}(X)^{\perp}$.

We close this section with some remarks on Proposition 2.1 and Corollary 2.1.

Remark 2.5. Suppose that $\mathcal{B}=\mathbb{R}^n$ or $\mathbb{C}^n$ equipped with the usual inner product $\langle x,y\rangle=x^{\intercal}y$. This is of course a special case of a Hilbert space, so we may consider the alternative definition of the cointegrating space, $\mathfrak{C}_{\mathcal{H}}(X)$, given in (2.11). If there exists a nonzero cointegrating vector, the long-run component $\Theta(1)$ from the Phillips–Solo decomposition in this setting is a reduced rank matrix, i.e., $\operatorname{rank}\Theta(1)=s<n$. If so, there are two full column rank $n\times s$ matrices Θ_1 and Θ_2 satisfying $\Theta(1)=\Theta_1\Theta_2^{\intercal}$ (see, e.g., Engle and Granger, 1987). As a result, $\mathfrak{C}_{\mathcal{H}}(X)$ is given by the collection of vectors that are orthogonal to the columns of Θ_1, which is an $(n-s)$-dimensional subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Remark 2.6 (Second-order cointegrating functionals). Under the summability requirement for the I(2) property in Definition 2.3, we may apply the Phillips–Solo device to $\Theta^{\star}(L)$ in (2.7), and obtain

$$\Theta(L)=\Theta(1)+\Delta\Theta^{\star}(1)+\Delta^2\Theta^{\star\star}(L), \tag{2.12}$$

where $\Theta^{\star\star}(L)=-\sum_{j=0}^{\infty}\theta_j^{\star\star}L^j$ and $\theta_j^{\star\star}=\sum_{k=j+1}^{\infty}\theta_k^{\star}$. We then may deduce from (2.12) that (2.6) with $d=2$ allows the following representation: for some τ_0 and τ_1,

$$X_t=\tau_0+\tau_1 t+\Theta(1)\sum_{r=1}^{t}\sum_{s=1}^{r}\varepsilon_s+\Theta^{\star}(1)\sum_{s=1}^{t}\varepsilon_s+\nu_t^{\star},\quad t\geq 0,$$

where $\{\nu_t^{\star}\}_{t\geq 0}$ is stationary and $\nu_t^{\star}=\Theta^{\star\star}(L)\varepsilon_t$ for each t. Note that for any $f\in\mathfrak{C}(X)=\operatorname{Ann}(\operatorname{ran}\Theta(1))$, we have $f(X_t)=f(\Theta^{\star}(1)\sum_{s=1}^{t}\varepsilon_s)+f(\nu_t^{\star})$ by assuming $f(\tau_0)=f(\tau_1)=0$. Then it may deduced from a nearly identical argument used to prove Proposition 2.1(i) that $\{f(X_t)\}_{t\geq 0}$ is stationary if and only if $f\in$

$\mathrm{Ann}(\mathrm{ran}\,\Theta^{\star}(1))$ also holds. To sum up, $\{f(X_t)\}_{t\geq 0}$ can be made stationary under a suitable choice of τ_0 and τ_1 for any $f \in \mathrm{Ann}(\mathrm{ran}\,\Theta^{\star}(1)) \cap \mathrm{Ann}(\mathrm{ran}\,\Theta(1))$. We call such f as a second-order cointegrating functional, which will be of interest to us in Section 3 as an important aspect of I(2) AR processes in $\mathcal{B}$.

3. REPRESENTATION OF I(1) AND I(2) AUTOREGRESSIVE PROCESSES

For fixed $p \in \mathbb{N}$, suppose that a sequence $\{X_t\}_{t\geq -p+1} \subset \mathfrak{L}^2(\mathcal{B})$ satisfies the following AR(p) law of motion:

$$\Phi(L)X_t = \varepsilon_t, \quad t \geq 1, \tag{3.1}$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}} \subset \mathfrak{L}^2(\mathcal{B})$ is an i.i.d. sequence with positive definite covariance C_{ε_0} and

$$\Phi(z) = I - \sum_{j=1}^{p} \phi_j z^j, \quad \phi_1, \ldots, \phi_p \in \mathcal{L}(\mathcal{B}).$$

We let the operator pencil $\Phi : \mathbb{C} \mapsto \mathcal{L}(\mathcal{B})$ be called the AR polynomial. The i.i.d. condition imposed on $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ can be replaced by the requirement that C_{ε_t} does not depend on t and $C_{\varepsilon_t,\varepsilon_s} = 0$ for all t and $s \neq t$ without affecting any of the results to be developed later; see Remark 2.1. For notational simplicity, it is convenient to expand $\Phi(z)$ around one as follows:

$$\Phi(z) = \sum_{j=0}^{p} \Phi_j (z-1)^j, \quad \Phi_j = \Phi^{(j)}(1)/j! = \begin{cases} I - \sum_{h=1}^{p} \phi_h, & \text{if } j = 0, \\ -\sum_{h=0}^{p-j} \binom{j+h}{j} \phi_{j+h}, & \text{if } j = 1, \ldots, p. \end{cases} \tag{3.2}$$

We hereafter say that Φ has a unit root if it satisfies Assumption 3.1, where the following notation is employed: $\sigma(\Phi)$ denotes the spectrum of Φ given by the set $\{z \in \mathbb{C} : \Phi(z) \text{ is not invertible}\}$ and D_r denotes the open disk with radius r centered at $0 \in \mathbb{C}$.

Assumption 3.1 (Unit root).

(a) $\sigma(\Phi) \cap D_{1+\eta} = \{1\}$ for some $\eta > 0$ and $\Phi_0 \neq 0$.

(b) $\mathrm{ran}\,\Phi_0$ and $\ker\Phi_0$ can be complemented.

Assumption 3.1(a) is similar to the standard unit root assumption given in a Hilbert/Euclidean space setting; see Franchi and Paruolo (2020, Sect. 3.1). Assumption 3.1(b) requires $\mathrm{ran}\,\Phi_0$ and $\ker\Phi_0$ to be closed and allow complementary subspaces, and we note that such complementary subspaces are not uniquely determined in general once they exist; see, e.g., Remark 2.3. Thus, for convenience, let $\mathsf{C}(\mathrm{ran}\,\Phi_0)$ (resp. $\mathsf{C}(\ker\Phi_0)$) denote the collection of all the complementary subspaces of $\mathrm{ran}\,\Phi_0$ (resp. $\ker\Phi_0$); that is, if $V_{\mathsf{C}} \in \mathsf{C}(\mathrm{ran}\,\Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\ker\Phi_0)$, we have $\mathcal{B} = \mathrm{ran}\,\Phi_0 \oplus V_{\mathsf{C}} = \ker\Phi_0 \oplus W_{\mathsf{C}}$. If $\mathcal{B}$ is a Hilbert space,

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then any closed subspace can be complemented by its orthogonal complement (Conway, 1994, pp. 35–36), hence these direct sums hold for $V_{\mathtt{C}} = [\operatorname{ran}\Phi_0]^{\perp}$ and $W_{\mathtt{C}} = [\ker\Phi_0]^{\perp}$ as long as $\operatorname{ran}\Phi_0$ is closed ($\ker\Phi_0$ is necessarily closed since $\Phi_0 \in \mathcal{L}(\mathcal{B})$). In the existing representation theorems developed in a Hilbert space setting, closedness of $\operatorname{ran}\Phi_0$ is implied by the employed assumptions, so Assumption 3.1(b) consequentially holds (see Remark 3.1 to appear in Section 3.1). In a general Banach space setting, on the other hand, Assumption 3.1(b) does not necessarily hold even when $\operatorname{ran}\Phi_0$ is closed; see the example given by Megginson (2012, Thm. 3.2.20). Nevertheless, Assumption 3.1(b) may not be restrictive in general and, moreover, is a fairly weaker requirement which is strictly implied by the regularity conditions on $\Phi(z)$ employed in the existing literature; a more detailed discussion will be given in Remark 3.1.

In this setting, what we seek are (i) a necessary and sufficient condition under which the AR(p) law of motion (3.1) allows I(1) or I(2) solutions, and (ii) a characterization of such solutions; in the case $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$, these issues are dealt with in Johansen's representation theory. Hereafter, we conveniently say that a sequence $\{X_t\}_{t\geq -p+1}$ from (3.1) allows the Johansen I(1) representation if it satisfies the following: for τ_0 depending on initial values of (3.1), a stationary sequence $\{\nu_t\}_{t\geq 0} \subset \mathfrak{L}^2(\mathcal{B})$, and $\Upsilon_{-1} \in \mathcal{L}(\mathcal{B})$,

$$X_t = \tau_0 + \Upsilon_{-1}\sum_{s=1}^{t}\varepsilon_s + \nu_t, \quad t \geq 0. \tag{3.3}$$

We also say that $\{X_t\}_{t\geq -p+1}$ allows the Johansen I(2) representation if it can be represented as follows: for τ_0 and τ_1 depending on initial values of (3.1), a stationary sequence $\{\nu_t\}_{t\geq 0} \subset \mathfrak{L}^2(\mathcal{B})$, and $\Upsilon_{-2}, \Upsilon_{-1} \in \mathcal{L}(\mathcal{B})$,

$$X_t = \tau_0 + \tau_1 t + \Upsilon_{-2}\sum_{r=1}^{t}\sum_{s=1}^{r}\varepsilon_s + \Upsilon_{-1}\sum_{s=1}^{t}\varepsilon_s + \nu_t, \quad t \geq 0. \tag{3.4}$$

In the case $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$, Johansen (1991, 1995) shows that a necessary and sufficient condition for the AR(p) law of motion (3.1) to allow I(1) solutions is given by that

$$\alpha_{\perp\!\!\!\perp}^{\mathsf{T}}\Phi_1\beta_{\perp\!\!\!\perp} \text{ is invertible,} \tag{3.5}$$

where $\alpha_{\perp\!\!\!\perp}$ (resp. $\beta_{\perp\!\!\!\perp}$) is a full-rank $n\times(n-r)$ matrix whose columns are orthogonal to α (resp. β) for some $r < n$, and α and β are full-rank $n\times r$ matrices satisfying $\Phi_0 = \alpha\beta^{\mathsf{T}}$; without loss of generality, we assume that $\alpha_{\perp\!\!\!\perp}^{\mathsf{T}}\alpha_{\perp\!\!\!\perp} = \beta_{\perp\!\!\!\perp}^{\mathsf{T}}\beta_{\perp\!\!\!\perp} = I_{n-r}$ (the identity matrix of dimension $n-r$). The condition given by (3.5) is called the Johansen I(1) condition, under which a sequence $\{X_t\}_{t\geq -p+1}$ from (3.1) allows the Johansen I(1) representation with a certain operator Υ_{-1}; see Johansen (1995, Thm. 4.2). A similar representation result for the I(2) case is also given by Johansen (1995, Thm. 4.6). Extending these results to a Banach space setting may not be done by a simple extension: due to the fact that $\mathcal{B}$ may not be equipped with an inner product and can be infinite dimensional, we cannot rely on some important

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geometrical properties and matrix algebraic results that are allowed in Euclidean space. We thus need a novel approach that relies neither on the geometrical properties induced by an inner product nor on the finite dimension of $\mathcal{B}$.

As observed in an early contribution by Schumacher (1991), the I(d) property of solutions to the AR(p) law of motion, characterized by a matrix pencil $\Phi(z)$, in $\mathbb{R}^n$ is determined by the behavior of the inverse $\Phi(z)^{-1}$ around $z = 1$. This is also true in our Banach space setting; hence, our approach to developing representation theory for I(1) and I(2) AR processes essentially boils down to examining the inverse of the AR polynomial around $z = 1$. As a way to achieve this goal, we first consider the companion form of (3.1) characterized by a linear operator pencil $\widetilde{\Phi}$ to be defined later, and study the behavior of $\widetilde{\Phi}(z)^{-1}$ around $z = 1$ based on the spectral theory of linear operator pencils given in, e.g., Kato (1995) and Gohberg, Goldberg, and Kaashoek (2013). We then recover the behavior of $\Phi(z)^{-1}$ around $z = 1$ from that of $\widetilde{\Phi}(z)^{-1}$ by generalizing Johansen's I(1) and I(2) conditions; this second step adds to earlier findings on the inversion of linear operator pencils given by Albrecht, Howlett, and Pearce (2011) and Albrecht, Howlett, and Verma (2019).

It will be convenient to fix standard notation and terminology, based on Appendix A.4 providing a brief introduction to operator pencils, for the subsequent discussions. For any operator pencil A and its spectrum $\sigma(A) = \{z \in \mathbb{C} : A(z)$ is not invertible$\}$, we let $\rho(A)$ denote the set $\mathbb{C} \setminus \sigma(A)$, which is called the resolvent set of A. Now, suppose that $A(z)$ permits a Laurent series expansion at $z = z_0$ as follows: for some $d \geq 0$,

$$A(z) = \sum_{j=-d}^{\infty} A_j (z - z_0)^j, \quad A_{-d} \neq 0. \tag{3.6}$$

If $d = 0$, we say that $A(z)$ is holomorphic (or equivalently, complex-differentiable) at $z = z_0$. In this case, (3.6) becomes the Taylor series of $A(z)$ at $z = z_0$, which is called the Maclaurin series of $A(z)$ if $z_0 = 0$. If $d \neq 0$, $A(z)$ is said to have an isolated singularity at $z = z_0$. An isolated singularity with $d < \infty$ is called a pole of order d. A pole of order 1 is said to be simple. If $d = \infty$, $A(z)$ is said to have an essential singularity at $z = z_0$. The sum of the leading terms indexed by $j = -d, \ldots, -1$ is called the principal part, and the sum of the remaining terms is called the holomorphic part.

3.1. Relationship to Earlier Literature

A few different versions of the Granger–Johansen representation theorem have been proposed in the recent literature on cointegrated functional time series taking values in a Hilbert space, such as Chang et al. (2016a), Hu and Park (2016), Beare et al. (2017), Beare and Seo (2020), and Franchi and Paruolo (2020). Compared to those papers, our versions are developed under a general Banach space setting without relying on the richer geometry of a Hilbert space, which can help us consider a greater variety of functional time series as subjects of the theory of cointegration, as illustrated in Section 2.1. Apart from such mathematical gains, we here briefly describe how our setting is related to the assumptions employed in the aforementioned papers by assuming $\mathcal{B} = \mathcal{H}$ (recall that $\mathcal{H}$ denotes an arbitrary separable complex Hilbert space).

We first focus on the I(1) case. Except for the paper by Chang et al. (2016a) providing a quite different representation result, the AR polynomial $\Phi(z)$ in the foregoing papers satisfies the following condition: for $z \in \mathbb{C}$,

$$\dim(\ker \Phi(z)) < \infty \text{ and } \dim([\operatorname{ran} \Phi(z)]^{\perp}) < \infty. \tag{3.7}$$

If (3.7) holds, $\Phi(z)$ is called a Fredholm operator. Fredholmness of $\Phi(z)$ can be more generally defined in our Banach space setting by replacing the latter condition in (3.7) with $\dim(\mathcal{B}/\operatorname{ran}\Phi(z)) < \infty$. The Fredholm property, combined with the unit root assumption (Assumption 3.1), produces some special behavior of $\Phi(z)^{-1}$ near $z = 1$, which becomes a crucial input to the existing theorems; see Beare and Seo (2020, Appx. A.1) and Franchi and Paruolo (2020, Appx. B). An important consequence of assuming (3.7) is that Υ_{-1} in (3.3) always becomes a finite rank operator, hence the random walk component $\Upsilon_{-1}\sum_{s=1}^{t}\varepsilon_s$ in (3.3) essentially boils down to a finite dimensional unit root process. As a result, the attractor space (resp. the cointegrating space) associated with the AR(p) law of motion is necessarily finite dimensional (resp. infinite dimensional). On the other hand, the version of Chang et al. (2016a) relies on the assumption that $\Phi_0(= \Phi(1))$ is compact, which turns out to result in the opposite case, where the cointegrating space is finite dimensional and the random walk component takes values in an infinite dimensional space unless $\mathcal{H}$ is finite dimensional. Their compactness assumption is not compatible with Fredholmness of $\Phi(z)$ in an infinite dimensional setting (Abramovich and Aliprantis, 2002, Lem. 4.41). We thus have two qualitatively different I(1) representation results depending on two generally incompatible regularity conditions on $\Phi(z)$.

To the best of the author's knowledge, the existing representation theorems for I(2) AR processes in a general Hilbert space setting were recently provided by Beare and Seo (2020) and Franchi and Paruolo (2020), where Fredholmness of $\Phi(z)$ is an essential assumption for their representation theory. Similar to the I(1) case, the Fredholm assumption makes Υ_{-2} and Υ_{-1} in (3.4) become finite rank operators, hence the random walk component $\Upsilon_{-2}\sum_{r=1}^{t}\sum_{s=1}^{r}\varepsilon_s + \Upsilon_{-1}\sum_{s=1}^{t}\varepsilon_s$ is intrinsically a finite dimensional unit root process. This requirement entailed by the Fredholm assumption not only compels the attractor space associated with I(2) solutions to be finite dimensional, but also places some more restrictions on their cointegrating behavior; a more detailed discussion will be given in Sections 3.4 and 3.5.

As discussed above, any regularity conditions imposed on $\Phi(z)$ may compel solutions to the AR(p) law of motion to have some specific characteristics. It is thus desirable to develop representation theory for I(1) and I(2) AR processes under minimal regularity conditions on $\Phi(z)$. We in this paper require weaker conditions

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on $\Phi(z)$ than either of Fredholmness or compactness, which naturally makes our representation theory place weaker restrictions on solutions to the AR(p) law of motion. In particular, the random walk component of I(1) or I(2) solutions can be either finite dimensional or infinite dimensional in our results, whereas the component is required to be exclusively finite dimensional or infinite dimensional depending on the employed regularity condition on $\Phi(z)$ in the recent literature.

Remark 3.1. As discussed, in the existing literature concerning the case $\mathcal{B} = \mathcal{H}$ of infinite dimension, $\Phi(z)$ is either Fredholm or compact but not both. Fredholm property (3.7) implies that $\operatorname{ran}\Phi_0$ is closed (Conway, 1994, Thm. 2.3, p. 350). We thus find that $\operatorname{ran}\Phi_0$ (resp. $\ker\Phi_0$) allows a finite (resp. an infinite) dimensional complementary subspace; this is also true in a more general situation where $\mathcal{B}$ is not necessarily a Hilbert space and $\Phi(z)$ is a Fredholm operator acting on $\mathcal{B}$ (see Remark 2.4). On the other hand, if $\Phi(z)$ is compact and satisfies Assumption 3.1(a), then Chang et al. (2016a, Lem. 1) showed that $\operatorname{ran}\Phi_0$ is necessarily finite dimensional (and thus closed); from an obvious extension of their proof, it can be shown that this result does not require the assumption that $\mathcal{B}$ is a Hilbert space. It is then obvious that $\operatorname{ran}\Phi_0$ (resp. $\ker\Phi_0$) allows an infinite (resp. a finite) dimensional complementary subspace in a general Banach space setting. Note that Fredholmness or compactness of $\Phi(z)$ places some specific dimensionality restrictions on the complementary subspaces, whereas no such restrictions are required by Assumption 3.1(b).

Remark 3.2. Suppose that $\mathcal{B}$ is infinite dimensional. As will be shown in Proposition 3.4, the random walk component in the I(1) case always takes values in $\ker\Phi_0$, whose dimension is finite (resp. infinite) if $\Phi(z)$ is Fredholm (resp. compact) under Assumption 3.1(a). This shows where the difference between the existing I(1) representation results about the dimensionality of the random walk component originates from.

3.2. Linearization of the AR Polynomial

Consider the product Banach space $\mathcal{B}^p$ equipped with the norm $\|(x_1,\dots,x_p)\|_{\mathcal{B}^p} = \sum_{j=1}^p \|x_j\|_{\mathcal{B}}$ for any $(x_1,\dots,x_p) \in \mathcal{B}^p$. We let I_p denote the identity map acting on $\mathcal{B}^p$. In fact, the AR(p) law of motion (3.1) may be understood as the following AR(1) law of motion in $\mathcal{B}^p$:

$$\widetilde{\Phi}(L)\widetilde{X}_t = \widetilde{\varepsilon}_t, \tag{3.8}$$

where $\widetilde{\Phi} : \mathbb{C} \mapsto \mathcal{L}(\mathcal{B}^p)$ is a linear operator pencil given by $\widetilde{\Phi}(z) = I_p - z\widetilde{\phi}_1$ and

$$\widetilde{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-p+1} \end{bmatrix}, \quad \widetilde{\phi}_1 = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \widetilde{\varepsilon}_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.9}$$

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Commonly, (3.8) is called the companion form of (3.1); see, e.g., Johansen (1995, p. 15) or Bosq (2000, p. 128). From a mathematical point of view, the behavior of $\Phi(z)^{-1}$ that we want to know may be obtained from that of $\widetilde{\Phi}(z)^{-1}$, and this is as described in Proposition 3.1, where the following notation is employed: $\Pi_p : \mathcal{B}^p \mapsto \mathcal{B}$ and $\Pi_p^* : \mathcal{B} \mapsto \mathcal{B}^p$ denote the maps defined by

$$\Pi_p(x_1, x_2, \ldots, x_p) = x_1, \qquad \Pi_p^*(x_1) = (x_1, 0, \ldots, 0). \tag{3.10}$$

PROPOSITION 3.1. *Under Assumption 3.1, the operator pencils $\widetilde{\Phi}$ and Φ satisfy the following.*

(i) $\sigma(\widetilde{\Phi}) = \sigma(\Phi)$ *and* $\Pi_p \widetilde{\Phi}(z)^{-1} \Pi_p^* = \Phi(z)^{-1}$.

(ii) *Under Assumption 3.1, if either of $\widetilde{\Phi}(z)^{-1}$ or $\Phi(z)^{-1}$ has a pole of order d (resp. essential singularity) at $z = 1$, then the other has a pole of order d (resp. essential singularity) at $z = 1$.*

Proposition 3.1(i) shows that $\widetilde{\Phi}(z)$ inherits the unit root property of $\Phi(z)$ given by Assumption 3.1, and $\Phi(z)^{-1}$ can be recovered from $\widetilde{\Phi}(z)^{-1}$ using the maps given in (3.10). Moreover, Proposition 3.1(ii) implies that we can obtain a necessary and sufficient condition for $\Phi(z)^{-1}$ to have a pole of order 1 or 2 at $z = 1$ by finding such a condition for $\widetilde{\Phi}(z)^{-1}$. These results will become useful in the development of our representation theorems for I(1) and I(2) AR processes.

3.3. Representation of I(1) Autoregressive Processes

In Section 3.3.1, we develop our representation theory for I(1) AR processes resorting to the companion form AR(1) representation (3.8); this is done by studying the spectral properties of $\widetilde{\Phi}(z)$ under Assumption 3.1. We then discuss on how the results obtained via the companion form can be reformulated in terms of the behavior of the AR polynomial $\Phi(z)$ in Section 3.3.2.

3.3.1. *Representation via the Companion Form.* Resorting to the companion form (3.8), we in this section provide a necessary and sufficient condition for the AR(p) law of motion (3.1) to admit I(1) solutions and a characterization of such solutions.

Under Assumption 3.1(a), we know from the results given in Appendix A.4 (especially, see (A.1)) that $\widetilde{\Phi}(z)^{-1}$ can be written as the following Laurent series: for $d \in \mathbb{N} \cup \{\infty\}$,

$$\widetilde{\Phi}(z)^{-1} = -\sum_{j=-d}^{-1} \mathrm{N}_j (z-1)^j - \sum_{j=0}^{\infty} \mathrm{N}_j (z-1)^j. \tag{3.11}$$

For notational convenience, we let

$$\mathrm{P} := \mathrm{N}_{-1} \widetilde{\phi}_1, \tag{3.12}$$

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and expand $\widetilde{\Phi}(z)$ around one, obtaining

$$\widetilde{\Phi}(z) = \widetilde{\Phi}_0 + (z-1)\widetilde{\Phi}_1, \quad \widetilde{\Phi}_0 = (I_p - \widetilde{\phi}_1), \quad \widetilde{\Phi}_1 = -\widetilde{\phi}_1. \tag{3.13}$$

The operator P given above turns out to be a projection under Assumption 3.1 (Lemma B.1(ii)) and has a crucial role in the subsequent discussion. What we first pursue for the development of our representation theory is a necessary and sufficient condition under which the AR(p) law of motion (3.1) admits I(1) solutions (or equivalently, $\widetilde{\Phi}(z)^{-1}$ has a simple pole at $z = 1$). If $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$, it is well known that the Johansen I(1) condition given by (3.5) is such a condition, and this condition plays an essential role in Johansen's representation theory for I(1) AR processes. Beare et al. (2017), who studied the same issue for the case $\mathcal{B} = \mathcal{H}$, revisited the Johansen I(1) condition and provided its geometric reformulation given by a certain nonorthogonal direct sum of $\mathcal{H}^p$; see Remark 3.4. Inspired by their direct sum condition that can be applied for $\mathcal{H}$ of an arbitrary dimension, we propose the following condition.

I(1) condition: $\mathcal{B}^p = \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0$.

Some remarks on the I(1) condition are given as follows.

