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THE SLICE-BENNEQUIN INEQUALITY FOR THE FRACTIONAL DEHN TWIST COEFFICIENT

PETER FELLER
ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
(peter.feller@math.ch)

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Abstract We characterize the fractional Dehn twist coefficient (FDTC) on the n-stranded braid group as the unique homogeneous quasimorphism to $\mathbb{R}$ of defect at most 1 that equals 1 on the positive full twist and vanishes on the $(n-1)$-stranded braid subgroup. In a different direction, we establish that the slice-Bennequin inequality holds with the FDTC in place of the writhe. In other words, we establish an affine linear lower bound for the smooth slice genus of the closure of a braid in terms of the braid's FDTC. We also discuss connections between these two seemingly unrelated results. In the appendix, we provide a unifying framework for the slice-Bennequin inequality and its counterpart for the FDTC.

1. Introduction

A *quasimorphism* on a group G is a function f from G to the real numbers $\mathbb{R}$ such that $\sup_{a,b\in G}|f(ab)-f(a)-f(b)|<\infty$, where $\sup_{a,b\in G}|f(ab)-f(a)-f(b)|$ is called the *defect* of f and is denoted by D_f. A function $f\colon G\to\mathbb{R}$ is said to be *homogeneous* if $f(g^k)=kf(g)$ for all $g\in G$ and integers k. In this article, we focus on the *fractional Dehn twist coefficient (FDTC)*, a certain homogeneous quasimorphism on the braid group on n strands. The FDTC appears in several contexts concerning different aspects of low-dimensional topology; see, for example, Gabai–Oertel, Malyutin and Honda–Kazez–Matić [GO89, Mal04, HKM07, HKM08].

A characterization of the FDTC as the homogeneous quasimorphism of smallest defect

For a fixed integer $n\geq 1$, we denote by

$$B_n=\langle a_1,\dots,a_{n-1}\mid a_ia_j=a_ja_i \text{ for } |i-j|\geq 2, a_ia_{i+1}a_i=a_{i+1}a_ia_{i+1}\rangle,$$

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Artin's braid group [Art25]. For the entire text, we identify $B_{n-1} \subset B_n$ as a subgroup via the inclusion $\iota\colon B_{n-1} \to B_n, a_i \mapsto a_{i+1}$ whenever $n \geq 2$. We delay an explicit definition of the FDTC. However, we recall that the FDTC, denoted by $\omega\colon B_n \to \mathbb{R}$, is known to be a homogeneous quasimorphism of defect ≤ 1 (in fact it is know to have defect 1 when $n \geq 3$; compare Lemma 10) that satisfies $\omega(B_{n-1}) = \{0\}$ and $\omega\left(\Delta^2\right) = 1$; see [Mal04]. Here, Δ^2 denotes $(a_1a_2\cdots a_{n-1})^n \in B_n$, which is known as the *positive full twist* and, for $n \geq 3$, generates the center of B_n. We establish that these properties characterize the FDTC.

Theorem 1. *For every integer $n \geq 3$, there exists a unique homogeneous quasimorphism $\omega\colon B_n \to \mathbb{R}$ with defect at most 1 that satisfies the following properties:*

(i) $\omega\left(\Delta^2\right) = 1$ *and*

(ii) $\omega(\beta) = 0$ *for all* $\beta \in B_{n-1} \subset B_n$.

We exclude considerations for $n = 2$ (and $n = 1$) as it is clear that there is at most one homogeneous quasimorphism on the infinite cyclic group B_2 (and the trivial group B_1) that sends Δ^2 to a given value since homogeneous quasimorphisms on Abelian groups are group homomorphisms. In contrast to this and to Theorem 1, for $n \geq 3$ there are many homogeneous quasimorphisms on B_n.

Proposition 2. *For every integer $n \geq 3$ and every $\varepsilon > 0$, there exist continuum-many linearly independent homogeneous quasimorphisms $f\colon B_n \to \mathbb{R}$ with defect at most $1+\varepsilon$ that satisfy (i) $f\left(\Delta^2\right) = 1$ and (ii) $f(B_{n-1}) = \{0\}$.*

We briefly comment on the two assumptions (i) and (ii).

Equation (i) can be understood as a normalization condition. In other words, Theorem 1 says that the homogeneous quasimorphisms $f\colon B_n \to \mathbb{R}$ that satisfy

$$D_f \leq \left|f(\Delta^2)\right| \text{ and } f(B_{n-1}) = \{0\}$$

form a one-dimensional $\mathbb{R}$-subspace of the vector space of functions from B_n to $\mathbb{R}$, while Proposition 2 says that the $\mathbb{R}$-subspace generated by homogeneous quasimorphisms $f\colon B_n \to \mathbb{R}$ that satisfy

$$D_f \leq (1+\varepsilon)\left|f(\Delta^2)\right| \text{ and } f(B_{n-1}) = \{0\}$$

has uncountably infinite dimension.

Every homogeneous quasimorphism $f\colon B_n \to \mathbb{R}$ can be written as the sum of a homogeneous quasimorphism that satisfy Equation (ii) and a homogeneous quasimorphism that is determined by the homogeneous quasimorphism $B_{n-1} \to \mathbb{R}, \beta \mapsto f(\iota(\beta))$ [Mal09, Theorem 2]. So, informally speaking, understanding homogeneous quasimorphisms on B_n amounts to understanding homogeneous quasimorphisms that satisfy Equation (ii) on B_n and homogeneous quasimorphisms on B_{n-1}.

The slice-Bennequin inequality for the FDTC

For a link L – a non-empty oriented closed smooth 1-submanifold of the 3-sphere S^3 – denote by $\chi_4(L)$ the largest integer among the Euler characteristics of smooth oriented

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surfaces in the 4-ball B^4 without closed components and oriented boundary $L \subset \partial B^4 = S^3$. In particular, for a knot K – a connected link – one has $2g_4(K) = 1 - \chi_4(K)$, where g_4 denotes the *slice genus*. The *slice-Bennequin inequality* states that

$$|\mathrm{wr}(\beta)| \leq -\chi_4\left(\widehat{\beta}\right) + n \quad \text{for all } \beta \in B_n \quad \text{[Rud93, KM93]}, \tag{1}$$

where $\mathrm{wr}\colon B_n \to \mathbb{Z}$ denotes the *writhe*, the group homomorphism with $\mathrm{wr}(a_i) = 1$, and $\widehat{\beta}$ denotes the link obtained as the closure of β. For β with closure a knot, Equation (1) reads $|\mathrm{wr}(\beta)| \leq 2g_4\left(\widehat{\beta}\right) + n - 1$. As before, we only consider $n \geq 3$ as $n \leq 2$ yields no new insight.

One may wonder which other maps $f\colon B_n \to \mathbb{R}$ satisfy a similar inequality. Concretely, [HKK+21, Question 1.6] asks whether, for each $n \geq 3$, there exist constants $A(n)$ and $C(n)$ such that $\omega(\beta) \leq A(n) g_4\left(\widehat{\beta}\right) + C(n)$ for all $\beta \in B_n$ with closure a knot. We answer affirmatively with $A(n)$ independent of n; concretely, $A(n) = 2$, which is optimal (e.g., by the examples from [HKK+21, Prop. 4.7]).