Remark 3.3. The I(1) condition is given as the direct sum of $\mathcal{B}^p$ by two fixed subspaces $\operatorname{ran}\widetilde{\Phi}_0$ and $\ker\widetilde{\Phi}_0$, and this specific direct sum condition will be shown to be necessary and sufficient for the existence of I(1) solutions. In the case where our I(1) condition holds, it is worth noting that a unique projection whose range is $\ker\widetilde{\Phi}_0$ and kernel is $\operatorname{ran}\widetilde{\Phi}_0$ is well defined (Megginson, 2012, Thm. 3.2.11).

Remark 3.4. In the case where $\mathcal{B} = \mathcal{H}$ of an arbitrary dimension and $\phi_1, \ldots, \phi_p$ are compact operators, Beare et al. (2017) showed that the nonorthogonal direct sum $\mathcal{H}^p = \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0$ is a sufficient condition for the AR(p) law of motion (3.1) to admit I(1) solutions; however, its necessity was not discussed in their paper. They showed that their condition becomes equivalent to the Johansen I(1) condition if $\mathcal{H} = \mathbb{R}^n$ or $\mathbb{C}^n$. The reader is referred to the results given in Section 4 (and the proofs of those) of their paper.

Our first result in this section not only shows that the I(1) condition is a necessary and sufficient condition for $\widetilde{\Phi}(z)^{-1}$ to have a simple pole at $z = 1$, but also characterizes the principal part of its Laurent series.

PROPOSITION 3.2. *Suppose that Assumption 3.1 holds. The following conditions are equivalent.*

(i) $\widetilde{\Phi}(z)^{-1}$ *has a simple pole at* $z = 1$.
(ii) P *is the projection onto* $\ker\widetilde{\Phi}_0$ *along* $\operatorname{ran}\widetilde{\Phi}_0$.
(iii) *The I(1) condition holds.*

Under any of these conditions, the following holds: for some $\eta > 0$,

$$(1-z)\widetilde{\Phi}(z)^{-1} = \mathrm{P} + (1-z)H(z), \quad z \in D_{1+\eta}, \tag{3.14}$$

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where $H(z)$ denotes the holomorphic part of the Laurent series of $\widetilde{\Phi}(z)^{-1}$ around $z = 1$. Moreover, each Maclaurin series of $(1-z)\widetilde{\Phi}(z)^{-1}$ and $H(z)$ is convergent on $D_{1+\eta}$.

Examples of the use of Proposition 3.2 for verifying that $\widetilde{\Phi}(z)^{-1}$ has a simple pole at $z = 1$ will be given later in this section. Proposition 3.2 extends the results given by Beare et al. (2017), which are briefly reviewed in Remark 3.4, in the sense that it provides a necessary and sufficient condition for the existence of I(1) solutions without requiring either of a Hilbert space structure or compactness of $\phi_1, \ldots, \phi_p$. In addition, combined with Propositions 3.1, the local behavior of $\widetilde{\Phi}(z)^{-1}$ around $z = 1$ given by (3.14) provides a characterization of solutions to the AR(p) law of motion (3.1), which leads to our first version of the Granger–Johansen representation theorem for I(1) AR processes given below.

PROPOSITION 3.3. *Suppose that Assumption 3.1 holds. Under the I(1) condition, a sequence $\{X_t\}_{t\geq -p+1}$ satisfying (3.1) allows the Johansen I(1) representation (3.3) with*

$$\Upsilon_{-1} = \Pi_p \mathrm{P} \Pi_p^*, \qquad \nu_t = \Pi_p H(L) \Pi_p^* \varepsilon_t = \sum_{j=0}^{\infty} (-1)^j \Pi_p \widetilde{\Phi}_1^j (I_p - \mathrm{P}) \Pi_p^* \varepsilon_{t-j}, \quad \textbf{(3.15)}$$

where P *and $H(z)$ are given in Proposition 3.2, and Π_p and Π_p^* are given in (3.10). Moreover, the AR(p) law of motion (3.1) does not allow I(1) solutions if the I(1) condition is not satisfied.*

Proposition 3.3 shows that, under our I(1) condition, solutions to the AR(p) law of motion (3.1) can be represented as (3.3) similar to the Beveridge–Nelson decomposition (2.8) of an I(1) cointegrated linear process. For such a solution $\{X_t\}_{t\geq 0}$, we may deduce from our discussion in Section 2.4 that $\{f(X_t)\}_{t\geq 0}$ can be made stationary under a suitable initial condition if and only if $f \in \mathrm{Ann}(\Pi_p \mathrm{P} \Pi_p^*)$. Some more remarks on the results given by Proposition 3.3 are in order.

Remark 3.5. Proposition 3.3 may be viewed as an extension of Theorem 4.1 of Beare et al. (2017), which provides a version of the Granger–Johansen representation theorem in a Hilbert space setting resorting to the companion form representation of a given AR(p) law of motion.

Remark 3.6. In the case $\dim(\mathcal{B}) = \infty$, neither the attractor nor the cointegrating space associated with (3.1) is compelled to be finite dimensional in our representation results, whereas one of those subspaces is necessarily finite dimensional in the existing theorems developed in a Hilbert space setting; see Section 3.1. As a simple illustration, let $p = 1$ and ϕ_1 be an arbitrary projection. In this case, we may deduce from Propositions 2.1, 3.2, and 3.3 that the dimension of the attractor space (resp. the cointegrating space) associated with (3.1) is equal to $\dim(\mathrm{ran}\,\phi_1)$ (resp. $\dim(\ker \phi_1)$). Since ϕ_1 is an arbitrary projection, all the following cases are possible: (i) $\dim(\operatorname{ran}\phi_1) < \infty$ and $\dim(\ker\phi_1) = \infty$, (ii) $\dim(\operatorname{ran}\phi_1) = \infty$ and $\dim(\ker\phi_1) < \infty$, and (iii) $\dim(\operatorname{ran}\phi_1) = \infty$ and $\dim(\ker\phi_1) = \infty$.

In this section, the I(1) condition and solutions to the AR(p) law of motion (3.1) are characterized in terms of the behavior of $\widetilde{\Phi}(z)$ around $z = 1$. For this reason, our results do not clearly reveal how the proposed I(1) condition and the cointegrating behavior of I(1) solutions are related to the structure of the original AR polynomial $\Phi(z)$. This is a natural consequence resulting from that we resort to the companion form representation of (3.1) to obtain our main results given in this section. In the next section, we will discuss how these results can be recast in terms of the behavior of the AR polynomial $\Phi(z)$. By doing so, we obtain a more detailed characterization of I(1) solutions and find the connection between our representation results and those developed in a Hilbert/Euclidean space setting. We close this section with a few examples illustrating the use of Proposition 3.2 for verifying that $\widetilde{\Phi}(z)^{-1}$ has a simple pole at $z = 1$.

Example 3.1. In the example given in Section 2.1, $\mathcal{B} = C[0,1]$, $\widetilde{\Phi}(z) = I - z\phi_1$, and ϕ_1 is defined by $\phi_1 x(u) = x(1)$ for $u \in [0,1]$. Note that $\widetilde{\Phi}_0 \neq 0$ and $\widetilde{\Phi}_0 x(1) = 0$ for any arbitrary $x \in C[0,1]$. This implies that every $y \in \operatorname{ran}\widetilde{\Phi}_0$ must satisfy $y(1) = 0$, from which we find that $\widetilde{\Phi}_0$ is not invertible. Let C_0 and C_1 be defined as in Remark 2.3. Then, it can be shown that Assumption 3.1(a) is satisfied, $\ker\widetilde{\Phi}_0 = \mathrm{C}_0$, and $\operatorname{ran}\widetilde{\Phi}_0 = \mathrm{C}_1$ (see Appendix B.4). As discussed in Remark 2.3, we have $\mathcal{B} = \mathrm{C}_0 \oplus \mathrm{C}_1$, and thus find that the I(1) condition holds.

Example 3.2. Suppose that $\mathcal{B} = C[-1,1]$, $\widetilde{\Phi}(z) = I - z\phi_1$, and ϕ_1 is defined by

$$\phi_1 x(u) = x(u)/2 + x(-u)/2, \quad x \in \mathcal{B}, \quad u \in [-1,1].$$

One can easily show that (i) $\sigma(\widetilde{\Phi}) = \{1\}$ and (ii) ϕ_1 (resp. $I - \phi_1$) is the projection onto the space of even (resp. odd) functions along the space of odd (resp. even) functions, hence $\mathcal{B} = \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0$. Thus, it is concluded that $\widetilde{\Phi}(z)^{-1}$ has a simple pole at $z = 1$. In fact, we reach the same conclusion in cases where any arbitrary projection replaces ϕ_1 in the above.

Example 3.3. Let $\mathbf{c_0}$ be the space of complex sequences converging to zero equipped with the norm $\|a\| = \sup_i |a_i|$ for $a = (a_1, a_2, \ldots) \in \mathbf{c_0}$. The space $\mathbf{c_0}$ may be viewed as a natural generalization of a finite dimensional vector space equipped with the supremum norm, and also turns out to be a separable Banach space (Megginson, 2012, Exams. 1.2.13 and 1.12.6). Let ϕ_1 be defined by

$$\phi_1(a_1, a_2, a_3, a_4, \ldots) = (a_1, a_1 + a_2, \lambda a_3, \lambda^2 a_4, \ldots), \tag{3.16}$$

where $\lambda \in (0,1)$. In this case, $\widetilde{\Phi}(z) = I - z\phi_1$ satisfies Assumption 3.1(a) (see Appendix B.4), and $\operatorname{ran}\widetilde{\Phi}_0$ and $\ker\widetilde{\Phi}_0$ are given as follows:

$$\begin{aligned} \operatorname{ran}\widetilde{\Phi}_0 &= \{(0, b_1, b_2, \ldots) : b_j \in \mathbb{C}, \lim_{j\to\infty} b_j = 0\}, \\ \ker\widetilde{\Phi}_0 &= \{(0, b_1, 0, 0, \ldots) : b_1 \in \mathbb{C}\}. \end{aligned} \tag{3.17}$$

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The above subspaces can be complemented (see Example 3.4), hence Assumption 3.1(b) is also satisfied. However, (3.17) clearly shows that $\mathcal{B} \neq \operatorname{ran} \widetilde{\Phi}_0 \oplus \ker \widetilde{\Phi}_0$. We thus conclude that $\widetilde{\Phi}(z)^{-1}$ does not have a simple pole at $z = 1$; it will be shown in Section 3.4 that $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 at $z = 1$.

3.3.2. *Further Characterization of I(1) Solutions.* If $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$, the Johansen I(1) condition given by (3.5) is necessary and sufficient for the AR(p) law of motion (3.1) to admit I(1) solutions. If we let $\mathrm{P}_{[\operatorname{ran}\Phi_0]^\perp}$ (resp. $\mathrm{P}_{[\ker\Phi_0]^\perp}$) denote the orthogonal projection onto $[\operatorname{ran}\Phi_0]^\perp$ (resp. $[\ker\Phi_0]^\perp$), then (3.5) can be equivalently understood as that $\mathrm{P}_{[\operatorname{ran}\Phi_0]^\perp}\Phi_1(I - \mathrm{P}_{[\ker\Phi_0]^\perp})$ as a map from $\ker\Phi_0$ to $[\operatorname{ran}\Phi_0]^\perp$ is invertible. We will obtain a natural generalization of this condition to our Banach space setting, and further characterize I(1) solutions using such a condition.

Recall that $\mathsf{C}(\operatorname{ran}\Phi_0)$ (resp. $\mathsf{C}(\ker\Phi_0)$) denotes the collection of all the complementary subspaces of $\operatorname{ran}\Phi_0$ (resp. $\ker\Phi_0$). Under Assumption 3.1, $\mathsf{C}(\operatorname{ran}\Phi_0)$ and $\mathsf{C}(\ker\Phi_0)$ are nonempty sets. For any $V_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\ker\Phi_0)$, we have

$$\mathcal{B} = \operatorname{ran}\Phi_0 \oplus V_{\mathsf{C}} = \ker\Phi_0 \oplus W_{\mathsf{C}}. \tag{3.18}$$

If the direct sums given in (3.18) hold, we may define the projections $\mathrm{P}_{V_{\mathsf{C}}}$ and $\mathrm{P}_{W_{\mathsf{C}}}$ satisfying

$$\operatorname{ran}\mathrm{P}_{V_{\mathsf{C}}} = V_{\mathsf{C}}, \quad \ker\mathrm{P}_{V_{\mathsf{C}}} = \operatorname{ran}\Phi_0, \quad \operatorname{ran}\mathrm{P}_{W_{\mathsf{C}}} = W_{\mathsf{C}}, \quad \ker\mathrm{P}_{W_{\mathsf{C}}} = \ker\Phi_0; \tag{3.19}$$

these projections are uniquely defined for any $V_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\Phi_0)$ (Megginson, 2012, Thm. 3.2.11). We then define the operator $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}}) \in \mathcal{L}(\mathcal{B})$ as follows:

$$\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}}) := \mathrm{P}_{V_{\mathsf{C}}}\Phi_1(I - \mathrm{P}_{W_{\mathsf{C}}}). \tag{3.20}$$

Note that our construction of $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}})$ depends on $V_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\Phi_0)$, which are arbitrary elements of $\mathsf{C}(\operatorname{ran}\Phi_0)$ and $\mathsf{C}(\ker\Phi_0)$, respectively. We thus may understand $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}})$ as an operator-valued *function* of the *variables* V_{C} and W_{C} hereafter. Suppose that $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$, $V_{\mathsf{C}} = [\operatorname{ran}\Phi_0]^\perp$, and $W_{\mathsf{C}} = [\ker\Phi_0]^\perp$. In this case, $\mathrm{P}_{V_{\mathsf{C}}}$ (resp. $I - \mathrm{P}_{W_{\mathsf{C}}}$) becomes the orthogonal projection onto $[\operatorname{ran}\Phi_0]^\perp$ (resp. $\ker\Phi_0$), then it follows from our earlier discussion that the Johansen I(1) condition given by (3.5) is equivalent to that $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}})$ as a map from $\ker\Phi_0$ to V_{C} is invertible. One may naturally generalize this condition to a Banach space setting by not requiring V_{C} (resp. W_{C}) to be the orthogonal complement to $\operatorname{ran}\Phi_0$ (resp. $\ker\Phi_0$), and our next result given below shows that this is in fact an equivalent reformulation of the I(1) condition given in Section 3.3.1. Using this result, we moreover obtain a useful characterization of Υ_{-1} in terms of the operators defined above.

PROPOSITION 3.4. *Suppose that Assumption 3.1 holds. Then the following conditions are equivalent.*

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(i) The I(1) condition holds.

(ii) $\Lambda_1(V_{\mathtt{C}}, W_{\mathtt{C}}) : \ker\Phi_0 \mapsto V_{\mathtt{C}}$ *is invertible for some* $V_{\mathtt{C}} \in \mathtt{C}(\operatorname{ran}\Phi_0)$ *and* $W_{\mathtt{C}} \in \mathtt{C}(\ker\Phi_0)$.

(iii) $\Lambda_1(V_{\mathtt{C}}, W_{\mathtt{C}}) : \ker\Phi_0 \mapsto V_{\mathtt{C}}$ *is invertible for any arbitrary* $V_{\mathtt{C}} \in \mathtt{C}(\operatorname{ran}\Phi_0)$ *and* $W_{\mathtt{C}} \in \mathtt{C}(\ker\Phi_0)$.

Let any of the above conditions hold. Then, for any $V_{\mathtt{C}} \in \mathtt{C}(\operatorname{ran}\Phi_0)$ *and* $W_{\mathtt{C}} \in \mathtt{C}(\ker\Phi_0)$*, a sequence* $\{X_t\}_{t\geq -p+1}$ *satisfying (3.1) allows the Johansen I(1) representation (3.3) with* Υ_{-1} *satisfying*

$$\mathrm{P}_{W_{\mathtt{C}}}\Upsilon_{-1} = \Upsilon_{-1}(I - \mathrm{P}_{V_{\mathtt{C}}}) = 0, \quad \Upsilon_{-1} : V_{\mathtt{C}} \mapsto \ker\Phi_0 = \Lambda_{1,R}(V_{\mathtt{C}}, W_{\mathtt{C}})^{-1}, \tag{3.21}$$

where $\Lambda_{1,R}(V_{\mathtt{C}}, W_{\mathtt{C}})$ *denotes the invertible map* $\Lambda_1(V_{\mathtt{C}}, W_{\mathtt{C}}) : \ker\Phi_0 \mapsto V_{\mathtt{C}}$.

It is interesting that a natural generalization of the Johansen I(1) condition is equivalent to our previous necessary and sufficient condition for the existence of I(1) solutions developed in a general Banach space setting. This shows that our operator-theoretic approach is in fact closely related to the conventional Johansen's approach. We know from the equivalence between the three conditions given in Proposition 3.4 that $V_{\mathtt{C}}$ (resp. $W_{\mathtt{C}}$) can be fixed to any arbitrary element of $\mathtt{C}(\operatorname{ran}\Phi_0)$ (resp. $\mathtt{C}(\ker\Phi_0)$) with no loss of generality; this, of course, implies that $V_{\mathtt{C}} = [\operatorname{ran}\Phi_0]^\perp$ and $W_{\mathtt{C}} = [\ker\Phi_0]^\perp$ can always be assumed in a Hilbert/Euclidean space setting. Moreover, (3.21) describes how the operator Υ_{-1} acts as a map from $\operatorname{ran}\Phi_0 \oplus V_{\mathtt{C}}$ to $\ker\Phi_0 \oplus W_{\mathtt{C}}$ for any $V_{\mathtt{C}} \in \mathtt{C}(\operatorname{ran}\Phi_0)$ and $W_{\mathtt{C}} \in \mathtt{C}(\ker\Phi_0)$; this, of course, provides a full characterization of Υ_{-1}, given that $\mathcal{B}$ allows the bipartite decompositions given in (3.18).

In Section 3.3.1, the I(1) condition and the cointegrating behavior of I(1) solutions are characterized in terms of some operators associated with $\widetilde{\Phi}(z)$ given in the companion form (3.8); however, one may be interested in characterizing those using operators associated with the original AR polynomial $\Phi(z)$. The conditions (ii) and (iii) in Proposition 3.4 are already given in such a manner, and those are equivalent reformulations of our I(1) condition. Moreover, our characterization of Υ_{-1}, given by (3.21), helps us characterize the cointegrating behavior in a desired way; see Remarks 3.7 and 3.8.

Remark 3.7. From (3.21), we know that the attractor space of I(1) solutions is given by $\operatorname{ran}\Upsilon_{-1} = \ker\Phi_0$. Let $W_{\mathtt{C}} \in \mathtt{C}(\ker\Phi_0)$, and let $\mathrm{P}_{W_{\mathtt{C}}}$ be the projection onto $W_{\mathtt{C}}$ along $\ker\Phi_0$. We then deduce from Proposition 2.1 that a cointegrating functional f can be written as $f = g\mathrm{P}_{W_{\mathtt{C}}}$ for some $g \in \mathcal{B}'$.

Remark 3.8. Using the results given in Proposition 3.4, we may obtain a stronger characterization of the cointegrating behavior of I(1) solutions than that given in Section 3.3.1. From the expression of Υ_{-1} given in (3.21), we find that a nonzero element $f \in \mathcal{B}'$ satisfies

$$\{f(X_t)\}_{t\geq 0} \text{ is I(0) if and only if } f \in \operatorname{Ann}(\ker\Phi_0); \tag{3.22}$$

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see Appendix B.4 for our proof of (3.22). The above characterization not only shows that the cointegrating space is given by $\mathrm{Ann}(\ker\Phi_0)$, but also establishes I(0)-ness of $\{f(X_t)\}_{t\geq 0}$ for any $f \in \mathrm{Ann}(\ker\Phi_0)$.

3.4. Representation of I(2) Autoregressive Processes

In this section, we suppose that the I(1) condition fails and develop our representation theory for I(2) AR processes. We know from Proposition 3.4 that the failure of the I(1) condition is understood as noninvertibility of $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}})$ as a map from $\ker\Phi_0$ to V_{C}, where V_{C} (resp. W_{C}) may be fixed to any arbitrary element of $\mathsf{C}(\mathrm{ran}\,\Phi_0)$ (resp. $\mathsf{C}(\ker\Phi_0)$) without loss of generality; especially, $V_{\mathsf{C}} = [\mathrm{ran}\,\Phi_0]^{\perp}$ and $W_{\mathsf{C}} = [\ker\Phi_0]^{\perp}$ can be assumed if $\mathcal{B}$ is a Hilbert space, which is helpful to compare the results to be developed with the existing I(2) results given in a Hilbert space setting. For such a fixed choice of V_{C} and W_{C}, we define $\mathrm{P}_{V_{\mathsf{C}}}$ and $\mathrm{P}_{W_{\mathsf{C}}}$ as in (3.19) and let

$$\mathsf{R} := \mathrm{P}_{V_{\mathsf{C}}}\Phi_1 \ker\Phi_0, \qquad \mathsf{K} := \{x \in \ker\Phi_0 : \Phi_1 x \in \mathrm{ran}\,\Phi_0\}.$$

Using the notation given above, we summarize the assumptions for our I(2) representation results as follows.

Assumption 3.2.

(a) Assumption 3.1 holds and $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}})$ is not invertible for some fixed $V_{\mathsf{C}} \in \mathsf{C}(\mathrm{ran}\,\Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\ker\Phi_0)$; if $\mathcal{B}$ is a Hilbert space, $V_{\mathsf{C}} = [\mathrm{ran}\,\Phi_0]^{\perp}$ and $W_{\mathsf{C}} = [\ker\Phi_0]^{\perp}$.

(b) R (resp. K) can be complemented in V_{C} (resp. $\ker\Phi_0$), i.e., for some $\mathsf{R}_{\mathsf{C}} \subset \mathcal{B}$ and $\mathsf{K}_{\mathsf{C}} \subset \mathcal{B}$,

$$V_{\mathsf{C}} = \mathsf{R} \oplus \mathsf{R}_{\mathsf{C}}, \qquad \ker\Phi_0 = \mathsf{K} \oplus \mathsf{K}_{\mathsf{C}}. \tag{3.23}$$

If R_{C} and K_{C} satisfying (3.23) exist, then they are not uniquely determined in general. For convenience, we hereafter let $\mathsf{C}(\mathsf{R})$ (resp. $\mathsf{C}(\mathsf{K})$) denote the collection of the complementary subspaces of R in V_{C} (resp. K in $\ker\Phi_0$). Our requirement for the existence of such complementary subspaces may not be restrictive in general and do not invalidate that our I(2) results to be developed can complement the earlier results developed in a general Hilbert space setting; the regularity condition imposed on $\Phi(z)$ for the existing I(2) results strictly implies the requirement and, moreover, places some certain restrictions on the dimensions of the complementary subspaces; see Remark 3.9.

Remark 3.9. Fredholmness of $\Phi(z)$ is an essential assumption in the existing I(2) representation theorems developed in the case $\mathcal{B} = \mathcal{H}$ (Beare and Seo, 2020; Franchi and Paruolo, 2020). In such a setting, we have $\mathcal{B} = \mathrm{ran}\,\Phi_0 \oplus V_{\mathsf{C}} = \ker\Phi_0 \oplus W_{\mathsf{C}}$ for some V_{C} and W_{C} (Remark 3.1), and both V_{C} and $\ker\Phi_0$ are finite dimensional. Since every finite dimensional space can be complemented (Remark 2.4), (3.23) holds for some R_{C} and K_{C}. This is also true in a more general situation where $\mathcal{B}$ is not necessarily a Hilbert space and $\Phi(z)$ is a Fredholm operator acting

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on $\mathcal{B}$, which can be seen from Remark 2.4. Note that Fredholmness of $\Phi(z)$ requires that V_{C}, $\ker\Phi_0$, R_{C}, and K_{C} are all finite dimensional (and thus W_{C} is infinite dimensional), which means that the assumption leads us to stronger conditions than those required by Assumption 3.2.

As in Section 3.3, we first develop our representation theory for I(2) AR processes resorting to the companion form representation (3.8), and then discuss how the results obtained via the companion form can be recast in terms of the behavior of the AR polynomial $\Phi(z)$. In the subsequent discussion, we will need the notion of a generalized inverse of a linear operator in our Banach space setting, which is introduced in Appendix A.3; the generalized inverse considered in this paper is a natural generalization of the well-known Moore–Penrose inverse, employed in Beare and Seo (2020) and Franchi and Paruolo (2020) for their I(2) representation results developed in a Hilbert space setting.

3.4.1. *Representation via the Companion Form.* As in Section 3.3.1, we first resort to linearization of $\Phi : \mathbb{C} \mapsto \mathcal{B}$, hence consider $\widetilde{\Phi} : \mathbb{C} \mapsto \mathcal{B}^p$ given in (3.8) and (3.9). Under Assumption 3.2, we have the Laurent series of $\widetilde{\Phi}(z)^{-1}$ near $z = 1$ as in (3.11), and also define P as in (3.12). Before stating our main results, we collect some preliminary results and fix notation.

Under Assumption 3.2, $\operatorname{ran}\widetilde{\Phi}_0$ and $\ker\widetilde{\Phi}_0$ can be complemented (Lemma B.2(i)) in $\mathcal{B}^p$. We let $\mathsf{C}(\operatorname{ran}\widetilde{\Phi}_0)$ (resp. $\mathsf{C}(\ker\widetilde{\Phi}_0)$) denote the collection of the complementary subspaces of $\operatorname{ran}\widetilde{\Phi}_0$ (resp. $\ker\widetilde{\Phi}_0$); that is, for any $\mathcal{V}_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\mathsf{C}} \in \mathsf{C}(\ker\widetilde{\Phi}_0)$, we have

$$\mathcal{B}^p = \operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{V}_{\mathsf{C}} = \ker\widetilde{\Phi}_0 \oplus \mathcal{W}_{\mathsf{C}}. \tag{3.24}$$

Under the direct sums given by (3.24), we may define the projections $\mathrm{P}_{\mathcal{V}_{\mathsf{C}}}$ and $\mathrm{P}_{\mathcal{W}_{\mathsf{C}}}$ satisfying

$$\operatorname{ran}\mathrm{P}_{\mathcal{V}_{\mathsf{C}}} = \mathcal{V}_{\mathsf{C}}, \quad \ker\mathrm{P}_{\mathcal{V}_{\mathsf{C}}} = \operatorname{ran}\widetilde{\Phi}_0, \quad \operatorname{ran}\mathrm{P}_{\mathcal{W}_{\mathsf{C}}} = \mathcal{W}_{\mathsf{C}}, \quad \ker\mathrm{P}_{\mathcal{W}_{\mathsf{C}}} = \ker\widetilde{\Phi}_0. \tag{3.25}$$

The projections $\mathrm{P}_{\mathcal{V}_{\mathsf{C}}}$ and $\mathrm{P}_{\mathcal{W}_{\mathsf{C}}}$ are uniquely defined for any $\mathcal{V}_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\mathsf{C}} \in \mathsf{C}(\ker\widetilde{\Phi}_0)$; see Megginson (2012, Thm. 3.2.11). For notational convenience, we let

$$\mathcal{K} = \operatorname{ran}\widetilde{\Phi}_0 \cap \ker\widetilde{\Phi}_0. \tag{3.26}$$

A crucial preliminary result is that $\mathcal{K} \neq \{0\}$ is required for $\widetilde{\Phi}(z)^{-1}$ to have a pole of order 2 at $z = 1$ (Lemma B.2(ii)); based on this result and the notation defined above, we propose our I(2) condition as follows.

I(2) condition: $\mathcal{K} \neq \{0\}$ and for some $\mathcal{V}_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\mathsf{C}} \in \mathsf{C}(\ker\widetilde{\Phi}_0)$,

$$\mathcal{B}^p = (\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0) \oplus \widetilde{\Phi}_0^g\mathcal{K}, \tag{3.27}$$

where $\widetilde{\Phi}_0^g$ is the generalized inverse of $\widetilde{\Phi}_0$ (see Remark 3.10 and Appendix A.3).

Some remarks on the I(2) condition are given as follows.

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Remark 3.10. For any $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\complement} \in \complement(\ker\widetilde{\Phi}_0)$, the generalized inverse $\widetilde{\Phi}_0^g$ is defined as the unique linear operator satisfying

$$\widetilde{\Phi}_0\widetilde{\Phi}_0^g\widetilde{\Phi}_0 = \widetilde{\Phi}_0, \quad \widetilde{\Phi}_0^g\widetilde{\Phi}_0\widetilde{\Phi}_0^g = \widetilde{\Phi}_0^g, \quad \widetilde{\Phi}_0\widetilde{\Phi}_0^g = I_p - \mathrm{P}_{\mathcal{V}_{\complement}}, \quad \widetilde{\Phi}_0^g\widetilde{\Phi}_0 = \mathrm{P}_{\mathcal{W}_{\complement}};$$

see Appendix A.3 for a more detailed discussion. In a Hilbert space setting, $\mathcal{V}_{\complement}$ (resp. $\mathcal{W}_{\complement}$) can be set to $[\operatorname{ran}\widetilde{\Phi}_0]^\perp$ (resp. $[\ker\widetilde{\Phi}_0]^\perp$), then $\widetilde{\Phi}_0^g$ becomes equivalent to the Moore–Penrose inverse of $\widetilde{\Phi}_0$.

Remark 3.11. The I(2) condition is given as the direct sum of $\mathcal{B}^p$ by two subspaces $\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$ and $\widetilde{\Phi}_0^g\mathcal{K}$, where the latter depends on $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\complement} \in \complement(\ker\widetilde{\Phi}_0)$ since the definition of $\widetilde{\Phi}_0^g$ does so (see Remark 3.10). However, it turns out that if (3.27) holds for some $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\complement} \in \complement(\ker\widetilde{\Phi}_0)$, then it holds for any arbitrary $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\complement} \in \complement(\ker\widetilde{\Phi}_0)$ (Lemma B.2(iii)). Thus, the choice of $\mathcal{V}_{\complement}$ and $\mathcal{W}_{\complement}$ does not affect the subsequent results, and also may be arbitrarily fixed without loss of generality; for example, in a Hilbert space setting, we may assume that $\mathcal{V}_{\complement} = [\operatorname{ran}\widetilde{\Phi}_0]^\perp$ and $\mathcal{W}_{\complement} = [\ker\widetilde{\Phi}_0]^\perp$.

Remark 3.12. If $\mathcal{B} = \mathcal{H}$ and $\widetilde{\Phi}(z)$ is Fredholm, we may assume that $\mathcal{V}_{\complement} = [\operatorname{ran}\widetilde{\Phi}_0]^\perp$ and $\mathcal{W}_{\complement} = [\ker\widetilde{\Phi}_0]^\perp$ in the I(2) condition with no loss of generality (Remark 3.11); in this case, $\widetilde{\Phi}_0^g$ is equal to the Moore–Penrose inverse $\widetilde{\Phi}_0^\dagger$ of $\widetilde{\Phi}_0$ (Remark 3.10). Then, our I(2) condition reduces to a necessary and sufficient condition for $\widetilde{\Phi}(z)^{-1}$ to have a pole of order 2 at $z = 1$ given by Beare and Seo (2020, Thm. 4.2 and Rem. 4.7).[4]

Our next result shows that the proposed I(2) condition is indeed necessary and sufficient for $\widetilde{\Phi}(z)^{-1}$ to have a pole of order 2 at $z = 1$, and provides a partial characterization of the principal part of the Laurent series; a more detailed characterization of the principal part can be obtained in terms of some operators associated with $\widetilde{\Phi}(z)$, but which is postponed to Appendix B.3.3 since more preliminary results to be developed in Appendix B are required.

PROPOSITION 3.5. *Suppose that Assumption 3.2 holds. Then $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 if and only if the I(2) condition holds. Under the I(2) condition, the following holds for some $\eta > 0$:*

$$(1-z)^2\widetilde{\Phi}(z)^{-1} = -\mathrm{N}_{-2} + (1-z)(\mathrm{N}_{-2} + \mathrm{P}) + (1-z)^2 H(z), \quad z \in D_{1+\eta}, \qquad \textbf{(3.28)}$$

where N_{-2} satisfies $\operatorname{ran}\mathrm{N}_{-2} = \mathcal{K}$, $H(z)$ is the holomorphic part of the Laurent series of $\widetilde{\Phi}(z)^{-1}$, and each Maclaurin series of $(1-z)^2\widetilde{\Phi}(z)^{-1}$ and $H(z)$ converges on $D_{1+\eta}$.

Examples of the use of Proposition 3.5 for verifying that $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 will be given at the end of this section. Proposition 3.5 provides a

[4] In fact, the direct sum condition given by Beare and Seo (2020, Thm. 4.2 and Rem. 4.7) is slightly different. However, with a simple algebra, it can be shown that the direct sum given in the I(2) condition is equivalent to $\mathcal{B} = (\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0) \oplus (I_p - \widetilde{\Phi}_0^g)\mathcal{K}$, which is exactly comparable with their direct sum condition.

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characterization of the local behavior of $\widetilde{\Phi}(z)^{-1}$ around $z = 1$ under the I(2) condition, which, together with Proposition 3.1, leads to our first version of the Granger–Johansen representation theorem for I(2) AR processes as follows.

PROPOSITION 3.6. *Suppose that Assumption 3.2 holds. Under the I(2) condition, a sequence $\{X_t\}_{t\geq -p+1}$ satisfying (3.1) allows the Johansen I(2) representation (3.4) with*

$$\Upsilon_{-2} = -\Pi_p \mathrm{N}_{-2}\Pi_p^*, \quad \Upsilon_{-1} = \Pi_p(\mathrm{N}_{-2}+\mathrm{P})\Pi_p^*,$$
$$\nu_t = \Pi_p H(L)\Pi_p^*\varepsilon_t = \sum_{j=0}^{\infty}(-1)^j \Pi_p \widetilde{\Phi}_1^j (I_p - \mathrm{P})\Pi_p^*\varepsilon_{t-j}, \tag{3.29}$$

where N_{-2}, P, and $H(z)$ are given in Proposition 3.5, and Π_p and Π_p^* are given in (3.10). Moreover, the AR(p) law of motion (3.1) does not allow I(2) solutions if the I(2) condition is not satisfied.

Note that the representation (3.4) with (3.29) is similar to the Beveridge–Nelson decomposition (2.8) of an I(2) cointegrated linear process. From our discussion in Section 2.4, we may deduce that $f \in \mathrm{Ann}(\Pi_p \mathrm{N}_{-2}\Pi_p^*)$ (resp. $f \in \mathrm{Ann}(\Pi_p \mathrm{N}_{-2}\Pi_p^*) \cap \mathrm{Ann}(\Pi_p \mathrm{P}\Pi_p^*)$) is a cointegrating functional (resp. a second-order cointegrating functional). Appendix B.3.3 provides characterizations of N_{-2} and P in terms of certain operators associated with $\widetilde{\Phi}(z)$, which complements the results given by Proposition 3.6.

In this section, we have shown that the AR(p) law of motion (3.1) admits I(2) solutions if and only if the I(2) condition holds, and provided a partial characterization of such solutions. All these results are obtained resorting to the companion form representation of (3.1), hence it is not clearly revealed how the I(2) condition and the cointegrating behavior of I(2) solutions are related to the structure of the original AR polynomial $\Phi(z)$. This issue will be addressed in the next section by providing a more detailed characterization of I(2) solutions in terms of operators associated with $\Phi(z)$. Before closing this section, we give examples of the use of Proposition 3.5 for verifying that $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2.

Example 3.4. Consider Example 3.3, where we showed that $\widetilde{\Phi}(z)^{-1}$ does not have a simple pole at $z = 1$, hence we know that Assumption 3.2(a) holds. Note that $\mathrm{ran}\,\widetilde{\Phi}_0$ (resp. $\ker\widetilde{\Phi}_0$) allows a complementary subspace $\mathcal{V}_{\mathsf{C}}$ (resp. $\mathcal{W}_{\mathsf{C}}$); specifically, $\mathcal{V}_{\mathsf{C}}$ and $\mathcal{W}_{\mathsf{C}}$ may be set to

$$\mathcal{V}_{\mathsf{C}} = \{(b_1, 0, 0, \ldots), b_1 \in \mathbb{C}\}, \qquad \mathcal{W}_{\mathsf{C}} = \{(b_1, 0, b_2, b_3, \ldots), : b_j \in \mathbb{C}, \lim_{j\to\infty} b_j = 0\}.$$

Observe that $\mathrm{ran}\,\Phi_0 \oplus \mathsf{R} = \mathrm{ran}\,\Phi_0 + \Phi_1 \ker\Phi_0 = \mathrm{ran}\,\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0 = \mathrm{ran}\,\widetilde{\Phi}_0$ and $\mathsf{K} = \ker\widetilde{\Phi}_0$, hence Assumption 3.2(b) is also satisfied for $\mathsf{R}_{\mathsf{C}} = \mathcal{V}_{\mathsf{C}}$ and $\mathsf{K}_{\mathsf{C}} = \{0\}$. For any $(b_1, 0, \ldots) \in \mathcal{V}_{\mathsf{C}}$, we find that $-\widetilde{\Phi}_0(b_1, 0, \ldots) = (0, b_1, 0, \ldots)$. Precomposing

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both sides of this equation with $\widetilde{\Phi}_0^g$, we obtain

$$(-b_1,0,\ldots)=\widetilde{\Phi}_0^g(0,b_1,0,\ldots), \tag{3.30}$$

where the equality holds since $(b_1,0,\ldots)\in\mathcal{W}_{\mathbb{C}}$ and $\widetilde{\Phi}_0^g\widetilde{\Phi}_0=\mathrm{P}_{\mathcal{W}_{\mathbb{C}}}$. From (3.17), (3.30), and the fact that $\mathcal{K}=\operatorname{ran}\widetilde{\Phi}_0\cap\ker\widetilde{\Phi}_0=\ker\widetilde{\Phi}_0$, we deduce that $\widetilde{\Phi}_0^g\mathcal{K}=\{(b_1,0,\ldots), b_1\in\mathbb{C}\}$, which is a complementary subspace of $\operatorname{ran}\widetilde{\Phi}_0+\ker\widetilde{\Phi}_0$. Thus, $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 at $z=1$.

Example 3.5. In the setting of Examples 3.3 and 3.4, we observed that $\ker\widetilde{\Phi}_0$ and $\mathcal{K}$ are finite dimensional under the I(2) condition. However, these subspaces may not be finite dimensional in general. To see this, we will consider slight modifications of ϕ_1; under any of the changes in ϕ_1 to be given below, it can be easily shown that Assumption 3.2 is still satisfied. We first replace (3.16) with the following:

$$\begin{aligned}\phi_1&(a_1,a_2,a_3,a_4,\ldots)\\ &=(a_1,a_2+a_1,a_3+a_1,a_4,a_5+a_4,a_6+a_4,a_7,a_8+a_7,a_9+a_7,\ldots).\end{aligned} \tag{3.31}$$

In this case, the I(2) condition holds (see Appendix B.4), and $\operatorname{ran}\widetilde{\Phi}_0$ and $\ker\widetilde{\Phi}_0$ are given by

$$\operatorname{ran}\widetilde{\Phi}_0=\{(0,b_1,b_1,0,b_2,b_2,0,b_3,b_3,\ldots): b_j\in\mathbb{C},\ \lim_{j\to\infty}b_j=0\}, \tag{3.32}$$

$$\ker\widetilde{\Phi}_0=\{(0,b_1,b_2,0,b_3,b_4,0,b_5,b_6,\ldots): b_j\in\mathbb{C},\ \lim_{j\to\infty}b_j=0\}. \tag{3.33}$$

Since $\mathcal{K}=\operatorname{ran}\widetilde{\Phi}_0$, it is obvious that $\ker\widetilde{\Phi}_0$ and $\mathcal{K}$ are infinite dimensional. Now, we replace (3.16) with the following:

$$\phi_1(a_1,a_2,a_3,a_4,\ldots)=(a_1,a_2+a_1,a_3,a_4,\ldots).$$

From similar arguments to those in Appendix B.4, we find that the I(2) condition holds. Note also that

$$\operatorname{ran}\widetilde{\Phi}_0=\{(0,b_1,0,0,\ldots): b_1\in\mathbb{C}\},\quad \ker\widetilde{\Phi}_0=\{(0,b_1,b_2,\ldots): b_j\in\mathbb{C},\ \lim_{j\to\infty}b_j=0\}, \tag{3.34}$$

and thus $\mathcal{K}=\operatorname{ran}\widetilde{\Phi}_0$. We know from (3.34) that $\ker\widetilde{\Phi}_0$ is infinite dimensional, but $\mathcal{K}$ is finite dimensional.

3.4.2. *Further Characterization of I(2) Solutions.* Suppose that $\mathcal{B}=\mathbb{R}^n$ or $\mathbb{C}^n$ and the Johansen I(1) condition fails. Continuing with the notation introduced for (3.5), we let ϖ and ϱ be full-rank $(n-r)\times s$ matrices satisfying $\alpha_{\perp}^{\intercal}\Phi_1\beta_{\perp}=\varpi\varrho^{\intercal}$, where $s<n-r$. Let $\varpi_{\perp}$ (resp. $\varrho_{\perp}$) be a full-rank $n\times(n-r-s)$ matrix whose columns are orthogonal to those of $(\alpha,\alpha_{\perp}\varpi)$ (resp. $(\beta,\beta_{\perp}\varrho)$). Without loss of generality, we may assume that $\varpi_{\perp}^{\intercal}\varpi_{\perp}=\varrho_{\perp}^{\intercal}\varrho_{\perp}=I_{n-r-s}$ (the identity matrix of dimension $n-r-s$). Johansen (1992, 1995) provides a necessary and sufficient

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condition for the AR(p) law of motion (3.1) to admit I(2) solutions, which is given by that

$$\varpi_{\perp\perp}^{\intercal}\left(\Phi_2-\Phi_1\Phi_0^{\dagger}\Phi_1\right)\varrho_{\perp\perp} \text{ is invertible,} \tag{3.35}$$

where $\Phi_0^{\dagger}$ is the Moore–Penrose inverse of Φ_0. To provide a natural generalization of the Johansen I(2) condition that can be applied to our Banach space setting, as we did in Section 3.3.2 for the I(1) case, we first review some preliminary results that hold under Assumption 3.2 and fix some notation.

We let $\mathrm{P}_{V_{\mathsf{C}}}$ and $\mathrm{P}_{W_{\mathsf{C}}}$ be defined as in (3.19) for $V_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\ker\Phi_0)$, which are fixed with no loss of generality in Assumption 3.2. We then let Φ_0^g denote the generalized inverse of Φ_0, which is the unique linear operator satisfying the following properties:

$$\Phi_0\Phi_0^g\Phi_0=\Phi_0, \quad \Phi_0^g\Phi_0\Phi_0^g=\Phi_0^g, \quad \Phi_0\Phi_0^g=I-\mathrm{P}_{V_{\mathsf{C}}}, \quad \Phi_0^g\Phi_0=\mathrm{P}_{W_{\mathsf{C}}}.$$

If $\mathcal{B}=\mathcal{H}$ and thus $V_{\mathsf{C}}=[\operatorname{ran}\Phi_0]^{\perp}$ and $W_{\mathsf{C}}=[\ker\Phi_0]^{\perp}$, then Φ_0^g is equivalent to the Moore–Penrose inverse $\Phi_0^{\dagger}$ of Φ_0; see Appendix A.3. Moreover, we note that the direct sum conditions given in Assumption 3.2 can be combined and equivalently formulated as follows:

$$\mathcal{B}=\operatorname{ran}\Phi_0\oplus\mathsf{R}\oplus\mathsf{R}_{\mathsf{C}}=W_{\mathsf{C}}\oplus\mathsf{K}_{\mathsf{C}}\oplus\mathsf{K}. \tag{3.36}$$

Then, for any $\mathsf{R}_{\mathsf{C}}\in\mathsf{C}(\mathsf{R})$ and $\mathsf{K}_{\mathsf{C}}\in\mathsf{C}(\mathsf{K})$, we may define the unique projections $\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}$ and P_{K} satisfying

$$\operatorname{ran}\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}=\mathsf{R}_{\mathsf{C}}, \quad \ker\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}=\operatorname{ran}\Phi_0\oplus\mathsf{R}, \quad \operatorname{ran}\mathrm{P}_{\mathsf{K}}=\mathsf{K}, \quad \ker\mathrm{P}_{\mathsf{K}}=W_{\mathsf{C}}\oplus\mathsf{K}_{\mathsf{C}}; \tag{3.37}$$

see Megginson (2012, Thm. 3.2.11). We then define the operator $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}})\in\mathcal{L}(\mathcal{B})$ as follows:

$$\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}}):=\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\left(\Phi_2-\Phi_1\Phi_0^g\Phi_1\right)\mathrm{P}_{\mathsf{K}}. \tag{3.38}$$

Our construction of $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}})$ depends on the choice of R_{C} and K_{C}, hence $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}})$ may be understood as an operator-valued function of the variables R_{C} and K_{C}. Suppose that $\mathcal{B}=\mathbb{R}^n$ or $\mathbb{C}^n$ (hence $V_{\mathsf{C}}=[\operatorname{ran}\Phi_0]^{\perp}$ and $W_{\mathsf{C}}=[\ker\Phi_0]^{\perp}$), $\mathsf{R}_{\mathsf{C}}=[\operatorname{ran}\Phi_0\oplus\mathsf{R}]^{\perp}$, and $\mathsf{K}_{\mathsf{C}}=[W_{\mathsf{C}}\oplus\mathsf{K}]^{\perp}$ in (3.36). In this case, the Johansen I(2) condition given by (3.35) can be equivalently understood as that $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}})$ as a map from K to R_{C} is invertible, then we know from Johansen (1995, Thm. 4.6) that the AR(p) law of motion (3.1) allows I(2) solutions. We will show in Proposition 3.7 that this result can be extended to our Banach space setting where the notion of an orthogonal complement is not generally allowed. To simplify mathematical expressions, we hereafter employ the following notation:

$$\mathrm{M}_1=\Lambda_1(V_{\mathsf{C}},W_{\mathsf{C}}), \quad \mathrm{M}_2=\Phi_2-\Phi_1\Phi_0^g\Phi_1, \quad \mathrm{M}_3=\Phi_3-\Phi_1\Phi_0^g\Phi_1\Phi_0^g\Phi_1,$$

where, unlike in Section 3.3.2, $\mathrm{M}_1=\Lambda_1(V_{\mathsf{C}},W_{\mathsf{C}})$ is understood as a fixed element of $\mathcal{L}(\mathcal{B})$ under Assumption 3.2. Moreover, as the last piece of our preliminary

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results for Proposition 3.7, we note that the generalized inverse M_1^g of M_1 is uniquely defined for any $\mathsf{R}_{\mathsf{C}} \in \mathsf{C}(\mathsf{R})$ and $\mathsf{K}_{\mathsf{C}} \in \mathsf{C}(\mathsf{K})$ under Assumption 3.2, and it satisfies $\operatorname{ran}\mathrm{M}_1^g = \mathsf{K}_{\mathsf{C}}$ and $\ker \mathrm{M}_1^g = \operatorname{ran}\Phi_0 \oplus \mathsf{R}_{\mathsf{C}}$ (Lemma B.3).

PROPOSITION 3.7. *Suppose that Assumption 3.2 holds. Then the following conditions are equivalent.*

(i) *The I(2) condition holds.*
(ii) $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}}) : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$ *is invertible for some* $\mathsf{R}_{\mathsf{C}} \in \mathsf{C}(\mathsf{R})$ *and* $\mathsf{K}_{\mathsf{C}} \in \mathsf{C}(\mathsf{K})$.
(iii) $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}}) : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$ *is invertible for any arbitrary* $\mathsf{R}_{\mathsf{C}} \in \mathsf{C}(\mathsf{R})$ *and* $\mathsf{K}_{\mathsf{C}} \in \mathsf{C}(\mathsf{K})$.

Let any of the above conditions hold. Then, for any choice of R_{C} *and* K_{C}*, a sequence* $\{X_t\}_{t\geq -p+1}$ *satisfying (3.1) allows the Johansen I(2) representation (3.4) with* Υ_{-2} *satisfying*

$$(I-\mathrm{P}_{\mathsf{K}})\Upsilon_{-2} = \Upsilon_{-2}(I-\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}) = 0, \qquad \Upsilon_{-2} : \mathsf{R}_{\mathsf{C}} \mapsto \mathsf{K} = \Lambda_{2,R}(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}})^{-1}, \tag{3.39}$$

and Υ_{-1} *satisfying*

$$(I-\mathrm{P}_{\mathsf{K}})\Upsilon_{-1}(I-\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}) = -\mathrm{M}_1^g,$$
$$(I-\mathrm{P}_{\mathsf{K}})\Upsilon_{-1}\mathrm{P}_{\mathsf{R}_{\mathsf{C}}} = \left(\Phi_0^g\Phi_1 + \mathrm{M}_1^g\mathrm{M}_2\right)\Upsilon_{-2},$$
$$\mathrm{P}_{\mathsf{K}}\Upsilon_{-1}(I-\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}) = \Upsilon_{-2}\left(\Phi_1\Phi_0^g + \mathrm{M}_2\mathrm{M}_1^g\right),$$
$$\mathrm{P}_{\mathsf{K}}\Upsilon_{-1}\mathrm{P}_{\mathsf{R}_{\mathsf{C}}} = \Upsilon_{-2}\left(\mathrm{M}_3 - \mathrm{M}_2\Phi_0^g\Phi_1 - \Phi_1\Phi_0^g\mathrm{M}_2 - \mathrm{M}_2\mathrm{M}_1^g\mathrm{M}_2\right)\Upsilon_{-2},$$

where $\Lambda_{2,R}(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}})$ *denotes the invertible map* $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}}) : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$.

Proposition 3.7 not only shows that the condition given by (3.38) is equivalent to our I(2) condition given in Section 3.4.1, but also implies that R_{C} and K_{C} can be arbitrarily chosen among possible candidates; this, of course, means that R_{C} and K_{C} can always be fixed to the relevant orthogonal complements in a Hilbert/Euclidean space setting without loss of generality. Moreover, Proposition 3.7 shows in detail how Υ_{-2} and Υ_{-1} act on $\mathcal{B}$ satisfying the tripartite decompositions given by (3.36), from which we obtain a more detailed characterization of the cointegrating behavior of I(2) solutions than that given in Section 3.4.1; see Remarks 3.13–3.15.