Theorem 3. *For all integers $n \geq 3$, we have that the FDTC $\omega\colon B_n \to \mathbb{R}$ satisfies*

$$|\omega(\beta)| \leq -\chi_4\left(\widehat{\beta}\right) + n \text{ for all } \beta \in B_n.$$

Theorem 3 provides the affirmative answer claimed above since it reads $|\omega(\beta)| \leq 2g_4\left(\widehat{\beta}\right) + n - 1$ for all $\beta \in B_n$ such that $\widehat{\beta}$ is a knot.

We provide more context for Theorem 3 in Section 2. The main input in the proof of Theorem 3 is that the FDTC can be expressed in terms of the so-called homogenization of an instance of upsilon. Here, upsilon is a knot invariant, introduced in [OSS17], that has being a lower bound for the slice genus as a key feature; see Section 5.

Comparing bounds on the defect and affine linear bounds for the slice genus

At this point, the reader may have wondered why Theorem 1 and Theorem 3 appear in the same text, given they are concerned with different aspects of the FDTC. A link between these aspects is provided in Proposition 5 below.

We ask whether satisfying the slice-Bennequin inequality as described in Theorem 3 characterizes the FDTC in the same way that having defect 1 does characterize the FDTC by Theorem 1.

Question 4. *Fix an integer $n \geq 3$. Is the FDTC the unique homogeneous quasimorphism $\omega\colon B_n \to \mathbb{R}$ that satisfies $\omega\left(\Delta^2\right) = 1$, $\omega(\beta) = 0$ for all $\beta \in B_{n-1} \subset B_n$ and for which there exists a constant C such that*

$$|\omega(\beta)| \leq 2g_4\left(\widehat{\beta}\right) + C \text{ for all } \beta \in B_n \text{ for which } \widehat{\beta} \text{ is a knot?}$$

While we are unable to answer this question, we provide the following connection between the defect of a quasimorphism f and the possible slopes of affine linear bounds for $g_4\left(\widehat{\beta}\right)$ in terms of $f(\beta)$.

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Proposition 5. *Fix $n \geq 3$, and let $f\colon B_n \to \mathbb{R}$ be a homogeneous quasimorphism. If there exist constants $A, C \in \mathbb{R}$ such that*

$$|f(\beta)| \leq A g_4\left(\widehat{\beta}\right) + C \text{ for all } \beta \in B_n \text{ with closure a knot,}$$

then the defect D_f of f satisfies $D_f \leq \frac{A}{2}(n-1)$.

In a first version of this article, the conclusion of Proposition 5 stated $D_f \leq A(n-1)$. The factor $1/2$ improvement was pointed out by Tetsuya Ito.

Proposition 5 can be understood as a first step towards affirmatively answering Question 4. Concretely, we have the following corollary.

Corollary 6. *Fix $n \geq 3$, and let $f\colon B_n \to \mathbb{R}$ be a homogeneous quasimorphism. that satisfies $f(B_{n-1}) = \{0\}$ and $f(\Delta^2) = 1$. If there exist constants $A, C \in \mathbb{R}$ such that $|f(\beta)| \leq A g_4\left(\widehat{\beta}\right) + C$ for all $\beta \in B_n$ with closure a knot, then $A > \frac{2}{n-1}$.*

Proof of Corollary 6. Assume towards a contradiction that $A \leq \frac{2}{n-1}$. Then $D_f \leq \frac{A}{2}(n-1) \leq 1$ by Proposition 5, hence $f = \omega$ by Theorem 1. However, for $f = \omega$, we have $A \geq 2$, for example, by the examples from [HKK+21, Prop. 4.7]. □

We do not expect Proposition 5 to be optimal. If the inequality for D_f in Proposition 5 can be strengthened, concretely, if the next question can be answered affirmatively, then Theorem 1 implies that Question 4 can be answered affirmatively (by arguing as in the proof of Corollary 6).

Question 7. *Fix $n \geq 3$, and let $f\colon B_n \to \mathbb{R}$ be a homogeneous quasimorphism that satisfies $f(B_{n-1}) = \{0\}$. If there exist constants $A, C \in \mathbb{R}$ such that*

$$|f(\beta)| \leq A g_4\left(\widehat{\beta}\right) + C \text{ for all } \beta \in B_n \text{ with closure a knot,}$$

does the defect D_f of f satisfy $D_f \leq A/2$?

Ingredients for the proofs and structure of the paper

In Section 2, we provide context for Theorem 3.

In Section 3, we establish Theorem 1. The main step in the proof of Theorem 1 is to show that every braid $\beta \in B_n$ that can be written as a braid word that contains at most l occurrences of a_1 and no a_1^{-1} can be decomposed as a particular product of full twists Δ^2 and at most l braids that are conjugate to braids in $B_{n-1} \subset B_n$.

In Section 4, we show that Proposition 2 is a rather immediate consequence of the work on group actions on δ-hyperbolic spaces that satisfy weak proper discontinuity (WPD) [BF02].

In Section 5, we establish Theorem 3. A key ingredient is the reinterpretation of the FDTC as a linear combination of the writhe and the homogenization of $\Upsilon(t)$ [FH19, Theorem 1.3]. Additionally, we use some facts about concordances between braid closures. The latter is also what we use in the proof of Proposition 5.

We conclude the paper with a perspective that allows to view both the slice-Bennequin (1) and Theorem 3 as instances of an observation concerning the homogenization of 1-Lipschitz concordance homomorphisms; see Appendix A.

2. Context for Theorem 3: Bennequin and slice-Bennequin inequalities

For simplicity of exposition, in this section all braids β are assumed to have a knot as their closure. All that is said translates to the general setup if g_k is replaced by $-(\chi_k - 1)/2$ for $k \in \{3,4\}$.

The Bennequin inequality

The slice-Bennequin inequality

$$\mathrm{wr}(\beta) \leq 2g_4\left(\widehat{\beta}\right) + n - 1 \text{ for all } \beta \in B_n \text{ [Rud93, KM93]},$$

was predated by the Bennequin inequality

$$\mathrm{wr}(\beta) \leq 2g_3\left(\widehat{\beta}\right) + n - 1 \text{ for all } \beta \in B_n \text{ [Ben83]},$$

where g_3 denotes the smallest genus among smooth surface in S^3 with boundary the knot.

There is a conceptual gap between these two results. The slice-Bennequin inequality needed a strong input from smooth 4-manifold theory: the so-called local-Thom conjecture as proven by Kronheimer and Mrowka [KM93, Corollary 1.3]. The 'smooth' is crucial here. Indeed, the analog statement in the locally flat setting (where g_4 is replaced with the topological slice genus) is well known to fail in many instances; see, for example, [Rud93, Rud84, BFLL18].

Concerning the FDTC, we point to a version of the Bennequin inequality by Ito [Ito11, Theorem 1.2]: $\omega(\beta) < 2g_3\left(\widehat{\beta}\right)\frac{2}{n+2} - \frac{2}{n+2} + \frac{3}{2}$ for all $\beta \in B_n$.[1] Meaning that the Bennequin inequality holds with a slope $A(n)$ for which it is known not to hold when g_3 is replaced by g_4.