Remark 3.13. From (3.39), we know that the attractor space associated with I(2) solutions is given by $\operatorname{ran}\Upsilon_{-2} = \mathsf{K}$. Then it follows from Proposition 2.1 that a cointegrating functional f satisfies $f = g(I-\mathrm{P}_{\mathsf{K}})$ for some $g \in \mathcal{B}'$, where P_{K} is the projection on K along $W_{\mathsf{C}} \oplus \mathsf{K}_{\mathsf{C}}$; see (3.37).

Remark 3.14. Using the results given in Proposition 3.7, we can obtain a stronger characterization of the cointegrating behavior of I(2) solutions as follows: for any nonzero element $f \in \mathcal{B}'$ and for some $r \in \{0,1\}$,

$$\{f(X_t)\}_{t\geq 0} \text{ is I}(r) \text{ if and only if } f \in \operatorname{Ann}(\mathsf{K}), \tag{3.40}$$

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$\{f(X_t)\}_{t\geq 0}$ is I(0) if and only if $f \in \mathrm{Ann}(\ker \Phi_0) \cap \mathrm{Ann}(\Phi_0^g \Phi_1 \mathsf{K})$. **(3.41)**

Obviously, (3.40) (resp. (3.41)) identifies the cointegrating space (the collection of second-order cointegrating functionals). A detailed discussion including our proofs of these results is given in Appendix B.4.

Remark 3.15 (Polynomial cointegration for an I(2) AR process). For any cointegrating functional $f \in \mathrm{Ann}(\mathsf{K})$, f may or may not satisfy $f \in \mathrm{Ann}(\ker \Phi_0)$ since $\mathsf{K} \subset \ker \Phi_0$. If $f \in \mathrm{Ann}(\ker \Phi_0)$ and one combines levels and first differences as in $f(X_t) - f(\Phi_0^g \Phi_1 \Delta X_t)$, then such a sequence is always I(0); this phenomenon does not occur if $f \notin \mathrm{Ann}(\ker \Phi_0)$. A detailed discussion including our proof of this result is given in Appendix B.4. The case where the sequence of $f(X_t)$ and that of $f(\Phi_0^g \Phi_1 \Delta X_t)$ are both I(1) may be understood as polynomial cointegration or multicointegration (Yoo, 1987; Granger and Lee, 1989; Engsted and Johansen, 1999; Kheifets and Phillips, 2021) in our setting; a more extensive discussion and development of this topic can be found in the recent paper by Kheifets and Phillips (2022), and a detailed treatment for the case $\mathcal{B} = \mathcal{H}$ is given by Franchi and Paruolo (2020, Sect. 4.2).

3.5. Representation in a Hilbert Space Setting

We here focus on the case $\mathcal{B} = \mathcal{H}$ and see how our representation results can be simplified in this setting. It is then clarified how such results are related to the existing ones developed in a Hilbert space setting.

3.5.1. *I(1) Case.* If $\mathcal{B} = \mathcal{H}$, Proposition 3.4 implies that the AR(p) law of motion (3.1) admits I(1) solutions if and only if

$$\Lambda_1(V_{\mathtt{C}}, W_{\mathtt{C}}) : \ker \Phi_0 \mapsto V_{\mathtt{C}} \text{ is invertible for } V_{\mathtt{C}} = [\mathrm{ran}\, \Phi_0]^{\perp} \text{ and } W_{\mathtt{C}} = [\ker \Phi_0]^{\perp}. \tag{3.42}$$

Beare and Seo (2020, Thm. 3.2) and Franchi and Paruolo (2020, Prop. 4.6) earlier showed that (3.42) is necessary and sufficient for the AR(p) law of motion (3.1) to admit I(1) solutions if the Fredholm property (3.7) holds. In this case, any solution to (3.1) allows the Johansen I(1) representation (3.3) with Υ_{-1} of finite rank. (Note that (3.21) implies $\mathrm{ran}\, \Upsilon_{-1} = \ker \Phi_0$ and $\ker \Phi_0$ is finite dimensional if $\Phi(z)$ is Fredholm.) This implies that the random walk component in (3.3) reduces to a finite dimensional unit root process. On the other hand, if $\Phi(z)$ satisfies Assumption 3.1 and Φ_0 is compact, then Φ_0 turns out to be a finite rank operator (Chang et al., 2016a, Lem. 1). In this case, (3.42) reduces to the condition given by Chang et al. (2016a, Thm. 2) as a sufficient condition for the existence of I(1) solutions, and $\ker \Phi_0$, where the random walk component takes values, is infinite dimensional unless $\mathcal{H}$ is finite dimensional. In our results for the case $\mathcal{B} = \mathcal{H}$, $\mathrm{ran}\, \Upsilon_{-1} = \ker \Phi_0$ is not required to be either finite or infinite dimensional; we see this from Remark 3.6. Thus, our I(1) representation result given by

Proposition 3.4 complements the earlier results developed in a Hilbert space setting.

Noting that $f \in \text{Ann}(V)$ is given by the map $\langle \cdot, v\rangle$ for some $v \in V^{\perp}$,[5] we find that our characterization of the cointegrating behavior (3.22) given in Remark 3.8 reduces to the following: for any $v \in \mathcal{H} \setminus \{0\}$,

$$\{\langle X_t, v\rangle\}_{t\geq 0} \text{ is I(0) if and only if } v \in [\ker \Phi_0]^{\perp}.$$

The above characterization was earlier found by Beare and Seo (2020) and Franchi and Paruolo (2020) for the case where $\Phi(z)$ is a Fredholm operator satisfying (3.7).

Remark 3.16. Suppose, in addition to (3.42), that $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$. In this setting, Υ_{-1} may be understood as an $n \times n$ matrix. Using the notation introduced for (3.5), the results given by (3.21) can be written as $\beta(\beta^{\intercal}\beta)^{-1}\beta^{\intercal}\Upsilon_{-1} = \Upsilon_{-1}\alpha(\alpha^{\intercal}\alpha)^{-1}\alpha^{\intercal} = 0$ and $\beta_{\perp\!\!\perp}^{\intercal}\Upsilon_{-1}\alpha_{\perp\!\!\perp} = (\alpha_{\perp\!\!\perp}^{\intercal}\Phi_1\beta_{\perp\!\!\perp})^{-1}$. Thus, Υ_{-1} can be written as

$$\Upsilon_{-1} = \beta_{\perp\!\!\perp}\left(\alpha_{\perp\!\!\perp}^{\intercal}\Phi_1\beta_{\perp\!\!\perp}\right)^{-1}\alpha_{\perp\!\!\perp}^{\intercal},$$

which is equivalent to the expression of Υ_{-1} given by Johansen (1995, Thm. 4.2).

3.5.2. *I(2) Case.* Suppose that $\mathcal{B} = \mathcal{H}$ and the complementary subspaces, V_{C}, W_{C}, R_{C}, and K_{C}, are set to the relevant orthogonal complements without loss of generality. In this case, Φ_0^g is equal to the Moore–Penrose inverse $\Phi_0^{\dagger}$ (Remark 3.10). Proposition 3.7 implies that the AR(p) law of motion (3.1) admits I(2) solutions if and only if

$$\begin{aligned}&\Lambda_2(\mathsf{R}_{\mathsf{C}}, \mathsf{K}_{\mathsf{C}}) : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}} \text{ is invertible for } \mathsf{R}_{\mathsf{C}} = [\operatorname{ran}\Phi_0 \oplus \mathsf{R}]^{\perp} \text{ and}\\ &\quad \mathsf{K}_{\mathsf{C}} = [[\ker\Phi_0]^{\perp} \oplus \mathsf{K}]^{\perp}. \end{aligned} \tag{3.43}$$

In the case where $\Phi(z)$ satisfies the Fredholm assumption (3.7), Beare and Seo (2020, Thm. 4.2) showed that (3.43) is a necessary and sufficient condition for the existence of I(2) solutions (see also Section 4 of Franchi and Paruolo, 2020). In this case, any solution to the AR(p) law of motion satisfies the Johansen I(2) representation (3.4) for Υ_{-2} and Υ_{-1} of finite rank; that is, the random walk component of I(2) solutions is intrinsically a finite dimensional unit root process. However, in our results for the case $\mathcal{B} = \mathcal{H}$, the random walk component is not required to have such a property; we see this by noting that $\operatorname{ran}\Upsilon_{-2} = \mathsf{K}$ and K can be infinite dimensional as in Example 3.5. Thus, Proposition 3.7 complements the earlier I(2) representation results developed in a Hilbert space setting.

Moreover, in this case, our characterization of the cointegrating behavior given in Remarks 3.14 and 3.15 can be reformulated as follows: for any $v \in \mathcal{H} \setminus \{0\}$ and for some $r \in \{0, 1\}$,

$$\{\langle X_t, v\rangle\}_{t\geq 0} \text{ is I}(r) \text{ if and only if } v \in \mathsf{K}^{\perp},$$

[5]To see why, observe that (i) any $f \in \mathcal{H}'$ is identified as the map $\langle \cdot, v\rangle$ for a unique element $v \in \mathcal{H}$ and (ii) $\langle x, v\rangle = 0$ for all $x \in V$ if and only if $v \in V^{\perp}$.

$\{\langle X_t, v\rangle\}_{t\geq 0}$ is I(0) if and only if $v \in [\ker \Phi_0]^{\perp} \cap [\Phi_0^{\dagger}\Phi_1 \mathsf{K}]^{\perp}$,

and for any $v \in \mathsf{K}^{\perp}$,

$\{\langle X_t - \Phi_0^{\dagger}\Phi_1 \Delta X_t, v\rangle\}_{t\geq 1}$ is I(0) if and only if $v \in [\ker \Phi_0]^{\perp}$.

These results reduce to the cointegration properties of I(2) solutions provided by Beare and Seo (2020, Rems. 4.5 and 4.6) and Franchi and Paruolo (2020, Rem. 4.10) for the case where $\Phi(z)$ is Fredholm.

Remark 3.17. Suppose further that $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$. Using the notation introduced for (3.35), (3.39) can be recast as $\varrho(\varrho^{\intercal}\varrho)^{-1}\varrho^{\intercal}\Upsilon_{-2} = \Upsilon_{-2}\varpi(\varpi^{\intercal}\varpi)^{-1}\varpi^{\intercal} = 0$ and $\varrho_{\perp\!\!\perp}^{\intercal}\Upsilon_{-2}\varpi_{\perp\!\!\perp} = (\varpi_{\perp\!\!\perp}^{\intercal}\left(\Phi_2 - \Phi_1\Phi_0^{\dagger}\Phi_1\right)\varrho_{\perp\!\!\perp})^{-1}$, from which we find that

$$\Upsilon_{-2} = \varrho_{\perp\!\!\perp}\left[\varpi_{\perp\!\!\perp}^{\intercal}\left(\Phi_2 - \Phi_1\Phi_0^{\dagger}\Phi_1\right)\varrho_{\perp\!\!\perp}\right]^{-1}\varpi_{\perp\!\!\perp}^{\intercal}. \tag{3.44}$$

Let $\alpha_1 = \alpha_{\perp\!\!\perp}\varpi$, $\beta_1 = \beta_{\perp\!\!\perp}\varrho$, $\overline{\alpha}_1 = \alpha_1(\alpha_1^{\intercal}\alpha_1)^{-1}$, and $\overline{\beta}_1 = \beta_1(\beta_1^{\intercal}\beta_1)^{-1}$. Then the operator M_1 can be written as $\mathrm{M}_1 = \alpha_{\perp\!\!\perp}\varpi\varrho^{\intercal}\beta_{\perp\!\!\perp}^{\intercal}$, and its Moore–Penrose inverse $\mathrm{M}_1^{\dagger}$ is given by $\mathrm{M}_1^{\dagger} = \overline{\beta}_1\overline{\alpha}_1^{\intercal}$. By replacing Υ_{-2} and M_1^{g} with (3.44) and $\mathrm{M}_1^{\dagger}$, respectively, in our characterization of Υ_{-1} given in Proposition 3.7, we can obtain Υ_{-1} characterized as an $n \times n$ matrix. These expressions for Υ_{-2} and Υ_{-1} are equivalent to those in Johansen's representation of I(2) AR processes (see, e.g., Johansen, 2008, Thm. 5). The case where $\mathcal{B} = \mathbb{R}^n$ or $\mathbb{C}^n$ was discussed in detail as a special case of a Hilbert space in the recent literature, so the results given in this remark were already noted in the earlier works; see, e.g., Beare and Seo (2020, Rem. 4.4).

4. CONCLUDING REMARKS

This paper introduces a concept and formulation of cointegration in Banach spaces and studies theoretical properties of the cointegrating space. We also extend the Granger–Johansen representation theorem to a potentially infinite dimensional Banach space setting. Compared to existing results, our representation theorems are derived under a weaker geometry of a Banach space and weaker regularity conditions on the AR polynomial. As a consequence, our representation theory not only can accommodate more general AR(p) laws of motion, but also does not place potentially strong restrictions on solutions to such a law of motion, for example, such as finite or infinite dimensionality of the random walk component.

To develop our representation theorems under weaker assumptions, this paper only focuses on the I(1) and I(2) cases. On the other hand, Franchi and Paruolo (2020) recently studied the general I(d) case for $d \geq 1$ and provided a complete characterization of the cointegrating behavior in a convenient form based on the geometry of a Hilbert space and the spectral properties of Fredholm operator pencils; moreover, their representation results can also be extended to the AR(∞)

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case without any further theoretical difficulties, whereas such an extension is not straightforward in the present paper resorting to the companion form representation. Therefore, extending their results directly to our Banach space setting, where non-Fredholm AR polynomials are allowed, seems to be nontrivial. This can certainly be further explored in future study.

It may also be of interest in the near future to develop statistical procedures for analyzing Banach-valued cointegrated time series to complement existing results on estimation, testing, and forecasting with stationary Banach-valued time series (see, e.g., Pumo, 1998; Bosq, 2002; Labbas and Mourid, 2002; Dehling and Sharipov, 2005; Ruiz-Medina and Álvarez-Liébana, 2019; Dette et al., 2020). We believe that the present paper, with a more recent article by Albrecht et al. (2021) containing a novel contribution to the theory of cointegration and the Granger–Johansen representation in a Banach space setting, can serve as a building block for studies in this direction.

APPENDIX

A. Preliminaries

A.1. *Quotient Spaces.* Let V be a subspace of an arbitrary separable complex Banach space $\mathcal{B}$ equipped with norm $\|\cdot\|_{\mathcal{B}}$. The cosets of V are defined as the collection of the following sets: $x+V=\{x+v : v\in V\}$, $x\in\mathcal{B}$. The quotient space of V, denoted by $\mathcal{B}/V$, is the vector space whose elements are equivalence classes of the cosets of V: two cosets $x+V$ and $y+V$ are in the same equivalence class if and only if $x-y\in V$. In the present paper, any quotient space $\mathcal{B}/V$ is mostly associated with a closed subspace V. For such V, the quotient map $\pi_{\mathcal{B}/V}$ is defined by the map $\pi_{\mathcal{B}/V}(x)=x+V$ for $x\in\mathcal{B}$, and the quotient norm $\|\cdot\|_{\mathcal{B}/V}$ is defined as $\|x+V\|_{\mathcal{B}/V}=\inf_{y\in V}\|x-y\|_{\mathcal{B}}$ for $x+V\in\mathcal{B}/V$. $\mathcal{B}/V$ equipped with the quotient norm $\|\cdot\|_{\mathcal{B}/V}$ is a Banach space (Megginson, 2012, Thm. 1.7.7).

A.2. *Random Elements in $\mathcal{B}$.* We briefly introduce Banach-valued random elements, called $\mathcal{B}$-random variables. The reader is referred to Bosq (2000, Chap. 1) for a more detailed discussion on this subject.

Let $(\Omega,\mathbb{F},\mathbb{P})$ be an underlying probability triple. A $\mathcal{B}$-random variable is a measurable map $X:\Omega\mapsto\mathcal{B}$, where $\mathcal{B}$ is understood to be equipped with the Borel σ-field. We say that X is integrable if $\mathbb{E}\|X\|_{\mathcal{B}}<\infty$. If X is integrable, it turns out that there exists a unique element $\mathbb{E}X\in\mathcal{B}$ such that, for all $f\in\mathcal{B}'$, $\mathbb{E}[f(X)]=f(\mathbb{E}X)$. Let $\mathfrak{L}^2(\mathcal{B})$ be the space of $\mathcal{B}$-random variables X with $\mathbb{E}X=0$ and $\mathbb{E}\|X\|^2<\infty$. The covariance operator C_X of $X\in\mathfrak{L}^2(\mathcal{B})$ is a map from $\mathcal{B}'$ to $\mathcal{B}$, defined by $C_X(f)=\mathbb{E}[f(X)X]$ for $f\in\mathcal{B}'$. For $X,Y\in\mathfrak{L}^2(\mathcal{B})$, the cross-covariance operator $C_{X,Y}$ is defined by $C_{X,Y}(f)=\mathbb{E}[f(X)Y]$.

A.3. *Generalized Inverse Operators.* Suppose that $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are separable complex Banach spaces, and $\widetilde{\mathcal{B}}=\operatorname{ran}A\oplus V$ and $\mathcal{B}=\ker A\oplus W$ hold for some $A\in\mathcal{L}(\mathcal{B},\widetilde{\mathcal{B}})$. We then may define the projection P_V (resp. P_W) onto V (resp. W) along $\operatorname{ran}A$ (resp. $\ker A$). Since $A_R=A:W\mapsto\operatorname{ran}A$ is invertible, $A_R^{-1}:\operatorname{ran}A\to W$ is well defined. The generalized inverse A^g of A is obtained by extending the domain (resp. codomain) of A_R^{-1} to $\mathcal{B}$ (resp. $\widetilde{\mathcal{B}}$); that is, A^g is the map given by

$$\mathcal{B} \ni x \quad \mapsto \quad A_R^{-1}(I - \mathrm{P}_V)x \in \widetilde{\mathcal{B}}.$$

It can be shown that A^g satisfies the following properties: $AA^gA = A$, $A^gAA^g = A^g$, $AA^g = (I - \mathrm{P}_V)$, and $A^gA = \mathrm{P}_W$. Note that V and W satisfying $\widetilde{\mathcal{B}} = \operatorname{ran} A \oplus V$ and $\mathcal{B} = \ker A \oplus W$ are not uniquely determined in general, hence the above definition of A^g depends on the choice of V and W; however, for any given choice of V and W, A^g is uniquely defined. In the case where $\widetilde{\mathcal{B}}$ and $\mathcal{B}$ are Hilbert spaces, V (resp. W) can be set to $[\operatorname{ran} A]^{\perp}$ (resp. $[\ker A]^{\perp}$), which makes A^g become equivalent to the Moore–Penrose inverse of A. For a more detailed discussion on generalized inverses, see Engl and Nashed (1981).

A.4. *Operator Pencils.* Let U be some open and connected subset of the complex plane $\mathbb{C}$. An operator pencil is an operator-valued map $A : U \to \mathcal{L}(\mathcal{B})$. An operator pencil A is said to be holomorphic on an open and connected set $U_0 \subset U$ if, for each $z_0 \in U_0$, the limit $A^{(1)}(z_0) := \lim_{z \to z_0}(A(z) - A(z_0))/(z - z_0)$ exists in the norm of $\mathcal{L}(\mathcal{B})$. It turns out that if an operator pencil A is holomorphic, for every $z_0 \in U_0$, we may represent A on U_0 in terms of a power series $A(z) = \sum_{j=0}^{\infty} A_j(z - z_0)^j$ for $z \in U_0$, where $A_0, A_1, \ldots$ is a sequence in $\mathcal{L}(\mathcal{B})$. If there exists k such that $A_j = 0$, for all $j \geq k$, then A is called a polynomial operator pencil. If $A_j = 0$, for all $j \geq 2$, then A is called a linear operator pencil. The collection of $z \in U$ at which the operator $A(z)$ is not invertible is called the spectrum of A, and denoted by $\sigma(A)$. It turns out that the spectrum is always a closed set, and if A is holomorphic on U, then $A(z)^{-1}$ is holomorphic on $U \setminus \sigma(A)$ (Markus, 2012, p. 56). $U \setminus \sigma(A)$ is called the resolvent set of A, and denoted $\rho(A)$. If A is holomorphic and z_0 is an isolated point of $\sigma(A)$, then $A(z)^{-1}$ allows the following Laurent series in a punctured neighborhood of $z = z_0$:

$$A(z)^{-1} = \sum_{j=-d}^{\infty} A_j(z - z_0)^j, \quad d \in \mathbb{N} \cup \{\infty\}, \quad A_j \in \mathcal{L}(\mathcal{B}). \tag{A.1}$$

By Cauchy's residue theorem, we have $A_j = -\frac{1}{2\pi i}\int_{\Gamma} \frac{A(z)^{-1}}{(z-1)^{j+1}} dz$, where $\Gamma \subset \rho(\widetilde{\Phi})$ is a clockwise-oriented contour around z_0 such that the only element of $\sigma(A)$ included inside the contour is z_0.

B. Mathematical Appendix

We provide mathematical proofs of the results given in Sections 2 and 3. It is sometimes convenient to consider $A \in \mathcal{L}(\mathcal{B})$ whose domain is restricted to $V \subset \mathcal{B}$, which is denoted by $A|_V$; that is, $A|_V = A : V \mapsto \mathcal{B}$.

B.1. Proofs of the Results Given in Sections 2 and 3.2

Proof of Proposition 2.1. To show (i), we take $0 \neq f \in \mathcal{B}'$ to both sides of (2.8) and obtain $f(\Delta^{d-1}X_t) = f(\tau_0) + f\Theta(1)(\sum_{s=1}^{t} \varepsilon_s) + f(\nu_t)$, $t \geq 0$. Then $\{f(\nu_t)\}_{t \geq 0}$ is stationary since f is Borel measurable and $\{\nu_t\}_{t \geq 0}$ is stationary. Because $\mathbb{E}[(f\Theta(1)\varepsilon_t)^2] = f\Theta(1)C_{\varepsilon_0}f\Theta(1)$, the second moment of $f\Theta(1)(\sum_{s=1}^{t} \varepsilon_s)$ is given by $tf\Theta(1)C_{\varepsilon_0}f\Theta(1)$, which increases without bound as t grows unless $f\Theta(1) = 0$. Therefore, for $\{f(\Delta^{d-1}X_t)\}_{t \geq 0}$ to be stationary, $f\Theta(1) = 0$ is required. In this case, a suitable initial condition on τ_0 can be obtained by

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letting $f(\tau_0) = 0$. Moreover, one can show without difficulty that $f\Theta(1) = 0$ if and only if $f \in \mathrm{Ann}(\mathrm{ran}\,\Theta(1))$, so $\mathfrak{C}(X) = \mathrm{Ann}(\mathfrak{A}(X))$.

To show (ii), suppose that $g \in \mathfrak{C}(X)$. Note that $x \in \mathcal{B}$ allows the following unique decomposition, $x = x_{\mathrm{cl}\,\mathfrak{A}(X)} + x_V$, where $x_{\mathrm{cl}\,\mathfrak{A}(X)} \in \mathrm{cl}\,\mathfrak{A}(X)$ and $x_V \in V$. From (i) and continuity of bounded linear functionals, we find that $\mathfrak{C}(X) = \mathrm{Ann}(\mathfrak{A}(X)) = \mathrm{Ann}(\mathrm{cl}\,\mathfrak{A}(X))$. Therefore, $g \in \mathfrak{C}(X)$ implies that $g(x) = g(x_V) = g \circ \mathrm{P}_V(x)$. Now, suppose that $g = f \circ \mathrm{P}_V$. Then, clearly, $g(X_t) = g(\tau_0) + g(\nu_t)$, $t \geq 0$. This can be made stationary by letting $g(\tau_0) = 0$. □

Proof of Corollary 2.1. Under the simplifications discussed in Section 2.4 for the case $\mathcal{B} = \mathcal{H}$, Proposition 2.1(ii) implies that $f \in \mathfrak{C}(X)$ is given by the map $\langle \cdot, \mathrm{P}_V y \rangle$ for $y \in \mathcal{H}$. Then the stated result follows. □

Proof of Proposition 3.1. Note that $\widetilde{\Phi}(z)$ may be viewed as the following block operator matrix,

$$\widetilde{\Phi}(z) = \left(\begin{array}{c|cccc} I - z\phi_1 & -z\phi_2 & -z\phi_3 & \cdots & -z\phi_p \\ \hline -zI & I & 0 & \cdots & 0 \\ 0 & -zI & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -zI & I \end{array}\right) =: \begin{pmatrix} \widetilde{\Phi}_{[11]}(z) & \widetilde{\Phi}_{[12]}(z) \\ \widetilde{\Phi}_{[21]}(z) & \widetilde{\Phi}_{[22]}(z) \end{pmatrix}, \tag{B.1}$$

where $\widetilde{\Phi}_{[22]}(z) : \mathcal{B}^{p-1} \mapsto \mathcal{B}^{p-1}$ is invertible for all $z \in \mathbb{C}$. Define the Schur complement of $\widetilde{\Phi}_{[22]}(z)$ as $\widetilde{\Phi}^{+}_{[11]}(z) := \widetilde{\Phi}_{[11]}(z) - \widetilde{\Phi}_{[12]}(z)\widetilde{\Phi}_{[22]}(z)^{-1}\widetilde{\Phi}_{[21]}(z)$. From a little algebra, we find that $\widetilde{\Phi}^{+}_{[11]}(z) = \Phi(z)$. When $\widetilde{\Phi}_{[22]}(z)$ is invertible, $\widetilde{\Phi}(z)$ is invertible if and only if $\widetilde{\Phi}^{+}_{[11]}(z)$ is invertible (Bart et al., 2007, Sect. 2.2), so $\sigma(\widetilde{\Phi}) = \sigma(\Phi)$. Furthermore, from the Schur formula in Bart et al. (2007, p. 29), we have

$$\widetilde{\Phi}(z)^{-1} = \begin{pmatrix} \Phi(z)^{-1} & -\Phi(z)^{-1}\widetilde{\Phi}_{[12]}(z)\widetilde{\Phi}_{[22]}(z)^{-1} \\ -\widetilde{\Phi}_{[22]}(z)^{-1}\widetilde{\Phi}_{[21]}(z)\Phi(z)^{-1} & \widetilde{\Phi}_{[22]}(z)^{-1} + \widetilde{\Phi}_{[22]}(z)^{-1}\widetilde{\Phi}_{[21]}(z)\Phi(z)^{-1}\widetilde{\Phi}_{[12]}(z)\widetilde{\Phi}_{[22]}(z)^{-1} \end{pmatrix}, \tag{B.2}$$

which shows $\Phi(z)^{-1} = \Pi_p \widetilde{\Phi}(z)^{-1} \Pi_p^*$. (iii) is deduced from (B.2) and invertibility of $\widetilde{\Phi}_{[22]}(z)$. □

B.2. Proofs of the Results Given in Section 3.3 (I(1) Representation)

B.2.1. *Preliminary Results.* We provide important preliminary results for the subsequent discussions. Hereafter, it should be noted that P and N_j may be alternatively expressed as the following contour integrals:

$$\mathrm{P} = \frac{-1}{2\pi i}\int_\Gamma (I_p - z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1 dz, \quad \mathrm{N}_j = \frac{-1}{2\pi i}\int_\Gamma (I_p - z\widetilde{\phi}_1)^{-1}(z-1)^{-j-1} dz, \quad j \in \mathbb{Z}, \tag{B.3}$$

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where $\Gamma \subset \rho(\widetilde{\Phi})$ is a clockwise-oriented contour around one such that the only element of $\sigma(\widetilde{\Phi})$ included inside the contour is one. From $\widetilde{\Phi}(z)^{-1}\widetilde{\Phi}(z) = I_p = \widetilde{\Phi}(z)\widetilde{\Phi}(z)^{-1}$, the identity map may be understood as the power series defined in a punctured neighborhood of one as follows:

$$\sum_{j=-\infty}^{\infty} \left(\mathrm{N}_{j-1}\widetilde{\phi}_1 - \mathrm{N}_j(I_p - \widetilde{\phi}_1)\right)(z-1)^j = I_p = \sum_{j=-\infty}^{\infty} \left(\widetilde{\phi}_1\mathrm{N}_{j-1} - (I_p - \widetilde{\phi}_1)\mathrm{N}_j\right)(z-1)^j. \tag{B.4}$$

The above identity expansions give us the following relationships:

$$\mathrm{N}_{-1}\widetilde{\phi}_1 - \mathrm{N}_0(I_p - \widetilde{\phi}_1) = I_p = \widetilde{\phi}_1\mathrm{N}_{-1} - (I_p - \widetilde{\phi}_1)\mathrm{N}_0, \tag{B.5}$$

$$\mathrm{N}_{j-1}\widetilde{\phi}_1 - \mathrm{N}_j(I_p - \widetilde{\phi}_1) = 0 = \widetilde{\phi}_1\mathrm{N}_{j-1} - (I_p - \widetilde{\phi}_1)\mathrm{N}_j, \quad j \neq 0. \tag{B.6}$$

The following lemma collects some essential spectral properties of $\widetilde{\Phi}(z)$.

LEMMA B.1. *Suppose that Assumption 3.1 holds. Then the following hold.*

(i) *If* $\widetilde{\Phi}(z)^{-1}\widetilde{\phi}_1$ *has a pole at* $z = 1$ *of order* ℓ*, then* $\mathrm{N}_{-m}\widetilde{\Phi}_0 = 0$ *for all* $m \geq \ell$.
(ii) $\mathrm{N}_j\widetilde{\phi}_1\mathrm{N}_k = (1 - \eta_j - \eta_k)\mathrm{N}_{j+k+1}$*, where* $\eta_j = 1\{j \geq 0\}$*. Moreover,* $\mathrm{N}_{-1}\widetilde{\phi}_1$ *and* N_{-1} *are projections.*
(iii) $\mathrm{N}_j\widetilde{\phi}_1 = \widetilde{\phi}_1\mathrm{N}_j$ *for all* $j \in \mathbb{Z}$.
(iv) $\widetilde{\Phi}(z)^{-1}$ *has a pole of order at most* ℓ *at* $z = 1$ *if and only if (a)* $n^{-1}\|G^{\ell-1}(I_p - G)^n\|_{\mathrm{op}} \to 0$ *for some* $\ell \in \mathbb{N}$ *and (b)* $\mathrm{ran}(G^m)$ *is closed for some* $m \in \mathbb{N}$ *satisfying* $m \geq \ell$*, where* $G = \widetilde{\Phi}_0\mathrm{P}$.

Proof. To show (i), we note that $I_p = (I_p - z\widetilde{\phi}_1)^{-1}(I_p - \widetilde{\phi}_1) - (z-1)(I_p - z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1$ in a punctured neighborhood of $z = 1$. It is then clear that $(I_p - z\widetilde{\phi}_1)^{-1}(I_p - \widetilde{\phi}_1)$ must have a pole of order $\ell - 1$ if $(I_p - z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1$ has a pole of order $\ell \geq 1$. We therefore have $\mathrm{N}_{-m}\widetilde{\Phi}_0 = 0$ for all $m \geq \ell$.

Our proof of (ii) is similar to those in Kato (1995, p. 38) and Amouch et al. (2015, p. 119). Let $\Gamma, \Gamma' \subset \rho(\Phi)$ be contours enclosing $z = 1$, and assume that Γ' is outside Γ. Using the generalized resolvent equation (Gohberg et al., 2013, p. 50), it can be shown that

$$\mathrm{N}_j\widetilde{\phi}_1\mathrm{N}_k = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'}\int_{\Gamma} \frac{(I_p - \lambda\widetilde{\phi}_1)^{-1} - (I_p - z\widetilde{\phi}_1)^{-1}}{(\lambda - z)(z-1)^{j+1}(\lambda - 1)^{k+1}} dz d\lambda. \tag{B.7}$$

From Kato (1995, p. 38), we may deduce that

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{(\lambda - z)^{-1}}{(z-1)^{j+1}} dz = \eta_j(\lambda - 1)^{-j-1}, \qquad \frac{1}{2\pi i}\int_{\Gamma'} \frac{(\lambda - z)^{-1}}{(\lambda - 1)^{k+1}} d\lambda = (1 - \eta_k)(z-1)^{-k-1}. \tag{B.8}$$

Since we may evaluate the integral in any order, the right-hand side of (B.7) can be written as

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$$\underbrace{\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'}\int_{\Gamma} \frac{(I_p-\lambda\widetilde{\phi}_1)^{-1}}{(\lambda-z)(z-1)^{j+1}(\lambda-1)^{k+1}}dzd\lambda}_{①}$$
$$-\underbrace{\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma}\int_{\Gamma'} \frac{(I_p-z\widetilde{\phi}_1)^{-1}}{(\lambda-z)(z-1)^{j+1}(\lambda-1)^{k+1}}d\lambda dz}_{②}.$$

From (B.8) and Cauchy's residue theorem, we deduce that ① $= -\eta_j \mathrm{N}_{j+k+1}$ and ② $= (\eta_k-1)\mathrm{N}_{j+k+1}$, from which we find that $\mathrm{N}_j\widetilde{\phi}_1\mathrm{N}_k = (1-\eta_j-\eta_k)\mathrm{N}_{j+k+1}$. By putting $j=-1$ and $k=-1$, we obtain $\mathrm{N}_{-1}\widetilde{\phi}_1\mathrm{N}_{-1} = \mathrm{N}_{-1}$, which implies that $\mathrm{N}_{-1}\widetilde{\phi}_1$ is a projection. We now let $U(z) = zI-\widetilde{\phi}_1$, and define $\mathrm{P}_U = \frac{1}{2\pi i}\int_\Gamma U(z)^{-1}dz$. Then P_U is a projection (Gohberg et al., 2013, Lem. 2.1 in Chap. I) and $\mathrm{P}_U = \mathrm{N}_{-1}$ holds (Beare and Seo, 2020, Rem. 3.11).

To show (iii), note that $\widetilde{\phi}_1$ and $I_p-z\widetilde{\phi}_1$ commute. This implies $(I_p-z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1 = \widetilde{\phi}_1(I_p-z\widetilde{\phi}_1)^{-1}$ (Kato, 1995, Thm. 6.5), and we thus have $\mathrm{N}_j\widetilde{\phi}_1 = \widetilde{\phi}_1\mathrm{N}_j$, for all $j\in\mathbb{Z}$.

To show (iv), we will first verify that the following holds for some $\eta>0$:

$$-(I_p-z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1 = \sum_{j=1}^{\infty} G^j(z-1)^{-1-j} + \mathrm{N}_{-1}\widetilde{\phi}_1(z-1)^{-1} + \mathrm{N}_H(z), \quad z\in D_{1+\eta}\setminus\{1\}, \tag{B.9}$$

where $\mathrm{N}_H(z)$ is the holomorphic part of the above Laurent series. We deduce that $\widetilde{\phi}_1\mathrm{N}_{-2} = \widetilde{\Phi}_0\mathrm{N}_{-1}$ from (B.6), and $\widetilde{\Phi}_0\mathrm{N}_{-1} = \widetilde{\Phi}_0\mathrm{N}_{-1}\widetilde{\phi}_1\mathrm{N}_{-1} = \widetilde{\Phi}_0\mathrm{P} = G$ from (ii) and (iii). We thus find that $G = \mathrm{N}_{-2}\widetilde{\phi}_1 = \widetilde{\phi}_1\mathrm{N}_{-2}$. It is also deduced from (ii) and (iii) that $\mathrm{N}_{-k}\widetilde{\phi}_1 = G^{k-1}$ and $G^{k-1} = \widetilde{\Phi}_0^{k-1}\mathrm{P}$ for $k\geq 2$, from which we find that (B.9) holds. In order for (B.9) to converge for $z\in D_{1+\eta}\setminus\{1\}$, $\lim_{k\to\infty}\|G^k\|_{\mathrm{op}}^{1/k}=0$ must hold (Kato, 1995, pp. 180–181). In this case, (a) and (b) are necessary and sufficient for G^ℓ to be zero (Laursen and Mbekhta, 1995, Lem. 3 and Cor. 7). If $G^\ell=0$, we know, from (B.9) and (i), that $-(I_p-z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1$ has a pole of order at most ℓ at $z=1$ and $\mathrm{N}_{-k}\widetilde{\Phi}_0=0$, for all $k\geq\ell$. Combining these results with (B.6), we have $\mathrm{N}_{-k-1}\widetilde{\phi}_1 = \mathrm{N}_{-k}\widetilde{\Phi}_0 = 0$, for all $k\geq\ell$. Since $\mathrm{N}_{-k-1} = \mathrm{N}_{-k-1}\widetilde{\Phi}_0 + \mathrm{N}_{-k-1}\widetilde{\phi}_1$, we find that $\mathrm{N}_{-k-1}=0$, for all $k\geq\ell$, so $(I_p-z\widetilde{\phi}_1)^{-1}$ has a pole of order at most ℓ at $z=1$. Conversely, if $(I_p-z\widetilde{\phi}_1)^{-1}$ has a pole of order at most ℓ at $z=1$, we have $\mathrm{N}_{-k-1}\widetilde{\phi}_1 = G^k = 0$, for $k\geq\ell$. For $G^{\ell+1}$ to be nilpotent, (a) and (b) must hold; see Laursen and Mbekhta (1995, Lem. 3 and Cor. 7). □

B.2.2. *Proofs of the Main Results.*

Proof of Proposition 3.2. Since (ii) ⇒ (iii) is trivial, we will show that (i) ⇒ (ii) and (iii) ⇒ (i). This completes our proof of the equivalence of (i)–(iii). Then we will verify (3.14).

We will show (i) ⇒ (ii). Since $\widetilde{\Phi}(z)^{-1}$ has a simple pole at $z=1$, we may deduce from (B.6) that $\widetilde{\Phi}_0\mathrm{N}_{-1}=0$. This implies that $\operatorname{ran}\mathrm{N}_{-1}\subset\ker\widetilde{\Phi}_0$. We then find that $\operatorname{ran}\mathrm{P}\subset\ker\widetilde{\Phi}_0$ since $\mathrm{P}=\mathrm{N}_{-1}\widetilde{\phi}_1$. Furthermore, $\ker\widetilde{\Phi}_0\subset\operatorname{ran}\mathrm{P}$ holds. To see this, note that if $x_k\in\ker\widetilde{\Phi}_0$,

$$\mathrm{P}x_k = -\frac{1}{2\pi i}\int_\Gamma (I_p-z\widetilde{\phi}_1)^{-1}\widetilde{\phi}_1 x_k dz = \frac{1}{2\pi i}\int_\Gamma (z-1)^{-1}x_k dz = x_k. \tag{B.10}$$

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We thus find that $\operatorname{ran}\mathrm{P} = \ker\widetilde{\Phi}_0$. Moreover, $\mathrm{N}_{-1}\widetilde{\Phi}_0 = 0$ is deduced from (B.6) and $I_p - \mathrm{P} = I_p - \widetilde{\phi}_1\mathrm{N}_{-1}$ is deduced from Lemma B.1(iii). Since $\mathrm{N}_{-1}\operatorname{ran}\widetilde{\Phi}_0 = \{0\}$, we have $(I_p - \mathrm{P})\operatorname{ran}\widetilde{\Phi}_0 = \operatorname{ran}\widetilde{\Phi}_0$, which implies that $\operatorname{ran}\widetilde{\Phi}_0 \subset \operatorname{ran}(I_p - \mathrm{P})$. On the other hand, for any $x \in \operatorname{ran}(I_p - \mathrm{P})$, we have $x = (I_p - \widetilde{\phi}_1\mathrm{N}_{-1})x$ since $I_p - \mathrm{P}$ is a projection and $\mathrm{P} = \widetilde{\phi}_1\mathrm{N}_{-1}$. We know from (B.5) that $\widetilde{\phi}_1\mathrm{N}_{-1} = \widetilde{\Phi}_0\mathrm{N}_0 + I_p$. Therefore, $x = (I_p - \widetilde{\phi}_1\mathrm{N}_{-1})x = -\widetilde{\Phi}_0\mathrm{N}_0 x$, which implies that $x \in \operatorname{ran}\widetilde{\Phi}_0$. Hence, $\operatorname{ran}(I_p - \mathrm{P}) \subset \operatorname{ran}\widetilde{\Phi}_0$ holds, from which we conclude that $\operatorname{ran}(I_p - \mathrm{P}) = \operatorname{ran}\widetilde{\Phi}_0$ since $\operatorname{ran}\widetilde{\Phi}_0 \subset \operatorname{ran}(I_p - \mathrm{P})$ was already shown. To sum up, $\operatorname{ran}\mathrm{P} = \ker\widetilde{\Phi}_0$ and $\operatorname{ran}(I_p - \mathrm{P}) = \operatorname{ran}\widetilde{\Phi}_0$, which means that P is the projection onto $\ker\widetilde{\Phi}_0$ along $\operatorname{ran}\widetilde{\Phi}_0$.

To prove (iii) $\Rightarrow$ (i), we first show that $\widetilde{\Phi}(z)^{-1}$ has a pole of order at most 2 at $z = 1$. Due to Lemma B.1(iv), it suffices to show that $n^{-1}\|G(I_p - G)^n\|_{\mathrm{op}} \to 0$ and $\operatorname{ran}(G^2)$ is closed for $G = \widetilde{\Phi}_0\mathrm{P}$. Since $\widetilde{\phi}_1$ and P commute, $G^2 = \widetilde{\Phi}_0^2\mathrm{P} = \mathrm{P}\widetilde{\Phi}_0^2$. Moreover, it can be shown that $\operatorname{ran}(\mathrm{P}\widetilde{\Phi}_0^2) = \operatorname{ran}\mathrm{P} \cap \operatorname{ran}\widetilde{\Phi}_0^2$. To see this, note that for $x \in \operatorname{ran}(\mathrm{P}\widetilde{\Phi}_0^2)$, there exists $y \in \mathcal{B}^p$ such that $x = \mathrm{P}\widetilde{\Phi}_0^2 y = \widetilde{\Phi}_0^2\mathrm{P}y$, where the second equality results from commutativity of P and $\widetilde{\Phi}_0^2$. This shows that $x \in \operatorname{ran}\mathrm{P} \cap \operatorname{ran}\widetilde{\Phi}_0^2$, hence $\operatorname{ran}(\mathrm{P}\widetilde{\Phi}_0^2) \subset \operatorname{ran}\mathrm{P} \cap \operatorname{ran}\widetilde{\Phi}_0^2$. The reverse inclusion is trivial, so we omit its proof. Since $\operatorname{ran}(\mathrm{P}\widetilde{\Phi}_0^2) = \operatorname{ran}\mathrm{P} \cap \operatorname{ran}(\widetilde{\Phi}_0^2)$ and $\operatorname{ran}\mathrm{P}$ is closed, $\operatorname{ran}(\mathrm{P}\widetilde{\Phi}_0^2)$ is closed if $\operatorname{ran}(\widetilde{\Phi}_0^2)$ is closed. Under the I(1) condition, $\widetilde{\Phi}_0\mathcal{B}^p = \widetilde{\Phi}_0[\operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0] = \widetilde{\Phi}_0\operatorname{ran}\widetilde{\Phi}_0$ holds; that is, $\operatorname{ran}(\widetilde{\Phi}_0^2) = \operatorname{ran}\widetilde{\Phi}_0$, which is closed. It remains to show that $n^{-1}\|G(I_p - G)^n\|_{\mathrm{op}} \to 0$. Note that $(I_p - G)^n = (I_p - \mathrm{P}) + \widetilde{\phi}_1^n\mathrm{P}$. Since $\widetilde{\phi}_1$ and P commute,

$$n^{-1}\|G(I_p - G)^n\|_{\mathrm{op}} \leq n^{-1}\|\widetilde{\Phi}_0\widetilde{\phi}_1^n\|_{\mathrm{op}} \leq n^{-1}\|\widetilde{\phi}_1^n|_{\operatorname{ran}\widetilde{\Phi}_0}\|_{\mathrm{op}}\|\widetilde{\Phi}_0\|_{\mathrm{op}}. \tag{B.11}$$

Under Assumption 3.1, we may deduce, from nearly identical arguments used in Beare et al. (2017) to prove a similar statement, that there exists $k \in \mathbb{N}$ such that, for all $n \geq k$, $\|\widetilde{\phi}_1^n|_{\operatorname{ran}\widetilde{\Phi}_0}\| < a^n$ for some $a \in (0,1)$. Hence, the upper bound in (B.11) vanishes to zero, and we conclude that $\widetilde{\Phi}(z)^{-1}$ has a pole of at most 2 at $z = 1$ under the direct sum $\mathcal{B}^p = \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0$. From (B.5) and (B.6), we have $\mathrm{N}_{-2}\widetilde{\Phi}_0 = 0$ and $\mathrm{N}_{-2}\widetilde{\phi}_1 = \mathrm{N}_{-1}\widetilde{\Phi}_0$. The former (resp. the latter) shows that $\mathrm{N}_{-2}|_{\operatorname{ran}\widetilde{\Phi}_0} = 0$ (resp. $\mathrm{N}_{-2}|_{\ker\widetilde{\Phi}_0} = 0$). Since $\mathcal{B}^p = \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0$, we conclude that $\mathrm{N}_{-2} = 0$. Hence, $\widetilde{\Phi}(z)^{-1}$ has a simple pole at $z = 1$.

It remains only for us to show that (3.14) holds. Let $H(z)$ denote the holomorphic part of the Laurent series given in (3.11). Note that if Assumption 3.1 holds and $\widetilde{\Phi}(z)^{-1}$ has a simple pole, the Maclaurin series of $(1-z)\widetilde{\Phi}(z)^{-1} = \mathrm{P} + (1-z)H(z)$ is convergent on $D_{1+\eta}$. Then, from Lemma 4.1 of Johansen (1995) (or its obvious extension allowing power series with operator coefficients), it may be deduced that the Maclaurin series of $H(z)$ is convergent on $D_{1+\eta}$. Now, we will show $\mathrm{P} = \mathrm{N}_{-1}$ to complete our proof. One can deduce from our proof of (i) $\Rightarrow$ (iii) that $\operatorname{ran}\mathrm{P} = \operatorname{ran}\mathrm{N}_{-1}$. Moreover, Lemma B.1(ii) implies that N_{-1} is a projection. Therefore, N_{-1} is clearly a projection whose range is equal to $\operatorname{ran}\mathrm{P} = \ker\widetilde{\Phi}_0$. Then we find that $\mathrm{P} = \mathrm{N}_{-1}\widetilde{\phi}_1 = \widetilde{\phi}_1\mathrm{N}_{-1} = \mathrm{N}_{-1}$, where the second equality is from the fact that $\widetilde{\phi}_1$ and N_{-1} commute, and the last equality is from that $\widetilde{\phi}_1|_{\ker\widetilde{\Phi}_0} = I_p|_{\ker\widetilde{\Phi}_0}$ and $\operatorname{ran}\mathrm{N}_{-1} = \ker\widetilde{\Phi}_0$. □

Proof of Proposition 3.3. From Propositions 3.1 and 3.2, we have $(1-z)\Phi(z)^{-1} = \Pi_p\mathrm{N}_{-1}\Pi_p^* + (1-z)\Pi_p H(z)\Pi_p^*$. Applying the linear filter induced by $(1-z)\Phi(z)^{-1}$ to (3.1), we obtain $\Delta X_t = \Pi_p\mathrm{N}_{-1}\Pi_p^*\varepsilon_t + \Delta\nu_t$, where $\nu_t = \Pi_p H(L)\Pi_p^*\varepsilon_t$ and $H(z) = \sum_{j=0}^{\infty} H_j z^j$ with $H_j = H^{(j)}/j!$ is convergent on $D_{1+\eta}$. Clearly, the process given by

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$\Pi_p \mathrm{N}_{-1} \Pi_p^* \sum_{s=1}^t \varepsilon_s + \nu_t$ is a solution, which is completed by adding a time invariant component τ_0 given as the solution to the homogeneous equation $\Delta X_t = 0$.

We then verify the claimed expression of ν_t in (3.15). Once we show that

$$H_0 = I_p - \mathrm{P}, \quad H_j = \widetilde{\phi}_1 H_{j-1}, \quad j \geq 1, \tag{B.12}$$

then the claimed expression given in (3.15) may be easily verified. First, it can be shown that $H(z) = H(z)(I_p - \mathrm{P})$ holds. To see this, note that $H(z) = -\sum_{j=0}^{\infty} \mathrm{N}_j (z-1)^j$. Since Lemma B.1(ii) implies that $\mathrm{N}_j(I_p - \mathrm{P}) = \mathrm{N}_j - \mathrm{N}_j \widetilde{\phi}_1 \mathrm{N}_{-1} = \mathrm{N}_j$, for $j \geq 0$, we find that $H(z)(I_p - \mathrm{P}) = -\sum_{j=0}^{\infty} \mathrm{N}_j (I_p - \mathrm{P})(z-1)^j = -\sum_{j=0}^{\infty} \mathrm{N}_j (z-1)^j$. This also shows that $\widetilde{\Phi}(z)^{-1}(I_p - \mathrm{P}) = H(z)$, and

$$I_p - \mathrm{P} = \widetilde{\Phi}(z) \widetilde{\Phi}(z)^{-1} (I_p - \mathrm{P}) = \widetilde{\Phi}(z) H(z). \tag{B.13}$$

We then easily deduce that $H_0 = I_p - \mathrm{P}$ from (B.13) evaluated at $z = 0$. Furthermore, (B.13) can be rewritten as $(I_p - \widetilde{\phi}_1) H(z) - (z-1)\widetilde{\phi}_1 H(z) = I_p - \mathrm{P}$, from which we have

$$H^{(j)}(z) - j \widetilde{\phi}_1 H^{(j-1)}(z) - z \widetilde{\phi}_1 H^{(j)}(z) = 0, \quad j \geq 1. \tag{B.14}$$

Evaluating (B.14) at $z = 0$, we obtain $H^{(j)}(0) = j \widetilde{\phi}_1 H^{(j-1)}(0) = j! \widetilde{\phi}_1 H_{j-1}$, which verifies (B.12).