The slice-Bennequin inequality for (quasi-)positive braids

Recall that the key input for Rudolph's proof of the slice-Bennequin inequality (1) is that it holds (in fact with equality) for positive braids with closure a torus knot (by the local Thom conjecture [KM93, Corollary 1.3]) and hence, as observed by Rudolph, also for positive (actually also quasipositive) braids. Then Equation (1) follows using that, for all $\beta \in B_n$ and generators a_i, $\mathrm{wr}(\beta a_i) - \mathrm{wr}(\beta) \geq 1$ and there exists a cobordism with Euler characteristic -1 between the closures of βa_i and β. For the FDTC, combining $\omega(\beta) \leq \mathrm{wr}(\beta) - 1$ for quasipositive braids $\beta \neq 1$ (this follows readily by using that $\omega(a_i) = 0$, which is implied by $\omega(B_{n-1}) = \{0\}$, and that ω is homogeneous quasimorphism of defect

[1] We note that [Ito11, Theorem 1.2] is phrased for the Dehornoy floor (compare Section 3). However, since the argument only uses properties of the Dehornoy floor that are also satisfied by the FDTC, it translates to a statement about the FDTC.

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at most 1) with Equation (1) yields

$$\omega(\beta) \leq -\chi_4\left(\widehat{\beta}\right) + n - 1 \quad \text{for all quasipositive } \beta \neq 1 \text{ in } B_n; \tag{2}$$

see [HKK+21, Theorem 1.5]. However, the strategy of reducing to the statement for positive (or quasipositive) braids cannot be carried over to establish the slice-Bennequin inequality for the FDTC since $\omega(\beta a_i) - \omega(\beta) \geq 1$ does not hold in general. Also, there is no bound on $\omega(\beta)$ in terms of an expression depending on $\mathrm{wr}(\beta)$ that holds for all braids $\beta \in B_n$.

Does a version of the slice-Bennequin inequality hold for all quasimorphisms?

In light of the writhe and FDTC satisfying the slice-Bennequin inequality, one might wonder whether, for each braid group B_n with $n \geq 3$, there exists a quasimorphism $f\colon B_n \to \mathbb{R}$ which does not satisfy a version of the slice-Bennequin inequality. We state this precisely.

Question 8. *Fix $n \geq 3$. Does there exist a quasimorphism $f\colon B_n \to \mathbb{R}$ such that, for every $A, C \in \mathbb{R}$, there exists a $\beta \in B_n$ with closure a knot such that $|f(\beta)| > Ag_4\left(\widehat{\beta}\right) + C$?*

We note that, since every quasimorphism is at bounded distance of a homogeneous one, asking the question for homogeneous quasimorphisms or quasimorphisms amounts to the same.

We suspect that many (if not most) quasimorphisms do not satisfy a version of the slice-Bennequin inequality; why, after all, would there be such a connection between quasimorphisms and concordance? For example, we suspect that many Brooks-like quasimorphisms as provided in [BF02] (a construction that, as far as we know, is devoid of connections to concordance) are quasimorphisms as asked for in Question 8. However, we are not able to confirm this at this point.

To answer Question 8 in the positive, the reader might be tempted (the author certainly was) to construct a homogeneous quasimorphism $f\colon B_n \to \mathbb{R}$ that is nonzero on some braid β with the property that $\lim_{k\to\infty} \frac{\chi_4\left(\widehat{\beta^k}\right)}{k} = 0$. Indeed, one readily checks that such an f is as asked for in Question 8. However, the only braids we are able to find with the property $\lim_{k\to\infty} \frac{\chi_4\left(\widehat{\beta^k}\right)}{k} = 0$ are braids that are (up to taking conjugates) of the form $\alpha\overline{\alpha^{-1}}$ for some braid α, and a small calculation reveals that, for all $\alpha \in B_n$, all homogeneous quasimorphisms $f\colon B_n \to \mathbb{R}$ vanish on $\alpha\overline{\alpha^{-1}}$. (Here, $\overline{\gamma} \in B_n$ is the result of changing all $a_i^{\pm 1}$ in a braid word for $\gamma \in B_n$ to $a_{n-i}^{\pm 1}$.)

3. The proof of Theorem 1

Definition of the FDTC via the Dehornoy order

We fix an integer $n \geq 2$. A braid β is said to be *Dehornoy positive*, denoted by $\beta \succ_{\mathrm{Deh}} 1$, if it can be written as a braid word that, for some integer $1 \leq i < n$, contains a braid generator a_i but no a_i^{-1} or any generators $a_j^{\pm 1}$ for $j < i$. We write $\beta \succeq_{\mathrm{Deh}} 1$ if $\beta \succ_{\mathrm{Deh}} 1$ or $\beta = 1$. Dehornoy showed that this gives a well-defined left-invariant total order

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$\succeq_{\text{Deh}}$ on B_n by setting $\beta \succeq_{\text{Deh}} \alpha$ to mean $\alpha^{-1}\beta \succeq_{\text{Deh}} 1$ [Deh94]. The *Dehornoy floor* $\lfloor \beta \rfloor$ is the unique integer m such that $(\Delta^2)^{m+1} \succ_{\text{Deh}} \beta \succeq_{\text{Deh}} (\Delta^2)^m$. For any $\beta \in B_n$, its *fractional Dehn twist coefficient* is $\omega(\beta) \coloneqq \lim_{k\to\infty} \frac{\lfloor \beta^k \rfloor}{k}$; see [Mal04]. In other words, ω equals the homogenization of the Dehornoy floor. We refer to [Mal04] for more details on this approach to the FDTC and how one derives its properties (e.g., being a homogeneous quasimorphism with defect at most 1).

Remark 9. It is essentially immediate from this definition that, if $\omega(\beta) > 0$, then β can be written as a braid word with at least one a_1 and no a_1^{-1} (and, in particular, $\beta \succ 1$). Indeed, if β can be written as a braid word without a_1 or a_1^{-1}, then $(\Delta^2) \succ_{\text{Deh}} \beta \succ_{\text{Deh}} (\Delta^2)^{-1}$. Hence, $1 \geq \lfloor \beta^k \rfloor \geq -1$, which implies $\omega(\beta) = 0$. Therefore, β can be written as braid word that either contains only a_1 or only a_1^{-1}. If it were the latter, then $1 \succ_{\text{Deh}} \beta^k$, hence $0 \geq \lfloor \beta^k \rfloor$, which implies $0 \geq \omega(\beta)$.

Proof of Theorem 1

The proof below uses relations in the braid group, general properties of homogeneous quasimorphisms and the property of the FDTC discussed in Remark 9.

Proof of Theorem 1. We fix $n \geq 3$ and let $\omega_* \colon B_n \to \mathbb{R}$ be any homogeneous quasimorphism that satisfies the assumptions.

Assume towards a contradiction that $\omega_* \neq \omega$. Pick $\beta_{\text{w}} \in B_n$ with $\omega(\beta_{\text{w}}) - \omega_*(\beta_{\text{w}}) \neq 0$. Since ω_* and ω are homogeneous, there exists $k_1 \in \mathbb{Z}$ such that

$$\omega\left(\beta_{\text{w}}^{k_1}\right) - \omega_*\left(\beta_{\text{w}}^{k_1}\right) = k_1(\omega(\beta_{\text{w}}) - \omega_*(\beta_{\text{w}})) > 1.$$

Since $f(ab) = f(a) + f(b)$ for all homogeneous quasimorphisms $f \colon G \to \mathbb{R}$ and commuting $a,b \in G$, and since $\omega\left(\Delta^2\right) = \omega_*\left(\Delta^2\right) = 1$, there exists $k_2 \in \mathbb{Z}$ such that

$$\omega\left(\beta_{\text{w}}^{k_1}(\Delta^2)^{k_2}\right) = \omega\left(\beta_{\text{w}}^{k_1}\right) + k_2 > 0 > \omega_*(\beta_{\text{w}}^{k_1}) + k_2 = \omega_*\left(\beta_{\text{w}}^{k_1}(\Delta^2)^{k_2}\right).$$

We define $\beta \coloneqq \beta_{\text{w}}^{k_1}(\Delta^2)^{k_2}$. Since $\omega(\beta) > 0$, we have that β can be given by a braid word with no occurrences of a_1^{-1} but at least one a_1 by Remark 9.

We proceed by showing $\omega_*(\beta) \geq 0$, which contradicts $\omega_*(\beta) < 0$. We may and do assume that the number of occurrences of a_1 in the braid word without a_1^{-1} we picked for β is even. (Indeed, otherwise we consider $\beta\beta$, which also satisfies $0 > \omega_*(\beta\beta)$ since $\omega_*(\beta\beta) = \omega_*(\beta) + \omega_*(\beta)$.) Hence, we have that

$$\beta = \prod_{i=1}^{2l}(a_1\beta_i),$$

where l is a positive integer and the β_i are (possibly trivial) n-braids in $B_{n-1} \subset B_n$.

Next, we observe that β maybe conjugated to a braid of the form $\Delta^{2l}\prod_{i=1}^{l} L_iR_i$, where R_i is an n-braid given by a braid word without $a_1^{\pm 1}$ (in other words an element of $B_{n-1} \subset B_n$) and L_i an n-braid given by a braid word without $a_{n-1}^{\pm 1}$. To describe L_i and R_i, we

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consider the element

$$\Delta \coloneqq \prod_{i=1}^{n-1} a_1 a_2 \cdots a_{n-i} = \prod_{i=1}^{n-1} a_{n-1} a_{n-2} \cdots a_{i+1} a_i \in B_n,$$

known as the (positive) half-twist since $\Delta\Delta = \Delta^2$. We denote by $\Delta_L \in B_n$ and $\Delta_R \in B_n$ the half-twist on the first $n-1$ strands and the last $n-1$ strands, respectively. In other words, Δ_R is the image of the half twist $\Delta \in B_{n-1}$ under the inclusion $\iota\colon B_{n-1} \to B_n, a_i \mapsto a_{i+1}$, while Δ_L is the image of the half twist $\Delta \in B_{n-1}$ under the inclusion $a_i \mapsto a_i$. We also denote by $\overline{\beta}$ the braid obtained from β by replacing $a_i^{\pm 1}$ with $a_{n-i}^{\pm 1}$ and recall that $\Delta^{\pm 1}\beta = \overline{\beta}\Delta^{\pm 1}$. With this we see

$$\begin{aligned} a_1\beta_{2i-1}a_1\beta_{2i} &= \Delta^2\Delta^{-2}a_1\beta_{2i-1}a_1\beta_{2i} \\ &= \Delta^2 \underbrace{\Delta^{-1}}_{\Delta_R^{-1}(a_1^{-1}a_2^{-1}\cdots a_{n-2}^{-1}a_{n-1}^{-1})} a_{n-1}\overline{\beta_{2i-1}}\overbrace{\Delta^{-1}}^{\Delta_L^{-1}(a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}a_1^{-1})} a_1\beta_{2i} \\ &= \Delta^2\Delta_R^{-1}\left(a_1^{-1}a_2^{-1}\cdots a_{n-2}^{-1}\right)\overline{\beta_{2i-1}}\Delta_L^{-1}\left(a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}\right)\beta_{2i}. \end{aligned}$$

Hence,

$$\begin{aligned} \beta &= \prod_{i=1}^{l} \Delta^2\Delta_R^{-1}\left(a_1^{-1}a_2^{-1}\cdots a_{n-2}^{-1}\right)\overline{\beta_{2i-1}}\Delta_L^{-1}\left(a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}\right)\beta_{2i} \\ &= \Delta_R^{-1}\left(\prod_{i=1}^{l}\Delta^2\left(a_1^{-1}a_2^{-1}\cdots a_{n-2}^{-1}\right)\overline{\beta_{2i-1}}\Delta_L^{-1}\left(a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}\right)\beta_{2i}\Delta_R^{-1}\right)\Delta_R \\ &= \Delta_R^{-1}\left(\Delta^{2l}\prod_{i=1}^{l}\left(a_1^{-1}a_2^{-1}\cdots a_{n-2}^{-1}\right)\overline{\beta_{2i-1}}\Delta_L^{-1}\left(a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}\right)\beta_{2i}\Delta_R^{-1}\right)\Delta_R, \end{aligned}$$

meaning that β is conjugate to

$$\beta' = \Delta^{2l}\prod_{i=1}^{l}\underbrace{\left(a_1^{-1}a_2^{-1}\cdots a_{n-2}^{-1}\right)\overline{\beta_{2i-1}}\Delta_L^{-1}}_{L_i}\underbrace{\left(a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}\right)\beta_{2i}\Delta_R^{-1}}_{R_i}.$$

Finally, using that homogeneous quasimorphisms are constant on conjugation classes, $\omega_*(\Delta^2\alpha) = 1 + \omega_*(\alpha)$ (since $\omega_*(\alpha\beta) = \omega_*(\alpha) + \omega_*(\beta)$ for commuting α and β and a homogeneous quasimorphism ω_*), and $\omega_*(\alpha\beta\gamma) \geq \omega_*(\alpha\gamma) + \omega_*(\beta) - 1$ for all n-braids α, β, γ (which follows from ω_* having defect at most 1 and being constant on conjugation classes), we calculate

$$\begin{aligned} \omega_*(\beta) = \omega_*(\beta') = \omega_*\left(\Delta^{2l}\prod_{i=1}^{l}L_iR_i\right) &= l + \omega_*\left(\prod_{i=1}^{l}L_iR_i\right) \\ &\geq l + \omega_*(L_1) + \omega_*\left(R_1\prod_{i=2}^{l}L_iR_i\right) - 1 \end{aligned}$$

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$$\dots$$

$$\geq l + \omega_*(L_1) + \omega_*(L_2) + \cdots + \omega_*(L_l) + \omega_*\left(\prod_{i=1}^{l} R_i\right) - l$$

$$= \omega_*(L_1) + \omega_*(L_2) + \cdots + \omega_*(L_l) + \omega_*\left(\prod_{i=1}^{l} R_i\right).$$

Since ω_* vanishes on braids that can be written without $a_1^{\pm 1}$ (which include $\prod_{i=1}^{l} R_i$), and thus (by conjugation invariance) also on braids without $a_{n-1}^{\pm 1}$ (which include L_i), we have $\omega_*(\beta) \geq 0 > \omega_*(\beta)$. □

The defect of ω is known to be 1 for $n \geq 3$. We provide an argument, which is of the same flavour (but much simpler) than the above proof.