It remains to show that if the I(1) condition is not satisfied, then the AR(p) law of motion (3.1) does not allow I(1) solutions. This immediately follows from Propositions 3.1 and 3.2. □

Proof of Proposition 3.4. Throughout this proof, we write the Laurent series of $\Phi(z)^{-1}$ near $z = 1$ as follows: for $d \in \mathbb{N} \cup \{\infty\}$, $\Phi(z)^{-1} = -\sum_{j=-d}^{\infty} \ddot{\mathrm{N}}_j (z-1)^j$. Since it is obvious that (iii) $\Rightarrow$ (ii), we will only show that (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i). The whole proof is divided into several parts.

1. (i) $\Rightarrow$ (iii): Let $V_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran} \Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\ker \Phi_0)$ be arbitrarily chosen. We know that $d = 1$, $\ddot{\mathrm{N}}_{-1} = \Pi_p \mathrm{P} \Pi_p^*$, and $\operatorname{ran} \mathrm{P} = \ker \widetilde{\Phi}_0$ under the I(1) condition (Propositions 3.1 and 3.2). Since $(x_1, \ldots, x_p) \in \ker \widetilde{\Phi}_0$ implies that $x_1 = \cdots = x_p \in \ker \Phi_0$, $\operatorname{ran} \ddot{\mathrm{N}}_{-1} = \operatorname{ran} \Pi_p \mathrm{P} \Pi_p^* \subset \Pi_p \ker \widetilde{\Phi}_0 = \ker \Phi_0$ holds. From the coefficients of $(z-1)^{-1}$ and $(z-1)^0$ in the identity expansion $\Phi(z)^{-1}\Phi(z) = I$, we know that $\ddot{\mathrm{N}}_{-1}$ satisfies $\ddot{\mathrm{N}}_{-1}\Phi_0 = 0$ and $\ddot{\mathrm{N}}_{-1}\Phi_1 + \ddot{\mathrm{N}}_0 \Phi_0 = -I$. From these equations, we observe that $-\ddot{\mathrm{N}}_{-1} \mathrm{P}_{V_{\mathsf{C}}} \Phi_1 (I - \mathrm{P}_{W_{\mathsf{C}}}) = I - \mathrm{P}_{W_{\mathsf{C}}}$, hence $\ker \Phi_0 \subset \operatorname{ran} \ddot{\mathrm{N}}_{-1}$. We thus find that $\operatorname{ran} \ddot{\mathrm{N}}_{-1} = \ker \Phi_0$, so $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}}) : \ker \Phi_0 \mapsto V_{\mathsf{C}}$ is an injection. Moreover, from the coefficients of $(z-1)^{-1}$ and $(z-1)^0$ in the expansion $\Phi(z)\Phi(z)^{-1} = I$, we find that $\Phi_0 \ddot{\mathrm{N}}_{-1} = 0$ and $\Phi_1 \ddot{\mathrm{N}}_{-1} + \Phi_0 \ddot{\mathrm{N}}_0 = -I$, which implies that $-\mathrm{P}_{V_{\mathsf{C}}} \Phi_1 \ddot{\mathrm{N}}_{-1} = \mathrm{P}_{V_{\mathsf{C}}}$. Since $\operatorname{ran} \ddot{\mathrm{N}}_{-1} = \ker \Phi_0$ was already shown, it is concluded that $\Lambda_1(V_{\mathsf{C}}, W_{\mathsf{C}}) : \ker \Phi_0 \mapsto V_{\mathsf{C}}$ is also a surjection, i.e., it is a bijection. The above arguments do not depend on the choice of V_{C} and W_{C}. Thus, (i) $\Rightarrow$ (iii).

2. (ii) $\Rightarrow$(i): For given $V_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran} \Phi_0)$ and $W_{\mathsf{C}} \in \mathsf{C}(\ker \Phi_0)$, we define $\mathcal{Q}_{\mathsf{C}} : \mathcal{B}^p \to \mathcal{B}^p$ as follows:

$$\mathcal{Q}_{\mathsf{C}} = \left(\begin{array}{c|cccc} \mathrm{P}_{V_{\mathsf{C}}} & \mathrm{P}_{V_{\mathsf{C}}} \sum_{j=2}^p \phi_j & \mathrm{P}_{V_{\mathsf{C}}} \sum_{j=3}^p \phi_j & \cdots & \mathrm{P}_{V_{\mathsf{C}}} \phi_p \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right). \tag{B.15}$$

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Then $\mathcal{Q}_{\complement}$ is a projection on $\mathcal{B}^p$ and $\mathcal{Q}_{\complement}\widetilde{\Phi}_0 = 0$. The latter may be verified by noting that

$$\widetilde{\Phi}_0 = \begin{pmatrix} \Phi_0 & \widetilde{\Phi}_{[12]}(1) \\ 0 & \widetilde{\Phi}_{[22]}(1) \end{pmatrix} \begin{pmatrix} I & 0 \\ \widetilde{\Phi}_{[22]}(1)^{-1}\widetilde{\Phi}_{[21]}(1) & I_{p-1} \end{pmatrix}, \qquad \mathcal{Q}_{\complement}\begin{pmatrix} \Phi_0 & \widetilde{\Phi}_{[12]}(1) \\ 0 & \widetilde{\Phi}_{[22]}(1) \end{pmatrix} = 0,$$

where $\widetilde{\Phi}_{[12]}(\cdot)$, $\widetilde{\Phi}_{[21]}(\cdot)$, and $\widetilde{\Phi}_{[22]}(\cdot)$ are defined in (B.1). We thus find that $\ker \mathcal{Q}_{\complement} \supset \operatorname{ran}\widetilde{\Phi}_0$. It can also be shown that $\ker \mathcal{Q}_{\complement} \subset \operatorname{ran}\widetilde{\Phi}_0$. To see why, note that $\mathrm{P}_{V_{\complement}}(x_1 + \sum_{j=2}^{p}\phi_j x_2 + \sum_{j=3}^{p}\phi_j x_3 + \cdots + \phi_p x_p) = 0$ holds for any $x = (x_1,\ldots,x_p) \in \ker \mathcal{Q}_{\complement}$, and $\widetilde{\Phi}_0(0, x_2, \ldots, \sum_{j=2}^{p} x_j) = (-\sum_{j=2}^{p}\phi_j x_2 - \sum_{j=3}^{p}\phi_j x_3 - \cdots - \phi_p x_p, x_2, \ldots, x_p)$. Let $y_1 = -\sum_{j=2}^{p}\phi_j x_2 - \sum_{j=3}^{p}\phi_j x_3 - \cdots - \phi_p x_p$. With some algebra, we obtain the following results:

$$(x, 0, \ldots, 0) \in \operatorname{ran}\widetilde{\Phi}_0, \text{ for } x \in \operatorname{ran}\Phi_0, \tag{B.16}$$

$$y_1 = \mathrm{P}_{V_{\complement}} y_1 + (I - \mathrm{P}_{V_{\complement}}) y_1, \quad \mathrm{P}_{V_{\complement}} y_1 = \mathrm{P}_{V_{\complement}} x_1. \tag{B.17}$$

Using (B.16) and (B.17), we find that $(\mathrm{P}_{V_{\complement}} x_1, x_2, \ldots, x_p) \in \operatorname{ran}\widetilde{\Phi}_0$. Combining this result with (B.16), we conclude that $x = (x_1, \ldots, x_p) \in \operatorname{ran}\widetilde{\Phi}_0$, so $\ker \mathcal{Q}_{\complement} \subset \operatorname{ran}\widetilde{\Phi}_0$.

We have shown that $\ker \mathcal{Q}_{\complement} = \operatorname{ran}\widetilde{\Phi}_0$, hence $\mathcal{Q}_{\complement}$ is a projection onto some $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$. Let $x_k = (x_{1,k}, \ldots, x_{p,k}) \in \ker \widetilde{\Phi}_0$, then it may be easily shown that $x_{1,k} = \cdots = x_{p,k}$ and $x_{1,k} \in \ker \Phi_0$. With a little algebra and from the fact that $x_{1,k} = (I - \mathrm{P}_{W_{\complement}}) x_{1,k}$, we obtain

$$\mathcal{Q}_{\complement} x_k = \left(-\mathrm{P}_{V_{\complement}}\Phi_1(I - \mathrm{P}_{W_{\complement}}) x_{1,k}, 0, \ldots, 0\right). \tag{B.18}$$

Moreover, from the definition of $\mathcal{Q}_{\complement}$, we find that $\operatorname{ran}\mathcal{Q}_{\complement} \subset \{(x_1, 0, \ldots, 0) \in \mathcal{B}^p : x_1 \in V_{\complement}\}$. Combining this result with (B.18) and invertibility of $\Lambda_1(V_{\complement}, W_{\complement}) : \ker \Phi_0 \mapsto V_{\complement}$, we find that $\mathcal{Q}_{\complement,R} = \mathcal{Q}_{\complement} : \ker \widetilde{\Phi}_0 \mapsto \mathcal{V}_{\complement}$ is invertible. This implies that $\mathcal{B} = \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker \widetilde{\Phi}_0$, which is deduced from Fact 4.3 of Fabian et al. (2010) and the fact that the map $D : \operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{V}_{\complement} \mapsto \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker \widetilde{\Phi}_0$, given by $D = \begin{pmatrix} I_p & 0 \\ 0 & \mathcal{Q}_{\complement,R}^{-1} \end{pmatrix}$, is invertible.

3. Formula for Υ_{-1}: For $V_{\complement} \in \complement(\operatorname{ran}\Phi_0)$ and $W_{\complement} \in \complement(\ker \Phi_0)$, we found that $-\ddot{\mathrm{N}}_{-1}\mathrm{P}_{V_{\complement}}\Phi_1(I - \mathrm{P}_{W_{\complement}}) = I - \mathrm{P}_{W_{\complement}}$ and $\Lambda_1(V_{\complement}, W_{\complement}) : \ker \Phi_0 \mapsto V_{\complement}$ is invertible, from which (3.21) follows immediately. □

B.3. Proofs the Results Given in Section 3.4 (I(2) Representation)

B.3.1. *Preliminary Results.* We first collect some preliminary results that are useful for the subsequent discussion.

LEMMA B.2. *Let everything be as in Section 3.4.*

(i) *Under Assumption 3.1,* $\operatorname{ran}\widetilde{\Phi}_0$ *and* $\ker \widetilde{\Phi}_0$ *can be complemented in* $\mathcal{B}^p$.
(ii) *Under Assumption 3.2,* $\mathcal{K} \neq \{0\}$ *is necessary for* $\widetilde{\Phi}(z)^{-1}$ *to have a pole of order 2.*
(iii) *Under Assumption 3.2, the I(2) condition is equivalent to the following:* $\mathcal{K} \neq \{0\}$ *and* $\mathcal{B}^p = (\operatorname{ran}\widetilde{\Phi}_0 + \ker \widetilde{\Phi}_0) \oplus \widetilde{\Phi}_0^g \mathcal{K}$ *holds for any arbitrary* $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ *and* $\mathcal{W}_{\complement} \in \complement(\ker \widetilde{\Phi}_0)$.

Let $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ *and* $\mathcal{W}_{\complement} \in \complement(\ker \widetilde{\Phi}_0)$ *be arbitrarily chosen and let* $\mathcal{R} = \mathrm{P}_{\mathcal{V}_{\complement}} \ker \widetilde{\Phi}_0$.

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(iv) *Under Assumption 3.2, the following direct sums hold for some* $\mathcal{R}_{\complement} \subset \mathcal{B}^p$ *and* $\mathcal{K}_{\complement} \subset \mathcal{B}^p$*,*

$$\mathcal{B}^p = \operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{R} \oplus \mathcal{R}_{\complement} = \mathcal{W}_{\complement} \oplus \mathcal{K}_{\complement} \oplus \mathcal{K}. \tag{B.19}$$

(v) *Under Assumption 3.2, the operator* $Q = \mathrm{P}_{\mathcal{V}_{\complement}}(I_p - \mathrm{P}_{\mathcal{W}_{\complement}})$ *allows the generalized inverse* Q^g *satisfying* $\operatorname{ran} Q^g = \mathcal{K}_{\complement}$ *and* $\ker Q^g = \operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{R}_{\complement}$*, where* $\mathcal{R}_{\complement}$ *and* $\mathcal{K}_{\complement}$ *satisfy (B.19).*

Proof. Under Assumption 3.1, we may define the projection $\mathcal{Q}_{\complement}$ given in (B.15). Since $\ker \mathcal{Q}_{\complement} = \operatorname{ran}\widetilde{\Phi}_0$, $\operatorname{ran}\widetilde{\Phi}_0$ is complemented by $\operatorname{ran}\mathcal{Q}_{\complement}$. Moreover, $\ker\widetilde{\Phi}_0$ is also complemented. To see this, we note that $x = (x_1,\ldots,x_p) \in \ker\widetilde{\Phi}_0$ implies that $x_1 = x_2 = \cdots = x_p$ and $x_1 \in \ker\Phi_0$. Let $\mathcal{T}_{\complement}$ be the $(p\times p)$ operator matrix whose entries of the first column are all equal to $I - \mathrm{P}_{\mathcal{W}_{\complement}}$ and all the other entries are equal to zero. Then it can be easily shown that this is a projection defined on $\mathcal{B}^p$ and its range is equal to $\ker\widetilde{\Phi}_0$; hence, $\ker\widetilde{\Phi}_0$ is complemented by the kernel of this projection. This completes our proof of (i).

To show (ii), suppose that $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 at $z = 1$ and $\mathcal{K} = \{0\}$. It can be shown that $\ker\widetilde{\Phi}_0 \subset \operatorname{ran}\mathrm{P} \subset \ker(\widetilde{\Phi}_0^2)$ holds under Assumption 3.1, where the first inclusion follows from (B.10). To see why the second inclusion holds, we note that $\mathrm{N}_{-2} = \widetilde{\Phi}_0\mathrm{P}$ (Lemma B.1(iv)), and then deduce from (B.6) that $\widetilde{\Phi}_0\mathrm{N}_{-2} = 0$. From these results, we find that $\widetilde{\Phi}_0\mathrm{N}_{-2} = \widetilde{\Phi}_0^2\mathrm{P} = 0$, which proves the second inclusion. Since $\mathcal{K} = \{0\}$ implies that $\ker\widetilde{\Phi}_0 = \ker(\widetilde{\Phi}_0^2)$, we find that $\operatorname{ran}\mathrm{P} = \ker\widetilde{\Phi}_0$ and so $\mathrm{N}_{-2} = \widetilde{\Phi}_0\mathrm{P} = 0$. This contradicts our assumption that $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 at $z = 1$, so $\mathcal{K} \neq \{0\}$.

To show (iii), we let $\widetilde{\Phi}^g_{0,1}$ and $\widetilde{\Phi}^g_{0,2}$ be the generalized inverses depending on two different choices of $\mathcal{V}_{\complement} \in \complement(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\complement} \in \complement(\ker\widetilde{\Phi}_0)$. Let $S = \operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$, $S_1 = \widetilde{\Phi}^g_{0,1}\mathcal{K}$, and $S_2 = \widetilde{\Phi}^g_{0,2}\mathcal{K}$. We know from Megginson (2012, Thm. 1.7.14 and Cor. 3.2.16) that two complementary subspaces of S are isomorphic, implying that $AS_1 = S_2$ and $S_1 = A^{-1}S_2$ for some invertible map $A \in \mathcal{L}(S_1,S_2)$ and its inverse $A^{-1} \in \mathcal{L}(S_2,S_1)$. Let $D : S\oplus S_1 \mapsto S\oplus S_2$ be the map given by $D = \begin{pmatrix} I_p & 0 \\ 0 & A\end{pmatrix}$, which is obviously invertible and its inverse $D^{-1} : S\oplus S_2 \mapsto S\oplus S_1$ is given by $D^{-1} = \begin{pmatrix} I_p & 0 \\ 0 & A^{-1}\end{pmatrix}$. Note that $D(S) = S$, $D(S_1) = S_2$, $D^{-1}(S) = S$, and $D^{-1}(S_2) = S_1$. It then follows from Fact 4.3 of Fabian et al. (2010) that $\mathcal{B}^p = S\oplus S_1$ holds if and only if $\mathcal{B}^p = S\oplus S_2$ holds. This completes the proof.

To show (iv), note that $\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0 = \operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{R}$. We define $\mathcal{S}_{\complement} : \mathcal{B}^p \to \mathcal{B}^p$ as the block operator matrix obtained by replacing $\mathrm{P}_{\mathcal{W}_{\complement}}$ with $\mathrm{P}_{\mathsf{R}_{\complement}}$ in (B.15). Then it can be shown that $\mathcal{S}_{\complement}$ is a projection and $\mathcal{S}_{\complement}\widetilde{\Phi}_0 = 0$. If $x \in \ker\widetilde{\Phi}_0$, then $x = (x_k,\ldots,x_k)$ for some $x_k \in \ker\Phi_0$ and $\mathcal{S}_{\complement}x = -\mathrm{P}_{\mathsf{R}_{\complement}}\Phi_1 x_k = 0$ since $\ker\mathrm{P}_{\mathsf{R}_{\complement}} = \operatorname{ran}\Phi_0 \oplus \mathsf{R} = \operatorname{ran}\Phi_0 + \Phi_1\ker\Phi_0$. We thus find that $\ker\mathcal{S}_{\complement} \supset \operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$. Moreover, $\ker\mathcal{S}_{\complement} \subset \operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$ also holds. To see why, let $x = (x_1,\ldots,x_p) \in \ker\mathcal{S}_{\complement}$. Then x must satisfy $\mathrm{P}_{\mathsf{R}_{\complement}}(x_1 + \sum_{j=2}^p \phi_j x_2 + \sum_{j=3}^p \phi_j x_3 + \cdots + \phi_p x_p) = 0$. As in our proof of Proposition 3.4, if we let $y_1 = -\sum_{j=2}^p \phi_j x_2 - \sum_{j=3}^p \phi_j x_3 - \cdots - \phi_p x_p$, then $\mathrm{P}_{\mathsf{R}_{\complement}} y_1 = \mathrm{P}_{\mathsf{R}_{\complement}} x_1$ and $\widetilde{\Phi}_0(0, x_2, \ldots, \sum_{j=2}^p x_j) = (y_1, x_2, \ldots, x_p)$ hold. Note that $(x, 0, \ldots, 0) \in \operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$ if $x \in \operatorname{ran}\Phi_0 + \Phi_1\ker\Phi_0$, which is because, for any arbitrary $u \in \mathcal{B}$ and $w \in \ker\Phi_0$, we have $\widetilde{\Phi}_0(v-w, v-2w, \ldots, v-pw) + (w, w, \ldots, w) = (\Phi_0 v + \Phi_1 w, 0, \ldots, 0)$. Combining all these results with the fact that $\ker\mathrm{P}_{\mathsf{R}_{\complement}} = \operatorname{ran}\Phi_0 + \Phi_1\ker\Phi_0$, we find that $x \in \operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$. To sum up, $\mathcal{S}_{\complement}$ is

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a projection and $\ker \mathcal{S}_{\mathrm{C}} = \operatorname{ran} \widetilde{\Phi}_0 + \ker \widetilde{\Phi}_0 = \operatorname{ran} \widetilde{\Phi}_0 \oplus \mathcal{R}$, meaning that $\operatorname{ran} \widetilde{\Phi}_0 \oplus \mathcal{R}$ is complemented by $\operatorname{ran} \mathcal{S}_{\mathrm{C}}$. The latter direct sum of (B.19) clearly holds for $\mathcal{K}_{\mathrm{C}} = \ker \widetilde{\Phi}_0 \cap \mathcal{V}_{\mathrm{C}}$.

To show (v), we note that $\mathcal{B}^p = \operatorname{ran} \widetilde{\Phi}_0 \oplus \operatorname{ran} Q \oplus \mathcal{R}_{\mathrm{C}}$ under the former direct sum of (B.19). Moreover, from the latter direct sum of (B.19), we have $\mathcal{B}^p = \ker Q \oplus \mathcal{K}_{\mathrm{C}}$, where $\ker Q = \mathcal{W}_{\mathrm{C}} \oplus \mathcal{K}$. Therefore, the generalized inverse Q^g exists, and $\operatorname{ran} Q^g = \mathcal{K}_{\mathrm{C}}$ and $\ker Q^g = \operatorname{ran} \widetilde{\Phi}_0 \oplus \mathcal{R}_{\mathrm{C}}$; see Appendix A.3. □

LEMMA B.3. *Let Assumption 3.2 hold. For any arbitrary choice of* R_{C} *and* K_{C}, *the operator* $\mathrm{M}_1 = \mathrm{P}_{V_{\mathrm{C}}} \Phi_1 (I - \mathrm{P}_{W_{\mathrm{C}}})$ *allows the generalized inverse* M_1^g *satisfying (i)* $\mathrm{M}_1 \mathrm{M}_1^g = (I - \mathrm{P}_{\mathsf{R}_{\mathrm{C}}}) \mathrm{P}_{V_{\mathrm{C}}}$, *(ii)* $\mathrm{M}_1^g \mathrm{M}_1 = \mathrm{P}_{\mathsf{K}_{\mathrm{C}}}$, *(iii)* $\mathrm{M}_1^g = \mathsf{K}_{\mathrm{C}}$, *and (iv)* $\ker \mathrm{M}_1^g = \operatorname{ran} \Phi_0 \oplus \mathsf{R}_{\mathrm{C}}$, *where* $\mathrm{P}_{\mathsf{K}_{\mathrm{C}}}$ *is the projection onto* K_{C} *along* $W_{\mathrm{C}} \oplus \mathsf{K}$.

Proof. Under the direct sums given in (3.36), $\mathcal{B} = \operatorname{ran} \Phi_0 \oplus \operatorname{ran} \mathrm{M}_1 \oplus \mathsf{R}_{\mathrm{C}}$ holds. Note also that $\ker \mathrm{M}_1 = W_{\mathrm{C}} \oplus \mathsf{K}$, hence we have $\mathcal{B} = \ker \mathrm{M}_1 \oplus \mathsf{K}_{\mathrm{C}}$. We then know from Appendix A.3 that M_1 allows the generalized inverse M_1^g whose range is K_{C} and kernel is $\operatorname{ran} \Phi_0 \oplus \mathsf{R}_{\mathrm{C}}$, from which the desired result follows. □

B.3.2. *Proofs of the Main Results.*

Proof of Proposition 3.5. Under the direct sums (B.19) given in Lemma B.2(iv), we let $\mathrm{P}_{\mathcal{R}_{\mathrm{C}}}$ denote the projection onto $\mathcal{R}_{\mathrm{C}}$ along $\operatorname{ran} \widetilde{\Phi}_0 \oplus \mathcal{R}$, which is well defined. The whole proof is divided into several parts.