Lemma 10. *For $n \geq 3$ if a homogeneous quasimorphism $f\colon B_n \to \mathbb{R}$ satisfies $f(B_{n-1}) = \{0\}$, then the defect of f is bounded below by $\left|f\left(\Delta^2\right)\right|$, that is, $\left|f\left(\Delta^2\right)\right| \leq D_f$.*

Proof. First, we note that

$$f\left(a_1 a_2 \cdots a_{n-2} a_{n-1} a_{n-1} a_{n-2} \cdots a_2 a_1\right) = f(\Delta^2 \Delta_R^{-2}) = f(\Delta^2) + f(\Delta_R^{-2}) = f(\Delta^2),$$

where the first equality is due to equality of the braids, the second equality uses that Δ^2 is in the center and the last equality uses that f vanishes on $\Delta_R \in B_{n-1} \subset B_n$. Hence, for $\alpha = a_2 \cdots a_{n-2} a_{n-1} a_{n-1} a_{n-2} \cdots a_2 \in B_{n-1}$ and $\beta = a_1 a_1$, we find

$$D_f \geq |f(\alpha\beta) - f(\alpha) - f(\beta)| = \left|f(\Delta^2) - 0 - 0\right|,$$

where we used that f evaluates to the same on $a_1 a_2 \cdots a_{n-2} a_{n-1} a_{n-1} a_{n-2} \cdots a_1$ and its conjugate $\alpha\beta$. □

As an aside, we note that the proof of Lemma 10 shows that for $f = \omega$ the supremum D_f is attained when $n \geq 3$.

4. Constructions of quasimorphisms and the proof of Proposition 2

In this section, we discuss the existence of many homogeneous quasimorphisms on B_n for $n \geq 3$ as claimed in Proposition 2. We make use of a geometric group theory setup due to Bestvina and Fujiwara [BF02], which we do not recall in detail. Since this makes this section the least self-contained, we point out that skipping this section can be done at no cost of understanding the results from the introduction except, of course, Proposition 2.

Proposition 2 reduces to the following lemma.

Lemma 11. *Let $n \geq 3$. There exist an injective $\mathbb{R}$-linear map*

$$\ell^1 \to \{f\colon B_n \to \mathbb{R} \mid f \text{ is a homogeneous quasimorphism and } f(B_{n-1}) = \{0\}\}.$$

Here, ℓ^1 denotes the vector space of real-valued sequences $\{a_i\}_{n \in N}$ with $\sum_{n=1}^{\infty} |a_i| < \infty$. Dropping the condition $f(B_{n-1}) = \{0\}$, Lemma 11 is known by work of Bestvina and

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Fujiwara. Indeed, there exist an injective $\mathbb{R}$-linear map

$$\ell^1 \to \{f\colon B_n \to \mathbb{R} \mid f \text{ is a homogeneous quasimorphism}\}$$

by [BF02, Theorem 7 and Proposition 11]. In fact, inspection of their proof reveals that all elements in the image of the $\mathbb{R}$-linear map they construct vanish on B_{n-1}. We explain this using the setup, notations and results from [BF02]. We only make use of these in the proof of Lemma 11, and we only invoke Lemma 11 to prove Proposition 2.

Proof of Lemma 11. Bestvina and Fujiwara construct a large vector subspace of the vector space of homogeneous quasimorphism on a group G whenever the group G has an action on a δ-hyperbolic space X that satisfies weak proper discontinuity (WPD for short) [BF02, Theorem 7]. Actually, Bestvina and Fujiwara construct quasimorphisms that are in general not homogeneous and then consider the quotient of the vector space of quasimorphisms by bounded functions. However, this quotient is readily identified with the vector space of homogeneous quasimorphisms. This identification is given by taking the quasimorphisms h_ω from the construction of Bestvina and Fujiwara to their homogenizations $\widetilde{h_\omega}$. Under this identification, their construction translates to constructing a subspace of the vector space of homogeneous quasimorphism isomorphic to ℓ^1 given by $\{\sum_{n=1}^{\infty} a_i b_i \mid \sum_{n=0}^{\infty} |a_i| < \infty\}$, where the b_i are elements of the form $\widetilde{h_\omega}$.

From the construction in [BF02, Section 2] of the homogeneous quasimorphism h_ω it follows that if an element $r \in G$ has a fixed point $x_0 \in X$, then the homogeneous quasimorphism $\widetilde{h_\omega}$ vanishes on r. Indeed, choosing x_0 as the basepoint in their construction of the quasimorphism h_ω, we see that $h_\omega(r^k) = 0$ for all $k \in \mathbb{Z}$. In particular, the homogeneous quasimorphism $\widetilde{h_\omega}$ satisfies $\widetilde{h_\omega}(r) := \lim_{k\to\infty} \frac{h_\omega(r^k)}{k} = \lim_{k\to\infty} \frac{0}{k} = 0$.

For technical reasons, we choose our group G to be the quotient $G := B_n/\langle \Delta^2 \rangle$ rather than B_n. Of course, any quasimorphism on G gives rise to one on B_n by composing with the quotient map $\pi\colon B_n \to G$. Thus, by the last paragraph it remains to check that G has an action on a δ-hyperbolic space that satisfies WPD such that the elements of $\pi(B_{n-1}) \subset G$ have a fixed point. To do this, we identify B_n with the mapping class group of the n-punctured disc and we identify $G = B_n/\langle \Delta^2 \rangle$ with a finite index subgroup of the mapping class group of the $(n+1)$-punctured sphere. Then G naturally acts on the curve complex X of the $(n+1)$-punctured sphere. The curve complex X is δ-hyperbolic and the action of G on X satisfies WPD since the action of the full mapping class satisfies WPD [BF02, Proposition 11] and restricting an action that satisfies WPD to a finite index subgroup yields an action that satisfies WPD. We conclude the proof by noting that there exists a simple closed curve γ in the $(n+1)$-punctured sphere (in particular, $[\gamma] \in X$) such that $\pi(B_{n-1}) = \{[\phi] \in G \mid [\phi][\gamma] = [\gamma]\}$. For sake of completeness, we describe such a γ explicitly.

For this, we make the identification of B_n with the mapping class group of the n-punctured disc D explicit. Here, we taken D to be the closed unit disc in $\mathbb{C}$ with the punctures placed on the open interval $(-1,1)$ and ordered by the usual order on $(-1,1) \subset \mathbb{R}$. Namely, we chose an identification isomorphism that sends the generator a_i to the mapping class given by a positive half-twist that exchanges the i-th and $(i+1)$-th punctures and is the identity outside a small neighbourhood of the arc on the real line

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connecting the i-th and $(i+1)$-th puncture. We further identify the $(n+1)$-punctured sphere with the quotient D/S^1, where the punctures are as for D with one extra puncture: the point ∞ in the quotient corresponding to the collapsed S^1. This yields an explicit identification of G with the subgroup of the mapping class group of the $(n+1)$-punctured sphere given by those mapping classes that fix the puncture ∞. This identification is such that the quotient map $\pi\colon B_n \to G$ is identified with the group homomorphism between the mapping class groups induced by the quotient map $D \to D/S^1$. See, for example, [HK06, Bir74] for these identifications.

With this set up, we choose γ to be a simple closed curve in $D \setminus S^1 \subset D/S^1$ that is the boundary of a round disc in $D \setminus S^1$ that contains all but the first puncture. Then, indeed, $B_{n-1} \subset B_n$ is sent to mapping classes that have a representative that restricts to the identity on γ. □

Proof of Proposition 2. Fix $\varepsilon > 0$. And, for $r \in \mathbb{R}$, let f_r be the image of the r-th basis element of a chosen basis for ℓ^1 under an injective map guaranteed to exist by Lemma 11. Up to multiplication with a constant, we can arrange for f_r to satisfy $f_r(\Delta^2) \geq 0$ and $D_{f_r} < \varepsilon$. Define $g_r := \frac{1}{1+f_r(\Delta^2)}(\omega + f_r)$, and note that $g_r(\Delta^2) = 1$, $g_r(B_{n-1}) = \{0\}$, and $D_{g_r} \leq 1 + D_{f_r} < 1 + \varepsilon$. Hence, for all but at most one $a \in \mathbb{R}$, $\{g_r\}_{r \in \mathbb{R}\setminus\{a\}}$ is a basis of a subspace of

$$\{f\colon B_n \to \mathbb{R} \mid f \text{ is a homogeneous quasimorphism and } f(B_{n-1}) = \{0\}\}. \qquad \square$$

5. The proofs of Theorem 3 and Proposition 5

For the proof of Theorem 3, we use that the FDTC can be expressed in terms of the homogenization of the upsilon invariant. For all $\beta \in B_n$ and $t = \frac{2}{n-1}$, we have

$$\omega(\beta) = \frac{\widetilde{\Upsilon}_\beta(t)}{t} + \frac{\mathrm{wr}(\beta)}{2}, \tag{3}$$

by [FH19, Theorem 1.3]. Here, for each $\beta \in B_n$ and for $\delta := a_1 a_2 \cdots a_{n-1} \in B_n$,

$$\widetilde{\Upsilon_\beta} := \lim_{k\to\infty} \frac{\Upsilon_{\widehat{\beta^{nk}\delta}}(t)}{nk}, \tag{4}$$

where for a knot K and $t \in [0,1]$ we denote by $\Upsilon_K(t)$ the upsilon invariant introduced in [OSS17]. For more details on homogenization of knot invariants, compare [GG05, Bra11] and Appendix A. For Υ specifically see [FK17].

Recasting ω using Υ via Equation (3) allows us to make use of the following slice genus bound. For every knot K, we have

$$\Upsilon_K(t) \leq t g_4(K) \text{ for all } t \in [0,1] \quad \text{[OSS17, Theorem 1.11]}. \tag{5}$$

As a further input for the proof of Theorem 3, but also the proof of Proposition 5, we need cobordisms with small genera between knots and links arising as connected sums and arising as closures of compositions of braids.

Lemma 12. *Let α, β, and γ be in B_n.*

(a) *There exists a cobordism given by $(n-1)$ 1-handles between $\widehat{\alpha\beta}$ and a connected sum of $\widehat{\alpha}$ and $\widehat{\beta}$.*

(b) *If at least one of the braids α, β or γ is a pure braid, then there exists a cobordism given by $2(n-1)$ 1-handles between $\widehat{\alpha\beta\gamma}$ and $\widehat{\alpha\gamma\beta}$.*

We remark that, in Equation (a), we do not claim to control which connected sum of $\widehat{\alpha}$ and $\widehat{\beta}$ is involved. (Recall that for two links L_1 and L_2 the notion of connected sum $L_1\#L_2$ depends on a choice of component in each link.) We postpone the proof of Lemma 12 to after its application in the proofs of Theorem 3, where we use Equation (a), and Proposition 5, where we employ Equation (b).

For the proof of Theorem 3, we observe that there exists a cobordism consisting of $(n-1)nk$ 1-handles between $\widehat{\beta^{nk}\delta}$ and a nk-fold connected sum of $\widehat{\beta}$; we denote the latter by $nk\widehat{\beta}$. Indeed, by concatenation of nk cobordism as provided by Lemma 12(a), we find such a cobordism between $\widehat{\beta^{nk}\delta}$ and a connected sum of nk many $\widehat{\beta}$ and one $\widehat{\delta}$ (which is an unknot) as desired; compare also [FH19, Appendix A]. In particular, we have

$$1-\chi_4\left(\widehat{\beta^{nk}\delta}\right) \leq 1-\chi_4\left(nk\widehat{\beta}\right)+nk(n-1) \leq nk\left(1-\chi_4\left(\widehat{\beta}\right)\right)+nk(n-1), \tag{6}$$

where the second inequality follows from $1-\chi_4$ being subadditive under connected sum.

Proof of Theorem 3. Set $t=\frac{2}{n-1}$. For every $\beta \in B_n$, we have

$$\begin{aligned}
\omega(\beta) &\overset{(3)}{=} \frac{\widetilde{\Upsilon}_\beta(t)}{t}+\frac{\mathrm{wr}(\beta)}{2} \overset{(4)}{=} \lim_{k\to\infty}\frac{\Upsilon_{\widehat{\beta^{nk}\delta}}(t)}{nkt}+\frac{\mathrm{wr}(\beta)}{2} \\
&\overset{(5)}{\leq} \lim_{k\to\infty}\frac{g_4\left(\widehat{\beta^{nk}\delta}\right)}{nk}+\frac{\mathrm{wr}(\beta)}{2} = \lim_{k\to\infty}\frac{1-\chi_4\left(\widehat{\beta^{nk}\delta}\right)}{2nk}+\frac{\mathrm{wr}(\beta)}{2} \\
&\overset{(6)}{\leq} \lim_{k\to\infty}\frac{nk\left(1-\chi_4\left(\widehat{\beta}\right)\right)+nk(n-1)}{2nk}+\frac{\mathrm{wr}(\beta)}{2} = \frac{-\chi_4\left(\widehat{\beta}\right)+n}{2}+\frac{\mathrm{wr}(\beta)}{2} \\
&\overset{(1)}{\leq} \frac{-\chi_4\left(\widehat{\beta}\right)+n}{2}+\frac{-\chi_4\left(\widehat{\beta}\right)+n}{2} = -\chi_4\left(\widehat{\beta}\right)+n.
\end{aligned}$$

□

Proof of Proposition 5. Fix $\varepsilon>0$, and let α and β be n-braids such that $f(\alpha\beta)-f(\alpha)-f(\beta)\geq D_f-\varepsilon$. We first note that we can and do assume that α and β are pure braids. Indeed, if not, pick α' and β' such that $f(\alpha'\beta')-f(\alpha')-f(\beta')\geq D_f-\varepsilon/n$ and set $\alpha \coloneqq (\alpha')^n$ and $\beta \coloneqq (\beta')^n$. Combining

$$f((\alpha'\beta')^n)-f((\alpha')^n)-f((\beta')^n)=nf(\alpha'\beta')-nf(\alpha')-nf(\beta')\geq nD_f-\varepsilon$$

with $|f((ab)^n)-f(a^nb^n)|\leq (n-1)D_f$, which one checks by iteratively applying

$$\left|f(a^{k-1}b^{k-1})+f(ab)-f(a^kb^k)\right|=\left|f(b^{k-1}a^{k-1})+f(ab)-f(b^{k-1}a^{k-1}(ab))\right|\leq D_f,$$

we have that $f(\alpha\beta)-f(\alpha)-f(\beta)\geq D_f-\varepsilon$.