1. Necessity of the I(2) condition: If the I(2) condition is not satisfied, then it must be the case either

$$\left(\operatorname{ran} \widetilde{\Phi}_0 + \ker \widetilde{\Phi}_0\right) \cap \widetilde{\Phi}_0^g \mathcal{K} \neq \{0\} \quad \text{or} \quad \mathcal{B}^p \neq \operatorname{ran} \widetilde{\Phi}_0 + \ker \widetilde{\Phi}_0 + \widetilde{\Phi}_0^g \mathcal{K}. \tag{B.20}$$

It will be shown that (B.20) is false if $\widetilde{\Phi}(z)^{-1}$ has a pole of order 2 at $z = 1$. From (B.5) and (B.6), we have

$$\mathrm{N}_{-2} \widetilde{\Phi}_0 = 0 = \widetilde{\Phi}_0 \mathrm{N}_{-2}, \tag{B.21}$$

$$\mathrm{N}_{-2} \widetilde{\phi}_1 - \mathrm{N}_{-1} \widetilde{\Phi}_0 = 0 = \widetilde{\phi}_1 \mathrm{N}_{-2} - \widetilde{\Phi}_0 \mathrm{N}_{-1}, \tag{B.22}$$

$$\mathrm{N}_{-1} \widetilde{\phi}_1 - \mathrm{N}_0 \widetilde{\Phi}_0 = I_p = \widetilde{\phi}_1 \mathrm{N}_{-1} - \widetilde{\Phi}_0 \mathrm{N}_0. \tag{B.23}$$

From (B.21), we observe that

$$\mathrm{N}_{-2} = \widetilde{\phi}_1 \mathrm{N}_{-2} = \mathrm{N}_{-2} \widetilde{\phi}_1, \tag{B.24}$$

$$\mathrm{N}_{-2} = \mathrm{N}_{-2} \mathrm{P}_{\mathcal{V}_{\mathrm{C}}}. \tag{B.25}$$

Restricting both sides of (B.22) to $\ker \widetilde{\Phi}_0$, we find that

$$\mathrm{N}_{-2}|_{\ker \widetilde{\Phi}_0} = 0 \quad \Leftrightarrow \quad \mathrm{N}_{-2} \ker \widetilde{\Phi}_0 = \{0\}. \tag{B.26}$$

We note that $\widetilde{\Phi}_0^g$ exists, and then deduce from (B.22) and (B.24) that

$$\mathrm{N}_{-1}(I_p - \mathrm{P}_{\mathcal{V}_{\mathrm{C}}}) = \mathrm{N}_{-2} \widetilde{\Phi}_0^g, \quad \mathrm{P}_{\mathcal{W}_{\mathrm{C}}} \mathrm{N}_{-1} = \widetilde{\Phi}_0^g \mathrm{N}_{-2}. \tag{B.27}$$

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(B.25) implies that postcomposing both sides of the latter equation in (B.27) with $\mathrm{P}_{\mathcal{V}_\mathbb{C}}$ changes nothing, so

$$\mathrm{N}_{-1}\mathrm{P}_{\mathcal{V}_\mathbb{C}} = \widetilde{\Phi}_0^g \mathrm{N}_{-2} + (I_p - \mathrm{P}_{\mathcal{W}_\mathbb{C}})\mathrm{N}_{-1}\mathrm{P}_{\mathcal{V}_\mathbb{C}}. \tag{B.28}$$

From the former equation in (B.27) and (B.28), we find that

$$\mathrm{N}_{-1} = \mathrm{N}_{-2}\widetilde{\Phi}_0^g + \widetilde{\Phi}_0^g \mathrm{N}_{-2} + (I_p - \mathrm{P}_{\mathcal{W}_\mathbb{C}})\mathrm{N}_{-1}\mathrm{P}_{\mathcal{V}_\mathbb{C}}. \tag{B.29}$$

Restricting both sides of (B.29) to $\mathcal{K}$, we obtain $\mathrm{N}_{-1}|_\mathcal{K} = \mathrm{N}_{-2}\widetilde{\Phi}_0^g|_\mathcal{K}$, which is due to (B.26). Furthermore, $\mathrm{N}_{-1}|_\mathcal{K} = I_p|_\mathcal{K}$ is deduced by restricting both sides of (B.23) to $\mathcal{K}$, so we find that $\mathrm{N}_{-1}|_\mathcal{K} = \mathrm{N}_{-2}\widetilde{\Phi}_0^g|_\mathcal{K} = I_p|_\mathcal{K}$. Given $\mathrm{P}_{\mathcal{R}_\mathbb{C}}$, the former direct sum given in (B.19), equations (B.25) and (B.26), and the fact that $\operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{R} = \operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$, we conclude that $\mathrm{N}_{-2} = \mathrm{N}_{-2}\mathrm{P}_{\mathcal{R}_\mathbb{C}}$. Therefore,

$$\mathrm{N}_{-2}\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0^g|_\mathcal{K} = I_p|_\mathcal{K}. \tag{B.30}$$

This proves injectivity of $\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0^g : \mathcal{K} \mapsto \mathcal{R}_\mathbb{C}$, hence the former condition in (B.20) is impossible.

Precomposing both sides of the latter equation in (B.22) with $\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0^g$ and using (B.24), we have

$$\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0^g \mathrm{N}_{-2} = \mathrm{P}_{\mathcal{R}_\mathbb{C}}\mathrm{P}_{\mathcal{W}_\mathbb{C}}\mathrm{N}_{-1} = \mathrm{P}_{\mathcal{R}_\mathbb{C}}\mathrm{N}_{-1}, \tag{B.31}$$

where the last equality is established from the fact that the kernel of $\mathrm{P}_{\mathcal{R}_\mathbb{C}}$ is $\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$ and thus $\mathrm{P}_{\mathcal{R}_\mathbb{C}}(I_p - \mathrm{P}_{\mathcal{W}_\mathbb{C}}) = 0$. Similarly, precomposing both sides of the latter equation of (B.23) with $\mathrm{P}_{\mathcal{R}_\mathbb{C}}$,

$$\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\phi}_1 \mathrm{N}_{-1} = \mathrm{P}_{\mathcal{R}_\mathbb{C}} + \mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0 \mathrm{N}_0 = \mathrm{P}_{\mathcal{R}_\mathbb{C}}. \tag{B.32}$$

Note that $\mathrm{P}_{\mathcal{R}_\mathbb{C}}\mathrm{N}_{-1} = \mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0\mathrm{N}_{-1} + \mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\phi}_1\mathrm{N}_{-1}$, where the first term is 0. We therefore deduce from (B.32) that $\mathrm{P}_{\mathcal{R}_\mathbb{C}}\mathrm{N}_{-1} = \mathrm{P}_{\mathcal{R}_\mathbb{C}}$. Combining this with (B.31), we obtain $\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0^g\mathrm{N}_{-2} = \mathrm{P}_{\mathcal{R}_\mathbb{C}}$, which implies surjectivity of $\mathrm{P}_{\mathcal{R}_\mathbb{C}}\widetilde{\Phi}_0^g : \mathcal{K} \mapsto \mathcal{R}_\mathbb{C}$. Hence, the latter condition in (B.20) cannot hold.

2. Sufficiency of the I(2) condition: Suppose that $\widetilde{\Phi}(z)^{-1}$ has a pole of order $m \geq 3$ at $z = 1$. We deduce that $\mathrm{N}_{-m} = \mathrm{N}_{-m}\widetilde{\phi}_1 = \widetilde{\phi}_1\mathrm{N}_{-m}$ from (B.6), and also $\mathrm{N}_{-m} = G^{m-1} = \widetilde{\Phi}_0^{m-1}\mathrm{P}$ from Lemma B.1(iv). If the I(2) condition holds, any $x \in \mathcal{B}^p$ can be written as $x = x_{\text{ran}} + x_{\text{ker}} + \widetilde{\Phi}_0^g x_\mathcal{K}$, where $x_{\text{ran}} \in \operatorname{ran}\widetilde{\Phi}_0$, $x_{\text{ker}} \in \ker\widetilde{\Phi}_0$, and $x_\mathcal{K} \in \mathcal{K}$. From the definitions of $\operatorname{ran}\widetilde{\Phi}_0$ and $\mathcal{K}$, we also know that there exist $y_1 \in \mathcal{B}^p$ and $y_2 \in \mathcal{B}^p$ satisfying $x_{\text{ran}} = \widetilde{\Phi}_0 y_1$, $x_\mathcal{K} = \widetilde{\Phi}_0 y_2$, and $\widetilde{\Phi}_0^2 y_2 = 0$. We thus find that

$$\widetilde{\Phi}_0^{m-1}\mathrm{P}x = \mathrm{P}\widetilde{\Phi}_0^{m-1}(x_{\text{ran}} + x_{\text{ker}} + \widetilde{\Phi}_0^g x_\mathcal{K}) = \mathrm{P}\widetilde{\Phi}_0^m y_1 + \mathrm{P}\widetilde{\Phi}_0^{m-1}(x_{\text{ker}} + y_2) = 0, \tag{B.33}$$

where the first equality comes from commutativity of P and $\widetilde{\Phi}_0$. It is then deduced from (B.33) that $\widetilde{\Phi}_0^{m-1}\mathrm{P}\mathcal{B} = \{0\}$, so $\widetilde{\Phi}_0^{m-1}\mathrm{P} = 0$. That is, $\mathrm{N}_{-m} = 0$ is concluded, which, however, contradicts our assumption that $\widetilde{\Phi}(z)^{-1}$ has a pole of order $m \geq 3$. In addition, $\mathcal{K} \neq \{0\}$ implies that $\mathcal{B} \neq \operatorname{ran}\widetilde{\Phi}_0 \oplus \ker\widetilde{\Phi}_0$, which excludes the existence of a simple pole (see Proposition 3.2).

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3. Formula for N_{-1}: Note that $\mathrm{N}_{-1} = \widetilde{\Phi}_0\mathrm{N}_{-1} + \widetilde{\phi}_1\mathrm{N}_{-1}$. We also know from (B.24) that $\mathrm{N}_{-2}\widetilde{\phi}_1^2 = \mathrm{N}_{-2}\widetilde{\phi}_1 = \mathrm{N}_{-2}$. Combining this with (B.22), it is deduced that $\mathrm{N}_{-1} = \widetilde{\Phi}_0\mathrm{N}_{-1} + \widetilde{\phi}_1\mathrm{N}_{-1} = \mathrm{N}_{-2} + \mathrm{P}$.

4. $\operatorname{ran}\mathrm{N}_{-2} = \mathcal{K}$: The result immediately follows from (B.30) and invertibility of $\mathrm{P}_{\mathcal{R}_{\mathsf{C}}}\widetilde{\Phi}_0^g : \mathcal{K} \to \mathcal{R}_{\mathsf{C}}$.

5. Holomorphicity of $(1-z)^2\widetilde{\Phi}(z)^{-1}$ and $H(z)$ on $D_{1+\eta}$: We know that the Maclaurin series of $(1-z)^2\widetilde{\Phi}(z)^{-1}$ is convergent on $D_{1+\eta}$. Then, from an obvious extension of Lemma 4.1 of Johansen (1995), it may be deduced that the Maulaurin series of $H(z)$ is convergent on $D_{1+\eta}$. □

Proof of Proposition 3.6. From Propositions 3.1 and 3.5, we have $(1-z)^2\Phi(z)^{-1} = -\Pi_p\mathrm{N}_{-2}\Pi_p^* + \Pi_p(\mathrm{N}_{-2}+\mathrm{P})\Pi_p^*(1-z) + \Pi_p H(z)\Pi_p(1-z)^2$. Applying the linear filter induced by $(1-z)^2\Phi(z)^{-1}$ to (3.1), we obtain $\Delta^2 X_t = -\Pi_p\mathrm{N}_{-2}\Pi_p^*\varepsilon_t + \Pi_p(\mathrm{N}_{-2} + \mathrm{P})\Pi_p^*(\varepsilon_t - \varepsilon_{t-1}) + (\Delta\nu_t - \Delta\nu_{t-1})$. We then may deduce that solutions to (3.1) satisfy (3.4) for some τ_0 and τ_1. The claimed expression of ν_t can be verified from nearly identical arguments used in our proof of Proposition 3.3. Moreover, we may deduce from Propositions 3.1 and 3.5 that (3.1) does not allow I(2) solutions if the I(2) condition is not satisfied. □

Proof of Proposition 3.7. We write the Laurent series of $\Phi(z)^{-1}$ around $z = 1$ as follows: for $d \in \mathbb{N}\cup\{\infty\}$, $\Phi(z)^{-1} = -\sum_{j=-d}^{\infty}\ddot{\mathrm{N}}_j(z-1)^j$. Since it is obvious that (iii) ⇒ (ii), we will only show that (i) ⇒ (iii) and (ii) ⇒ (i). The whole proof is divided into several parts.

1. (i) ⇒ (iii): Let $\mathsf{R}_{\mathsf{C}} \in \mathsf{C}(\mathsf{R})$ and $\mathsf{K}_{\mathsf{C}} \in \mathsf{C}(\mathsf{K})$ be arbitrarily chosen. If $(x_1,\dots,x_p) \in \mathcal{K}$, we know from (4.40) of Beare et al. (2017) that $x_1 = \cdots = x_p$, and there exists $y_1 \in \mathcal{B}$ such that $\Phi_0 y_1 = -\Phi_1 x_1$. This implies that $x_1 \in \mathsf{K}$. Since $\operatorname{ran}\mathrm{N}_{-2} = \mathcal{K}$, $\operatorname{ran}\ddot{\mathrm{N}}_{-2} = \operatorname{ran}\Pi_p\mathrm{N}_{-2}\Pi_p^* \subset \mathsf{K}$ holds. Under the I(2) condition, we know that $\Phi(z)^{-1}$ has a pole of order 2 at $z = 1$ and deduce the following from the identity expansion $\Phi(z)^{-1}\Phi(z) = I = \Phi(z)\Phi(z)^{-1}$:

$$\ddot{\mathrm{N}}_{-2}\Phi_0 = 0 = \Phi_0\ddot{\mathrm{N}}_{-2}, \tag{B.34}$$

$$\ddot{\mathrm{N}}_{-2}\Phi_1 + \ddot{\mathrm{N}}_{-1}\Phi_0 = 0 = \Phi_1\ddot{\mathrm{N}}_{-2} + \Phi_0\ddot{\mathrm{N}}_{-1}, \tag{B.35}$$

$$\ddot{\mathrm{N}}_{-2}\Phi_2 + \ddot{\mathrm{N}}_{-1}\Phi_1 + \ddot{\mathrm{N}}_0\Phi_0 = -I = \Phi_2\ddot{\mathrm{N}}_{-2} + \Phi_1\ddot{\mathrm{N}}_{-1} + \Phi_0\ddot{\mathrm{N}}_0. \tag{B.36}$$

From some algebra similar to that in the proof of Theorem 4.2 of Beare and Seo (2020), we find that $\ddot{\mathrm{N}}_{-2}(I - \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}) = 0$ and $-\ddot{\mathrm{N}}_{-2}\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} = \mathrm{P}_{\mathsf{K}}$. This implies that $\mathsf{K} \subset \operatorname{ran}\ddot{\mathrm{N}}_{-2}$ (hence $\operatorname{ran}\ddot{\mathrm{N}}_{-2} = \mathsf{K}$ has been established) and thus $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}}) = \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$ is an injection.

Now, from the latter equation in (B.35) and the properties of Φ_0^g, we may deduce that

$$\Phi_1\ddot{\mathrm{N}}_{-1} = -\Phi_1\Phi_0^g\Phi_1\ddot{\mathrm{N}}_{-2} + \Phi_1(I - \mathrm{P}_{W_{\mathsf{C}}})\ddot{\mathrm{N}}_{-1}. \tag{B.37}$$

From the latter equation in (B.36) and (B.37), we have $\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\ddot{\mathrm{N}}_{-2} + \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\Phi_1(I - \mathrm{P}_{W_{\mathsf{C}}})\ddot{\mathrm{N}}_{-1} = -\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}$. Because $\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\Phi_1\ker\Phi_0 = \{0\}$, $\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\Phi_1(I - \mathrm{P}_{W_{\mathsf{C}}})\ddot{\mathrm{N}}_{-1} = 0$ and thus $\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\ddot{\mathrm{N}}_{-2} = -\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}$ hold. Given that $\operatorname{ran}\ddot{\mathrm{N}}_{-2} = \mathsf{K}$, this implies that $\Lambda_2(\mathsf{R}_{\mathsf{C}},\mathsf{K}_{\mathsf{C}}) =$

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$\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$ is also a surjection, i.e., it is a bijection. The above arguments do not depend on a specific choice of $\mathsf{R}_{\mathsf{C}} \in \mathsf{C}(\mathsf{R})$ and $\mathsf{K}_{\mathsf{C}} \in \mathsf{C}(\mathsf{K})$, and thus (i) $\Rightarrow$ (iii).

2. (ii) $\Rightarrow$ (i): It suffices to show that the I(2) condition holds for some choice of $\widetilde{\Phi}_0^g$ (Lemma B.2(iii)). Let

$$\widetilde{\Phi}_0^g = \begin{pmatrix} \Phi_0^g & -\Phi_0^g\widetilde{\Phi}_{[12]}(1)\widetilde{\Phi}_{[22]}(1)^{-1} \\ -\widetilde{\Phi}_{[22]}(1)^{-1}\widetilde{\Phi}_{[21]}(1)\Phi_0^g & \widetilde{\Phi}_{[22]}(1)^{-1} + \widetilde{\Phi}_{[22]}(1)^{-1}\widetilde{\Phi}_{[21]}(1)\Phi_0^g\widetilde{\Phi}_{[12]}(1)\widetilde{\Phi}_{[22]}(1)^{-1} \end{pmatrix}, \tag{B.38}$$

where $\widetilde{\Phi}_{[12]}(\cdot)$, $\widetilde{\Phi}_{[21]}(\cdot)$, and $\widetilde{\Phi}_{[22]}(\cdot)$ are defined in (B.1). From the factorization formula (2.3) of Bart et al. (2007), it may be easily shown that $\widetilde{\Phi}_0\widetilde{\Phi}_0^g = I_p - \mathcal{Q}_{\mathsf{C}}$ for $\mathcal{Q}_{\mathsf{C}}$, the projection whose kernel is equal to $\operatorname{ran}\widetilde{\Phi}_0$, given in (B.15). Moreover, $\widetilde{\Phi}_0^g\widetilde{\Phi}_0 = I_p - \mathcal{T}_{\mathsf{C}}$ for $\mathcal{T}_{\mathsf{C}}$, the projection whose range is equal to $\ker\widetilde{\Phi}_0$, given in our proof of Lemma B.2(i). Therefore, $\widetilde{\Phi}_0^g$ given in (B.38) is the generalized inverse of $\widetilde{\Phi}_0$ for some $\mathcal{V}_{\mathsf{C}} \in \mathsf{C}(\operatorname{ran}\widetilde{\Phi}_0)$ and $\mathcal{W}_{\mathsf{C}} \in \mathsf{C}(\ker\widetilde{\Phi}_0)$. For given R_{C} and K_{C}, we let $\mathcal{S}_{\mathsf{C}} : \mathcal{B}^p \to \mathcal{B}^p$ denote the block operator matrix obtained by replacing $\mathrm{P}_{V_{\mathsf{C}}}$ with $\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}$ in the definition of $\mathcal{Q}_{\mathsf{C}}$ given by (B.15), and let $\mathcal{S}_{\mathsf{C},[1,2]} : \mathcal{B}^{p-1} \mapsto \mathcal{B}$ denote the upper-right block of $\mathcal{S}_{\mathsf{C}}$. We already showed that $\mathcal{S}_{\mathsf{C}}$ is a projection onto a complementary subspace of $\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$ in our proof of Lemma B.2(iv). Then, using the formula (2.3) of Bart et al. (2007), we find that $\mathcal{S}_{\mathsf{C}}\widetilde{\Phi}_0^g = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}$, where $C_1 = \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\Phi_0^g - \mathcal{S}_{\mathsf{C},[12]}\widetilde{\Phi}_{[22]}(1)^{-1}\widetilde{\Phi}_{[21]}(1)\Phi_0^g$ and $C_2 = \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\Phi_0^g\widetilde{\Phi}_{[12]}(1)\widetilde{\Phi}_{[22]}(1)^{-1} + \mathcal{S}_{\mathsf{C},[12]}(\widetilde{\Phi}_{[22]}(1)^{-1} + \widetilde{\Phi}_{[22]}(1)^{-1}\widetilde{\Phi}_{[21]}(1)\Phi_0^g\widetilde{\Phi}_{[12]}(1)\widetilde{\Phi}_{[22]}(1)^{-1})$. If $x_k = (x_{1,k}, \ldots, x_{p,k}) \in \mathcal{K}$, then $x_{1,k} = \cdots = x_{p,k}$ and $x_{1,k} \in \mathsf{K}$. For such x_k, we obtain the following after a tedious algebra:

$$\mathcal{S}_{\mathsf{C}}\widetilde{\Phi}_0^g x_k = \left(-\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}}x_{1,k}, 0, \ldots, 0\right). \tag{B.39}$$

From the definition of $\mathcal{S}_{\mathsf{C}}$, it may be deduced that $\operatorname{ran}\mathcal{S}_{\mathsf{C}} \subset \{(x_1, 0, \ldots, 0) \in \mathcal{B}^p : x_1 \in \mathsf{R}_{\mathsf{C}}\}$. Combining this result with (B.39) and invertibility of the map $\Lambda_2(\mathsf{R}_{\mathsf{C}}, \mathsf{K}_{\mathsf{C}}) = \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$, we find that $\mathcal{S}_{\mathsf{C}} : \widetilde{\Phi}_0\mathcal{K} \mapsto \mathcal{R}_{\mathsf{C}}$ is invertible. This implies that $\widetilde{\Phi}_0^g\mathcal{K}$ is a complementary subspace of $\operatorname{ran}\widetilde{\Phi}_0 + \ker\widetilde{\Phi}_0$ by a similar argument that we used in our proof of Proposition 3.4. From the above proof, we know that $\mathcal{K} = \{0\}$ is impossible since it implies $\mathsf{K} = \{0\}$.

3. Formula for $\boldsymbol{\Upsilon_{-2}}$: We know $-\ddot{\mathrm{N}}_{-2}\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} = \mathrm{P}_{\mathsf{K}}$ holds under the I(2) condition. Since the map $\Lambda_2(\mathsf{R}_{\mathsf{C}}, \mathsf{K}_{\mathsf{C}}) : \mathsf{K} \mapsto \mathsf{R}_{\mathsf{C}}$ is invertible and $\Upsilon_{-2} = -\ddot{\mathrm{N}}_{-2}$, the desired results given by (3.39) are easily obtained.

4. Formula for $\boldsymbol{\Upsilon_{-1}}$: We first establish some preliminary results. According to the direct sums given in Assumption 3.2 and for any arbitrary choice of the complementary subspaces therein, we have

$$I = (I - \mathrm{P}_{V_{\mathsf{C}}}) + (I - \mathrm{P}_{\mathsf{R}_{\mathsf{C}}})\mathrm{P}_{V_{\mathsf{C}}} + \mathrm{P}_{\mathsf{R}_{\mathsf{C}}}. \tag{B.40}$$

Based on the identity (B.40), we will obtain explicit expressions of $\ddot{\mathrm{N}}_{-1}(I - \mathrm{P}_{V_{\mathsf{C}}})$, $\ddot{\mathrm{N}}_{-1}(I - \mathrm{P}_{\mathsf{R}_{\mathsf{C}}})\mathrm{P}_{V_{\mathsf{C}}}$, and $\ddot{\mathrm{N}}_{-1}\mathrm{P}_{\mathsf{R}_{\mathsf{C}}}$. In the sequel, we need (B.35), (B.36), and the following obtained from the coefficient of $(z-1)^1$ in the identity expansion $\Phi(z)^{-1}\Phi(z) = I = \Phi(z)\Phi(z)^{-1}$:

$$\ddot{\mathrm{N}}_{-2}\Phi_3 + \ddot{\mathrm{N}}_{-1}\Phi_2 + \ddot{\mathrm{N}}_0\Phi_1 = 0 = \Phi_3\ddot{\mathrm{N}}_{-2} + \Phi_2\ddot{\mathrm{N}}_{-1} + \Phi_1\ddot{\mathrm{N}}_0. \tag{B.41}$$

From (B.35), we find that

$$\ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{V_{\mathtt{C}}}) = -\ddot{\mathrm{N}}_{-2}\Phi_1\Phi_0^g. \tag{B.42}$$

From (B.36) and the identity $\ddot{\mathrm{N}}_{-1}\Phi_1 = \ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{V_{\mathtt{C}}})\Phi_1 + \ddot{\mathrm{N}}_{-1}\mathrm{P}_{V_{\mathtt{C}}}\Phi_1$, it follows that $[\ddot{\mathrm{N}}_{-2}\Phi_2 + \ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{V_{\mathtt{C}}})\Phi_1 + \ddot{\mathrm{N}}_{-1}\mathrm{P}_{V_{\mathtt{C}}}\Phi_1](I-\mathrm{P}_{W_{\mathtt{C}}}) = -(I-\mathrm{P}_{W_{\mathtt{C}}})$. Substituting (B.42) into this equation, we have

$$\ddot{\mathrm{N}}_{-1}\mathrm{M}_1 = -(I+\ddot{\mathrm{N}}_{-2}\mathrm{M}_2)(I-\mathrm{P}_{W_{\mathtt{C}}}). \tag{B.43}$$

Postcomposing both sides of (B.43) with M_1^g and using the fact that $\mathrm{M}_1\mathrm{M}_1^g = (I-\mathrm{P}_{\mathsf{R}_{\mathtt{C}}})\mathrm{P}_{V_{\mathtt{C}}}$ (Lemma B.3), we obtain that

$$\ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{\mathsf{R}_{\mathtt{C}}})\mathrm{P}_{V_{\mathtt{C}}} = -(I+\ddot{\mathrm{N}}_{-2}\mathrm{M}_2)\mathrm{M}_1^g. \tag{B.44}$$

Now, from (B.41), we have $[\ddot{\mathrm{N}}_{-2}\Phi_3+\ddot{\mathrm{N}}_{-1}\Phi_2+\ddot{\mathrm{N}}_0\Phi_1]\mathrm{P}_{\mathsf{K}} = 0$. We note from the definition of K that $\ddot{\mathrm{N}}_0\Phi_1\mathrm{P}_{\mathsf{K}} = \ddot{\mathrm{N}}_0(I-\mathrm{P}_{V_{\mathtt{C}}})\Phi_1\mathrm{P}_{\mathsf{K}}$, and deduce from (B.36) that $\ddot{\mathrm{N}}_0(I-\mathrm{P}_{V_{\mathtt{C}}}) = -\Phi_0^g - \ddot{\mathrm{N}}_{-2}\Phi_2\Phi_0^g - \ddot{\mathrm{N}}_{-1}\Phi_1\Phi_0^g$. Combining these results, we obtain

$$\ddot{\mathrm{N}}_{-1}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} = -[\ddot{\mathrm{N}}_{-2}\mathrm{M}_3 - \ddot{\mathrm{N}}_{-2}\mathrm{M}_2\Phi_0^g\Phi_1 - \Phi_0^g\Phi_1]\mathrm{P}_{\mathsf{K}}. \tag{B.45}$$