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Fix an even positive integer k. Using that f is homogeneous and that $f(ab)-f(a)-f(b)\leq D_f$ for all $a,b\in B_n$, we calculate

$$\begin{aligned} kD_f - k\varepsilon &\leq k(f(\alpha\beta)-f(\alpha)-f(\beta)) = f\left((\alpha\beta)^k\right)+f\left(\alpha^{-k}\right)+f\left(\beta^{-k}\right)\\ &\leq f\left((\alpha\beta)^k\alpha^{-k}\right)+D_f+f\left(\beta^{-k}\right)\\ &\leq f\left((\alpha\beta)^k\alpha^{-k}\beta^{-k}\right)+D_f+D_f\\ &\leq f\left((\alpha\beta)^k\alpha^{-k}\beta^{-k}\delta\right)-f(\delta)+D_f+D_f+D_f\\ &\leq Ag_4(K)+C-f(\delta)+D_f+D_f+D_f, \end{aligned}$$

where K denotes the closure of $(\alpha\beta)^k\alpha^{-k}\beta^{-k}\delta$ and, as above, $\delta = a_1\cdots a_{n-1}$. Note that K is a knot since $(\alpha\beta)^k\alpha^{-k}\beta^{-k}$ is a pure braid.

Next, we observe that there exists a cobordism of genus $\frac{k}{2}(n-1)$ between K and the closure of $\beta^k\alpha^k\alpha^{-k}\beta^{-k}\delta=\delta$.

For this, we write $(\alpha\beta)^k\alpha^{-k}\beta^{-k}$ as a product of $\frac{k}{2}$ commutators of pure braids, that is, $(\alpha\beta)^k\alpha^{-k}\beta^{-k}=[\alpha_1,\beta_1][\alpha_2,\beta_2]\cdots[\alpha_{\frac{k}{2}},\beta_{\frac{k}{2}}]$ for some pure braids $\alpha_i,\beta_i\in B_n$. This is possible by [Cal09, Proof of Lemma 2.24]; compare also [Bav91].[2]

By Lemma 12(b), for all $b\in B_n$ the closures of $[\alpha_i,\beta_i]b=\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}b$ and $\beta_i\alpha_i\alpha_i^{-1}\beta_i^{-1}b=b$ are related by a cobordism with $2(n-1)$ 1-handles, hence applying this $\frac{k}{2}$ times gives a cobordism between K and the closure of δ given by $k(n-1)$ 1-handles. In other words, we have a cobordism of genus $\frac{k}{2}(n-1)$ as desired.

Since δ has the unknot as its closure, we have $g_4(K)\leq\frac{k}{2}(n-1)$ by the last paragraph. We conclude that

$$kD_f-k\varepsilon\leq A\frac{k}{2}(n-1)+C-f(\delta)+3D_f,$$

which yields $D_f\leq\frac{A}{2}(n-1)$ by first dividing by k and taking the limit $k\to\infty$ and then letting ε tend to 0. □

Finally, we turn to the proof of Lemma 12. The idea of the proof is of a similar flavour as the arguments used in [Bra11] and [FH19, Appendix A], but to the best of our knowledge, the exact statement does not yet appear in the literature.

Proof of Lemma 12. To see Equation (a), consider a diagram for $\widehat{\alpha\beta}$ as depicted in Figure 1 A) (where γ is taken to be the trivial braid) and apply $(n-1)$ handle moves, starting with the one indicated by the blackboard framed dotted (green) arc, to find the diagram in Figure 1 B). The link given by this diagram is a connected sum of $\widehat{\alpha}$ and $\widehat{\alpha\beta}$ with respect to the indicated sphere (red); hence, we have that there exists a cobordism between $\widehat{\alpha\beta}$ and the connected sum of $\widehat{\alpha}$ and $\widehat{\alpha\beta}$ depicted in Figure 1 B).

We turn to Equation (b). Since $\alpha\beta\gamma$, $\beta\gamma\alpha$, and $\gamma\alpha\beta$ all are conjugate and hence have the same closure, we may and do assume that γ is a pure braid. Consider a diagram for $\widehat{\alpha\beta}$

[2] In the first version of this article, we use a different expression for $(\alpha\beta)^k\alpha^{-k}\beta^{-k}$. We are thankful to Tesuya Ito for reminding us that the stable commutator length of a commutator is at most 1/2, which improved the bound of Proposition 5 by a factor of $\frac{1}{2}$.

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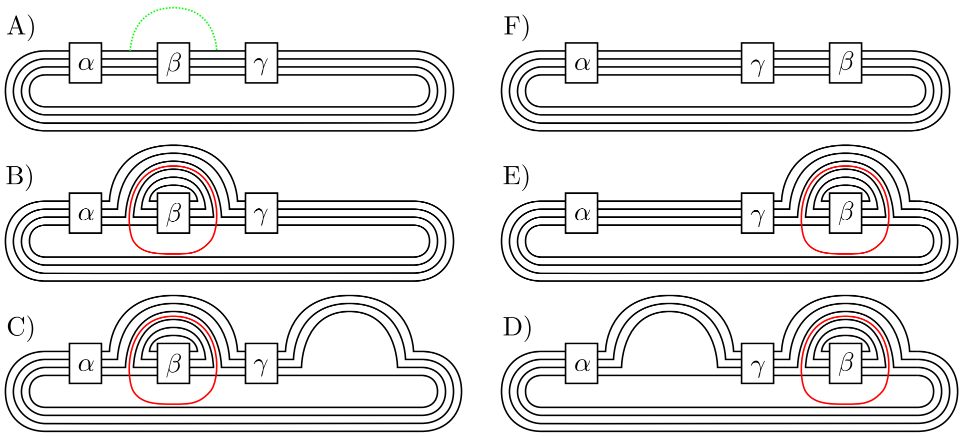

Figure 1. Isotopies and cobordisms proving Lemma 12. For readability of the diagrams the illustration is for $n = 4$.