Since $\ddot{\mathrm{N}}_{-1}\mathrm{P}_{\mathsf{R}_{\mathtt{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} = [\ddot{\mathrm{N}}_{-1}\mathrm{M}_2 - \ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{V_{\mathtt{C}}})\mathrm{M}_2 - \ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{\mathsf{R}_{\mathtt{C}}})\mathrm{P}_{V_{\mathtt{C}}}\mathrm{M}_2]\mathrm{P}_{\mathsf{K}}$, and $\ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{V_{\mathtt{C}}})$ and $\ddot{\mathrm{N}}_{-1}(I-\mathrm{P}_{\mathsf{R}_{\mathtt{C}}})\mathrm{P}_{V_{\mathtt{C}}}$ are given in (B.42) and (B.44), we have

$$\begin{aligned}\ddot{\mathrm{N}}_{-1}\mathrm{P}_{\mathsf{R}_{\mathtt{C}}}\mathrm{M}_2\mathrm{P}_{\mathsf{K}} = &-[\ddot{\mathrm{N}}_{-2}\mathrm{M}_3 - \ddot{\mathrm{N}}_{-2}\mathrm{M}_2\Phi_0^g\Phi_1 - \Phi_0^g\Phi_1]\mathrm{P}_{\mathsf{K}} + \ddot{\mathrm{N}}_{-2}\Phi_1\Phi_0^g\mathrm{M}_2\mathrm{P}_{\mathsf{K}}\\ &+ (I+\ddot{\mathrm{N}}_{-2}\mathrm{M}_2)\mathrm{M}_1^g\mathrm{M}_2\mathrm{P}_{\mathsf{K}}.\end{aligned} \tag{B.46}$$

By postcomposing both sides of (B.46) with $\ddot{\mathrm{N}}_{-2}$, a formula for $\ddot{\mathrm{N}}_{-1}\mathrm{P}_{\mathsf{R}_{\mathtt{C}}}$ is obtained. Combining this with (B.42), (B.44), and the fact that $\Upsilon_{-2} = -\ddot{\mathrm{N}}_{-2}$ and $\Upsilon_{-1} = \ddot{\mathrm{N}}_{-1}$, we obtain

$$\begin{aligned}\Upsilon_{-1} = &-\mathrm{M}_1^g + \left(\Phi_0^g\Phi_1 + \mathrm{M}_1^g\mathrm{M}_2\right)\Upsilon_{-2} + \Upsilon_{-2}\left(\Phi_1\Phi_0^g + \mathrm{M}_2\mathrm{M}_1^g\right)\\ &+\Upsilon_{-2}\left(\mathrm{M}_3 - \mathrm{M}_2\Phi_0^g\Phi_1 - \Phi_1\Phi_0^g\mathrm{M}_2 - \mathrm{M}_2\mathrm{M}_1^g\mathrm{M}_2\right)\Upsilon_{-2}.\end{aligned} \tag{B.47}$$

Using the fact that $\operatorname{ran}\mathrm{M}_1^g = \mathsf{K}_{\mathtt{C}}$ and $\Upsilon_{-2} = \mathrm{P}_{\mathsf{K}}\Upsilon_{-2}\mathrm{P}_{\mathsf{R}_{\mathtt{C}}}$, the desired results are obtained from (B.47). □

B.3.3. *Supplementary Results to Proposition 3.5*. We here characterize the principal part of the Laurent series $\widetilde{\Phi}(z)^{-1}$ in more detail. Let $\mathrm{P}_{\mathcal{W}_{\mathtt{C}}}$, $\mathrm{P}_{\mathcal{V}_{\mathtt{C}}}$, and $\widetilde{\Phi}_0^g$ be defined as in Section 3.4.1. Given the direct sum conditions in (B.19), we let $\mathrm{P}_{\mathcal{R}_{\mathtt{C}}}$ (resp. $\mathrm{P}_{\mathcal{K}}$) be the projection onto $\mathcal{R}_{\mathtt{C}}$ (resp. $\mathcal{K}$) along $\operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{R}$ (resp. $\mathcal{W}_{\mathtt{C}} \oplus \mathcal{K}_{\mathtt{C}}$). Let $\widetilde{\Lambda}$ be the map given by $\mathrm{P}_{\mathcal{R}_{\mathtt{C}}}\widetilde{\Phi}_0^g : \mathcal{K} \mapsto \mathcal{R}_{\mathtt{C}}$, and let $Q = \mathrm{P}_{\mathcal{V}_{\mathtt{C}}}(I_p - \mathrm{P}_{\mathcal{W}_{\mathtt{C}}})$. The generalized inverse Q^g is well defined and satisfies that $\operatorname{ran}Q^g = \mathcal{K}_{\mathtt{C}}$ and $\ker Q^g = \operatorname{ran}\widetilde{\Phi}_0 \oplus \mathcal{R}_{\mathtt{C}}$ (Lemma B.2(v)). We will show that $\widetilde{\Lambda} : \mathcal{K} \mapsto \mathcal{R}_{\mathtt{C}}$ is invertible and N_{-2} satisfies

$$(I_p - \mathrm{P}_{\mathcal{K}})\mathrm{N}_{-2} = \mathrm{N}_{-2}(I_p - \mathrm{P}_{\mathcal{R}_{\mathtt{C}}}) = 0, \qquad \mathrm{N}_{-2} : \mathcal{R}_{\mathtt{C}} \mapsto \mathcal{K} = \widetilde{\Lambda}^{-1}, \tag{B.48}$$

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and, moreover, $\mathrm{P} = (I_p - \Gamma_r)Q^g(I_p - \Gamma_\ell) + \Gamma_\ell(I_p - \Gamma_r) + \Gamma_r$, where $\Gamma_\ell = \widetilde{\Phi}_0^g \mathrm{N}_{-2}$ and $\Gamma_r = \mathrm{N}_{-2}\widetilde{\Phi}_0^g$.

1. Formula for $\mathbf{N_{-2}}$: Under the I(2) condition, we know from our proof of Proposition 3.5 that (B.30) holds, $\mathrm{N}_{-2} = \mathrm{N}_{-2}\mathrm{P}_{\mathcal{R}_\mathtt{C}}$, and $\widetilde{\Lambda} = \mathrm{P}_{\mathcal{R}_\mathtt{C}}\widetilde{\Phi}_0^g : \mathcal{K} \to \mathcal{R}_\mathtt{C}$ is bijective. From these results, (B.48) follows.

2. Formula for P: From the former direct sum in (B.19), we have

$$I_p = (I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}}) + (I_p - \mathrm{P}_{\mathcal{R}_\mathtt{C}})\mathrm{P}_{\mathcal{V}_\mathtt{C}} + \mathrm{P}_{\mathcal{R}_\mathtt{C}}. \tag{B.49}$$

Thus, $\mathrm{P} = \mathrm{P}(I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}}) + \mathrm{P}(I_p - \mathrm{P}_{\mathcal{R}_\mathtt{C}})\mathrm{P}_{\mathcal{V}_\mathtt{C}} + \mathrm{P}\mathrm{P}_{\mathcal{R}_\mathtt{C}}$, and we will obtain an expression for each summand.

Precomposing both sides of the former equation in (B.22) with $\widetilde{\phi}_1$, we obtain $\widetilde{\phi}_1\mathrm{N}_{-2}\widetilde{\phi}_1 = \widetilde{\phi}_1\mathrm{N}_{-1}\widetilde{\Phi}_0$. We then deduce the following from (B.24) and the fact that $\widetilde{\phi}_1\mathrm{N}_{-1} = \mathrm{P}$,

$$\mathrm{P}(I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}}) = \mathrm{N}_{-2}\widetilde{\Phi}_0^g. \tag{B.50}$$

Using the identity $I_p = (I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}}) + \mathrm{P}_{\mathcal{V}_\mathtt{C}}$ and postcomposing both sides of the former equation in (B.23) with $I_p - \mathrm{P}_{\mathcal{W}_\mathtt{C}}$, we find that $\mathrm{P}Q = (I_p - \mathrm{P}(I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}}))(I_p - \mathrm{P}_{\mathcal{W}_\mathtt{C}})$. Then, from (B.50), we obtain $\mathrm{P}Q = (I_p - \mathrm{N}_{-2}\widetilde{\Phi}_0^g)(I_p - \mathrm{P}_{\mathcal{W}_\mathtt{C}})$. Moreover, it may be deduced from Lemma B.2(v) that $QQ^g = (I_p - \mathrm{P}_{\mathcal{R}_\mathtt{C}})\mathrm{P}_{\mathcal{V}_\mathtt{C}}$, hence we find that

$$\mathrm{P}(I_p - \mathrm{P}_{\mathcal{R}_\mathtt{C}})\mathrm{P}_{\mathcal{V}_\mathtt{C}} = \left(I_p - \mathrm{N}_{-2}\widetilde{\Phi}_0^g\right)Q^g. \tag{B.51}$$

We obtain $\mathrm{P}\widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K} - \mathrm{N}_0(I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}})\mathrm{P}_\mathcal{K} = \widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K}$ by postcomposing both sides of the former equation in (B.23) with $\widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K}$. Note that $(I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}})\mathrm{P}_\mathcal{K} = \mathrm{P}_\mathcal{K}$. We also deduce from (B.6) that $\mathrm{N}_0\widetilde{\phi}_1 = \mathrm{N}_1\widetilde{\Phi}_0$, which implies that $\mathrm{N}_0\mathrm{P}_\mathcal{K} = 0$. Combining these results, we find that $\mathrm{P}\widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K} = \widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K}$. Then, from (B.49), we have

$$\mathrm{P}\mathrm{P}_{\mathcal{R}_\mathtt{C}}\widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K} = \widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K} - \mathrm{P}(I_p - \mathrm{P}_{\mathcal{V}_\mathtt{C}})\widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K} - \mathrm{P}(I_p - \mathrm{P}_{\mathcal{R}_\mathtt{C}})\mathrm{P}_{\mathcal{V}_\mathtt{C}}\widetilde{\Phi}_0^g\mathrm{P}_\mathcal{K}. \tag{B.52}$$

We then substitute (B.50) and (B.51) into (B.52), and then postcompose both sides with N_{-2}, noting that $\mathrm{P}_\mathcal{K}\mathrm{N}_{-2} = \mathrm{N}_{-2}$ due to (B.48). This gives us an explicit formula for $\mathrm{P}\mathrm{P}_{\mathcal{R}_\mathtt{C}}$. Combining this with (B.50) and (B.51), the claimed formula for P can be obtained.

B.4. Supplement to Examples and Remarks

Remark 2.3: Note that (i) $x(u) = x(1)$ holds for $u \in [0,1]$ if $x \in \operatorname{ran}\phi_1$, (ii) $(I - \phi_1)x = 0$ for any constant function $x \in C[0,1]$, and (iii) C_0 is closed; from these results, we find that $\operatorname{cl}\mathfrak{A}(X) = \operatorname{ran}\phi_1 = \mathrm{C}_0$.

Example 3.1: Assumption 3.1(a) is satisfied since $\widetilde{\Phi}(z)^{-1}x(u) = x(u) + zx(1)/(1-z)$, which is well defined for any $z \neq 1$ and $u \in [0,1]$. Moreover, since $\widetilde{\Phi}_0 = I - \phi_1$, we know from Section 2.1 and Remark 2.3 that $\ker\widetilde{\Phi}_0 = \mathrm{C}_0$ and $\operatorname{ran}\widetilde{\Phi}_0 \subset \mathrm{C}_1$. $\operatorname{ran}\widetilde{\Phi}_0 \supset \mathrm{C}_1$ follows from that $x = \widetilde{\Phi}_0 x$ holds for any $x \in \mathrm{C}_1$.

Example 3.3: We show that Assumption 3.1(a) is satisfied in this example. Note that $\widetilde{\Phi}(z)a = ((1-z)a_1, (1-z)a_2 - za_1, (1-z\lambda)a_3, (1-z\lambda^2)a_4, \ldots)$ for $a = (a_1, a_2, \ldots) \in \mathbf{c_0}$, from which it can be easily shown that $\widetilde{\Phi}(z)$ is injective on $\mathbf{c_0}$ for any $z \in D_{1+\eta} \setminus \{1\}$. Furthermore, for any sequence $b = (b_1, b_2, b_3, \ldots) \in \mathbf{c_0}$, we can find a sequence $a =$

$(a_1, a_2, a_3, \ldots) \in \mathbf{c_0}$ satisfying $\widetilde{\Phi}(z)a = b$ by setting $a_1 = b_1/(1-z)$, $a_2 = b_2/(1-z) + zb_1/(1-z)^2$, and $a_j = b_j/(1 - z\lambda^{j-2})$, for $j \geq 3$. This shows that $\widetilde{\Phi}(z)$ is also a surjection for $z \in D_{1+\eta} \setminus \{1\}$. Therefore, $\widetilde{\Phi}(z)$ is invertible on $D_{1+\eta} \setminus \{1\}$.

Example 3.5: $\mathcal{K} \neq \{0\}$ is obvious. With some algebra, it can be shown that (i) $\widetilde{\Phi}(z)$ is invertible for $z \neq 1$, (ii) $\operatorname{ran} \widetilde{\Phi}_0$ and $\ker \widetilde{\Phi}_0$ can be complemented, and (iii) $\mathcal{W}_{\mathsf{C}} = \{(b_1, 0, b_2, b_3, 0, b_4, b_5, 0, b_6, \ldots) : b_j \in \mathbb{C}, \lim_{j\to\infty} b_j = 0\}$ is a complementary subspace of $\ker \widetilde{\Phi}_0$. Note that $-\widetilde{\Phi}_0(b_1, 0, b_2, b_3, 0, b_4, b_5, \ldots) = (0, b_1, b_1, 0, b_3, b_3, \ldots)$ holds. Since $(b_1, 0, b_2, b_3, 0, b_4, \ldots) \in \mathcal{W}_{\mathsf{C}}$ and $\widetilde{\Phi}_0^g \widetilde{\Phi}_0 = \mathrm{P}_{\mathcal{W}_{\mathsf{C}}}$, we find that

$$-(b_1, 0, b_2, b_3, 0, b_4, b_5, \ldots) = \widetilde{\Phi}_0^g(0, b_1, b_1, 0, b_3, b_3, \ldots). \tag{B.53}$$

Equations (3.32), (3.33), and (B.53) imply that $\operatorname{ran} \widetilde{\Phi}_0 + \ker \widetilde{\Phi}_0 = \ker \widetilde{\Phi}_0$ and $\widetilde{\Phi}_0^g \mathcal{K} = \mathcal{W}_{\mathsf{C}}$, so (3.27) holds as desired.

Remark 3.8: We here verify (3.22). From (3.21), we find that $\operatorname{ran} \Upsilon_{-1} = \ker \Phi_0$. Therefore, for a nonzero $f \in \operatorname{Ann}(\ker \Phi_0)$, $f(X_t) = f(\tau_0) + f(\nu_t)$ holds for $t \geq 0$. Let τ_0 satisfy $f(\tau_0) = 0$. We know from Proposition 3.2 that $\Phi(z)^{-1} = -\ddot{\mathrm{N}}_{-1}(z-1)^{-1} - \sum_{j=0}^{\infty} \ddot{\mathrm{N}}_j (z-1)^j$, $H(z)$ is convergent on $D_{1+\eta}$ for $\eta > 0$ (and thus the coefficients of the Maclaurin series of $H(z)$ decay exponentially in norm) and $\ddot{\mathrm{N}}_0 = -\sum_{j=0}^{\infty} \Pi_p H_j \Pi_p^*$. We thus only need to show that $f\ddot{\mathrm{N}}_0 \neq 0$ to establish I(0)-ness of $\{f(\nu_t)\}_{t\geq 0}$. Under the I(1) condition, we know that $\ddot{\mathrm{N}}_{-1}\Phi_1 + \ddot{\mathrm{N}}_0 \Phi_0 = -I$, which implies that $\operatorname{ran} \ddot{\mathrm{N}}_{-1} + \operatorname{ran} \ddot{\mathrm{N}}_0 = \mathcal{B}$. In this case, for any $f \in \ker \Phi_0$, $f\ddot{\mathrm{N}}_0 = 0$ implies that $f = 0$, which contradicts our assumption that $f \neq 0$.

Remark 3.14: We here prove (3.40) and (3.41). Since $\operatorname{ran} \Upsilon_{-2} = \mathsf{K}$, $f \in \operatorname{Ann}(\mathsf{K})$ is a cointegrating functional. For any nonzero $f \in \operatorname{Ann}(\Upsilon_{-2})$, we may deduce from the formula of Υ_{-1} given in Proposition 3.7 that $f\Upsilon_{-1}$ is equal to

$$-f\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2}) + f\Phi_0^g \Phi_1 \Upsilon_{-2}. \tag{B.54}$$

Using the expression of $f\Upsilon_{-1}$ given by (B.54), we will show that the following holds:

$$f\Upsilon_{-1} = 0 \quad \Leftrightarrow \quad f \in \operatorname{Ann}(\Phi_0^g \Phi_1 \mathsf{K}) \cap \operatorname{Ann}(\ker \Phi_0). \tag{B.55}$$

Since $\operatorname{ran} \Upsilon_{-2} = \mathsf{K}$, the second term in (B.54) is zero if and only if $f \in \operatorname{Ann}(\Phi_0^g \Phi_1 \mathsf{K})$. It thus only remains to show that $f \in \operatorname{Ann}(\mathsf{K}_{\mathsf{C}})$ because $\ker \Phi_0 = \mathsf{K} \oplus \mathsf{K}_{\mathsf{C}}$ and $f \in \operatorname{Ann}(\mathsf{K})$. To see this, we first show that $\operatorname{ran}(\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})) = \mathsf{K}_{\mathsf{C}}$. Since $\operatorname{ran} \mathrm{M}_1^g = \mathsf{K}_{\mathsf{C}}$, $\operatorname{ran}(\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})) \subset \mathsf{K}_{\mathsf{C}}$. Note that for any $V \subset \mathcal{B}$, $\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})V \subset \mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})\mathcal{B} \subset \mathsf{K}_{\mathsf{C}}$ holds. Thus, if there is a subset V such that $\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})V = \mathsf{K}_{\mathsf{C}}$, then $\operatorname{ran}(\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})) = \mathsf{K}_{\mathsf{C}}$ holds. Let $V = (I - \mathrm{P}_{\mathsf{R}_{\mathsf{C}}})V_{\mathsf{C}}$. We then know from (3.39) that $\mathrm{M}_2\Upsilon_{-2}V = \{0\}$. Moreover, $\operatorname{ran} \mathrm{M}_1^g = \mathrm{M}_1^g V$ holds since $\ker \mathrm{M}_1^g = \operatorname{ran} \Phi_0 \oplus \mathsf{R}_{\mathsf{C}}$. From these results, we find that $\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})V = \mathrm{M}_1^g V = \mathsf{K}_{\mathsf{C}}$, so $\operatorname{ran}(\mathrm{M}_1^g(I - \mathrm{M}_2\Upsilon_{-2})) = \mathsf{K}_{\mathsf{C}}$. We thus conclude that $f \in \operatorname{Ann}(\mathsf{K}_{\mathsf{C}})$ if and only if the first term in (B.54) is zero.

We have shown that (B.55) holds for $f \in \operatorname{Ann}(\mathsf{K})$. We know from Proposition 3.7 that (ignoring τ_0 and τ_1 without loss of generality) a nonzero element $f \in \operatorname{Ann}(\mathsf{K})$ satisfies either of the following: (i) $f\Upsilon_{-1} \neq 0$ and $f(X_t) = f\Upsilon_{-1}\left(\sum_{s=1}^t \varepsilon_s\right) + f(\nu_t)$ or (ii) $f(X_t) = f(\nu_t)$. In case (i), $\{f(X_t)\}_{t\geq 0}$ is I(1) obviously. In case (ii), $f(X_t)$ is I(0) under our I(2) condition. To see this, note that we know from Proposition 3.5 that $\Phi(z)^{-1} = -\ddot{\mathrm{N}}_{-2}(z-1)^{-2} - \ddot{\mathrm{N}}_{-1}(z-1)^{-1} - \sum_{j=0}^{\infty} \ddot{\mathrm{N}}_j (z-1)^j$, $H(z)$ is convergent on $D_{1+\eta}$ for $\eta > 0$ (and thus the coefficients

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of the Maclaurin series of $H(z)$ decay exponentially in norm), and $\ddot{\mathrm{N}}_0 = -\sum_{j=0}^{\infty} \Pi_p H_j \Pi_p^*$. As in Appendix B.4, it suffices to show that $f\ddot{\mathrm{N}}_0 \neq 0$ to establish the desired I(0)-ness. Under the I(2) condition, we know from (B.36) that $\operatorname{ran}\ddot{\mathrm{N}}_{-2} + \operatorname{ran}\ddot{\mathrm{N}}_{-1} + \operatorname{ran}\ddot{\mathrm{N}}_0 = I$. Since $f\ddot{\mathrm{N}}_{-2} = f\ddot{\mathrm{N}}_{-1} = 0$ in case (ii), $f\ddot{\mathrm{N}}_0 = 0$ implies $f = 0$, which contradicts our assumption that $f \neq 0$. We thus find that $f\ddot{\mathrm{N}}_0 \neq 0$, so the cointegrating behavior of I(2) solutions is characterized as stated.

Remark 3.15: For a nonzero $f \in \operatorname{Ann}(\mathsf{K})$, we deduce from (B.54) that $f(X_t) - f(\Phi_0^g \Phi_1 \Delta X_t)$ is given by

$$-f\left(\mathrm{M}_1^g (I - \mathrm{M}_2 \Upsilon_{-2}) \sum_{s=1}^{t} \varepsilon_s\right) + f(\nu_t^\star), \tag{B.56}$$

where $\nu_t^\star = \nu_t - \Phi_0^g \Phi_1 \varepsilon_t + \Delta \nu_t$. As shown above, $\operatorname{ran}(\mathrm{M}_1^g (I - \mathrm{M}_2 \Upsilon_{-2})) = \mathsf{K}_{\mathsf{C}}$. Since $\ker \Phi_0 = \mathsf{K} \oplus \mathsf{K}_{\mathsf{C}}$ and $f \in \operatorname{Ann}(\mathsf{K})$, $f \notin \operatorname{Ann}(\ker \Phi_0)$ implies that $f \notin \operatorname{Ann}(\mathsf{K}_{\mathsf{C}})$. Thus, the sequence given in (B.56) cannot be stationary for $f \notin \operatorname{Ann}(\ker \Phi_0)$. On the other hand, if $f \in \operatorname{Ann}(\ker \Phi_0)$, then the first term in (B.56) is zero. Therefore, the desired result is established if $\{f(\nu_t^\star)\}_{t \geq 0}$ is I(0). The summability condition for the I(0) property is satisfied, which can be easily shown. We rewrite $\nu_t^\star$ as $\nu_t^\star = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}$, and find that $\sum_{j=0}^{\infty} \Psi_j = -\ddot{\mathrm{N}}_0 - \Phi_0^g \Phi_1 \ddot{\mathrm{N}}_{-1}$. If $f \sum_{j=0}^{\infty} \Psi_j$ is nonzero, then $\{f(\nu_t^\star)\}_{t \geq 0}$ is I(0). Suppose by contradiction that $f\ddot{\mathrm{N}}_0 = -f\Phi_0^g \Phi_1 \ddot{\mathrm{N}}_{-1}$. Precomposing (B.36) with $f\Phi_0^g$ and using the fact that $\Phi_0^g \Phi_0 = \mathrm{P}_{W_{\mathsf{C}}}$, we obtain

$$f\Phi_0^g \Phi_2 \ddot{\mathrm{N}}_{-2} + f\Phi_0^g \Phi_1 \ddot{\mathrm{N}}_{-1} + f\mathrm{P}_{W_{\mathsf{C}}} \ddot{\mathrm{N}}_0 = f\Phi_0^g \Phi_2 \ddot{\mathrm{N}}_{-2} - f\ddot{\mathrm{N}}_0 + f\mathrm{P}_{W_{\mathsf{C}}} \ddot{\mathrm{N}}_0 = -f\Phi_0^g.$$

In the above, $f\mathrm{P}_{W_{\mathsf{C}}} \ddot{\mathrm{N}}_0 = f\ddot{\mathrm{N}}_0$ since $f(I - \mathrm{P}_{W_{\mathsf{C}}}) = 0$ for any $f \in \operatorname{Ann}(\ker \Phi_0)$. We thus find that $f\Phi_0^g \Phi_2 \ddot{\mathrm{N}}_{-2} = -f\Phi_0^g$. From our expression of $\ddot{\mathrm{N}}_{-2} = -\Upsilon_{-2}$ given in (3.39), we know that $\ddot{\mathrm{N}}_{-2} \Phi_0 = 0$, so

$$0 = f\Phi_0^g \Phi_2 \ddot{\mathrm{N}}_{-2} x = -f\Phi_0^g x, \quad \text{for all } x \in \operatorname{ran} \Phi_0. \tag{B.57}$$

From the properties of Φ_0^g, we have $\Phi_0^g \operatorname{ran} \Phi_0 = W_{\mathsf{C}}$, so (B.57) holds if and only if $f \in \operatorname{Ann}(W_{\mathsf{C}})$. Since $f \in \operatorname{Ann}(\ker \Phi_0)$ and $\mathcal{B} = \ker \Phi_0 \oplus W_{\mathsf{C}}$, this means $f = 0$, which contradicts the assumption that $f \neq 0$.

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