as depicted in Figure 1 A) and apply $(n-1)$ handle moves to find the diagram in Figure 1 B) as in the proof of Equation (a). Figure 1 B) and Figure 1 C) depict isotopic links (in fact the diagrams are the same up to isotopy of the plane). Figure 1 C) and Figure 1 D) depict isotopic links. An isotopy is given by shrinking and moving the summand $\widehat{\beta}$ through γ (here, we invoke that γ is a pure braid). Finally, Figure 1 D) and Figure 1 E) depict isotopic links and $(n-1)$ handle moves turn the diagram given in Figure 1 E) into the one given in Figure 1 F). All in all, we find that there exists a cobordism given by $2(n-1)$ 1-handles between $\widehat{\alpha\beta\gamma}$ and $\widehat{\alpha\gamma\beta}$ as desired. □

Appendix A. The slice-Bennequin inequality for the homogenization of 1-Lipschitz concordance homomorphisms

In this appendix, we explain that the homogenization of a 1-Lipschitz concordance homomorphism satisfies a version of the slice-Bennequin inequality. This can be understood as providing a common framework for both the slice-Bennequin inequality (1) and Theorem 3; see Examples 14 and 15, respectively. What follows below is based on the same idea as the proof of Theorem 3, which was rather straightforward once the necessary preparations (like Lemma 12) are made. Still, we think it is worth making this explicit as the exact statement and perspective appear to be absent from the literature. What follows owes a lot to the ideas of homogenization of concordance homomorphisms as pursued in [GG05] for Tristram–Levine signatures and in general in [Bra11] and the idea of proof of the slice-Bennequin inequality as pioneered by Rudolph [Rud93].

We call a real-valued knot invariant $I\colon \mathfrak{Knots} \to \mathbb{R}$ a 1-*Lipschitz concordance homomorphism*, if $I(K\#J) = I(K) + I(J)$ and $|I(K)| \leq g_4(K)$ for all $K, J \in \mathfrak{Knots}$, where $\mathfrak{Knots}$ denotes the set of isotopy classes of knots. (As 'concordance homomorphism' in the name suggests, such I factor through the smooth concordance group and the induced map is a group homomorphism.) Writing $\delta \coloneqq a_1 a_2 \cdots a_{n-1} \in B_n$ as above, for each 1-Lipschitz

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concordance homomorphism I, we have that

$$\widetilde{I} \to \mathbb{R}, \quad \beta \mapsto \widetilde{I}(\beta) := \lim_{k\to\infty} \frac{I\left(\widehat{\beta^{nk}\delta}\right)(t)}{nk} \tag{7}$$

is a homogeneous quasimorphism with defect $D \leq \frac{n-1}{2}$; see [FH19, Lemma A.1].

Lemma 13. *Fix an integer $1 \geq n$. For all $\beta \in B_n$, we have $\widetilde{I}(\beta) \leq \frac{-\chi_4(\widehat{\beta})+n}{2}$.*

Proof. $\widetilde{I}(\beta) \overset{(7)}{=} \lim_{k\to\infty} \frac{I\left(\widehat{\beta^{nk}\delta}\right)}{nk} \leq \lim_{k\to\infty} \frac{g_4\left(\widehat{\beta^{nk}\delta}\right)}{nk} = \lim_{k\to\infty} \frac{1-\chi_4\left(\widehat{\beta^{nk}\delta}\right)}{2nk}$
$\overset{(6)}{\leq} \lim_{k\to\infty} \frac{nk\left(1-\chi_4\left(\widehat{\beta}\right)\right)+nk(n-1)}{2nk} = \frac{-\chi_4\left(\widehat{\beta}\right)+n}{2}$. □

In case $\beta \in B_n$ has a knot as its closure $\widehat{\beta}$, then $\left|I\left(\widehat{\beta}\right) - \widetilde{I}(\beta)\right| \leq \frac{n-1}{2}$ (this follows readily from Lemma 12; it is explicitly stated in [FH19, Lemma A.1]), hence

$$\widetilde{I}(\beta) \leq I\left(\widehat{\beta}\right) + \frac{n-1}{2} \leq g_4\left(\widehat{\beta}\right) + \frac{n-1}{2} = \frac{-\chi_4\left(\widehat{\beta}\right)+n}{2}. \tag{8}$$

Example 14. We consider the case when I is a *slice torus invariant* – a 1-Lipschitz concordance homomorphism I with $I(T_{p,p+1}) = g_4(T_{p,p+1}) = (p-1)p/2$ for positive integers p. Slice torus invariants include Ozsváth-Szabó's τ [OS03] and Rasumussen's s [Ras10]. In this case, we have $\widetilde{I} = \mathrm{wr}/2$; see, for example, [FH19, Lemma A.3]. Hence, for such I, Lemma 13 recovers Equation (1), and Equation (8) reads, for all $\beta \in B_n$ with closure a knot,

$$\mathrm{wr}(\beta) \leq 2I\left(\widehat{\beta}\right) + n - 1 \leq 2g_4\left(\widehat{\beta}\right) + n - 1.$$

We find this to be philosophically pleasing for the following reason. Observe that slice torus invariants are by definition the 1-Lipschitz concordance homomorphisms that are strong enough to reprove the local Thom conjecture (since the latter can be phrased as $g_4(T_{p,p+1}) = (p-1)p/2$ for all positive integers p [KM93, Corollary 1.3]). In fact, the existence of slice torus invariants can be seen to be equivalent to the statement of the local Thom conjecture without making use of any explicit construction of such an invariant [FLL22, Remark 4]. It is then fitting that, via homogenization, slice torus invariants recover the slice-Bennequin inequality, which Rudolph derived by elementary means using only the local Thom conjecture as an input.

Example 15. If $I(K) := \frac{\Upsilon_K(\frac{2}{n-1})}{n-1} + \frac{\tau(K)}{2}$, then $\widetilde{I} = 2\omega$ by Equation (3). Hence, Lemma 13 yields Theorem 3, and Equation (8) reads, for all $\beta \in B_n$ with closure a knot,

$$\omega(\beta) \leq \frac{2\Upsilon_{\widehat{\beta}}(\frac{2}{n-1})}{n-1} + \tau\left(\widehat{\beta}\right) + n - 1 \leq 2g_4\left(\widehat{\beta}\right) + n - 1.$$

In light of Equation (8), we wonder whether every homogeneous quasimorphism that satisfies a slice-Bennequin inequality does arise as a homogenization.

Question 16. *Fix $n \geq 3$, and let $f\colon B_n \to \mathbb{R}$ be a homogeneous quasimorphism. If there exist constants $A, C \in \mathbb{R}$ such that*

$$|f(\beta)| \leq Ag_4\left(\widehat{\beta}\right) + C \text{ for all } \beta \in B_n \text{ with closure a knot,}$$

does there exist a 1-Lipschitz concordance homomorphism I and $r \in \mathbb{R}$ such that $f = r\widetilde{I}$?

In light of the fact that Question 8, as far as we know, remains open, it seems that even a positive answer to the following is possible.

Question 17. *Fix $n \geq 3$, and let $f\colon B_n \to \mathbb{R}$ be a homogeneous quasimorphism. Does there exist a 1-Lipschitz concordance homomorphism I and $r \in \mathbb{R}$ such that $f = r\widetilde{I}$?*

This author strongly suspects that the answer to Question 17 is no, but is unable to provide a counterexample.

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