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COMBINATORIAL L^2-DETERMINANTS

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Abstract We show that the zeta function of a regular graph admits a representation as a quotient of a determinant over a L^2-determinant of the combinatorial Laplacian.

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1. Introduction

In [**2**] we showed that a geometric zeta function of a locally symmetric space of rank one admits a representation as a quotient

$$\frac{\det(\Delta + P(s))}{\det_\Gamma(\Delta + P(s))},$$

where P is a polynomial and Δ a generalized Laplacian, det refers to the determinant and $\det_\Gamma$ to the L^2-determinant. The combinatorial counterpart to these geometric zeta functions are zeta functions of finite graphs. In this paper we show an analogous formula for finite graphs. For the sake of conceptual clarity we will give the full proof only for regular graphs, i.e. graphs of constant valency, but the methods are easily seen to cover the general case as well. By the results of [**1**] we are reduced to a computation of the L^2-determinant of $\Delta + \lambda$, which cannot be calculated directly because of the combinatorial complexity. We circumvent this problem by using a technique developed in [**3**], which essentially gives a way of computing L^2-determinants as limits of ordinary determinants. See [**6**] for a similar assertion. Using the results of [**1**] or [**5**], the proof of the main theorem (3.3) then becomes easy.

If the valency of the graph is $q+1$ for q a prime power, the same result can be derived by means of the harmonic analysis of the locally compact group $G = SL_2(F)$ for a p-adic field F, since then the graph may be compared to a quotient of the Bruhat–Tits building of G. In [**4**] we generalized the notion of geometric zeta functions to higher rank Bruhat–Tits buildings. The question of whether there is an analogous formula in the higher rank case is as yet unanswered.

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2. The convergence theorem

Let (Y, d) be a discrete metric space that is *proper*, i.e. for any $y \in Y$ and any $R > 0$ the set of all $x \in Y$ with $d(x, y) \leqslant R$ is compact. By the discreteness this means the latter set is finite. Let Γ be a group of isometries of Y that acts freely and such that the quotient $\Gamma \backslash Y$ is finite. Note that the existence of Γ implies that Y is *uniformly proper*, i.e. for any $y \in Y$ and any $R > 0$ the number of points x with $d(x, y) \leqslant R$ is bounded by a constant depending only on R and not on y.

Let $\mathcal{F}$ be a sheaf of complex vector spaces over Y of dimension r. By the discreteness this just means that to each $y \in Y$ there is attached an r-dimensional vector space $\mathcal{F}(y)$. Assume the action of Γ lifts to $\mathcal{F}$ such that it preserves stalks and is linear on each stalk. For $R > 0$ and $M \subset Y$, let $U_R(M)$ be the R-neighbourhood of M, i.e. $U_R(M)$ is the set of $y \in Y$ with $d(y, M) < R$.

Let $\Gamma(\mathcal{F})$ be the space of global sections of $\mathcal{F}$ and let $\Gamma_{\mathrm{c}}(\mathcal{F})$ be the subspace of sections with compact support. Let $R > 0$, a linear operator $D : \Gamma_{\mathrm{c}}(\mathcal{F}) \to \Gamma(\mathcal{F})$ is said to have *finite propagation speed* $\leqslant R$ if for any $\varphi \in \Gamma_{\mathrm{c}}(\mathcal{F})$ we have $\operatorname{supp} D\varphi \subset U_R(\operatorname{supp} \varphi)$. An operator with finite propagation speed maps $\Gamma_{\mathrm{c}}(\mathcal{F})$ to itself.

Lemma 2.1. *Suppose the operator D maps $\Gamma_{\mathrm{c}}(\mathcal{F})$ to itself. Suppose further that D commutes with the action of Γ on $\Gamma_{\mathrm{c}}(\mathcal{F})$. Then D is of finite propagation speed.*

Proof. Let D be an operator stabilizing $\Gamma_{\mathrm{c}}(\mathcal{F})$. A δ-section at $y \in Y$ of the sheaf $\mathcal{F}$ is a section δ that vanishes outside $\{y\}$. Let $F_D(y)$ be the maximum distance $d(x, y)$ of a point $x \in Y$ with $D\delta(x) \neq 0$ for a δ-section at y. By the finite dimensionality of $\mathcal{F}(y)$, this maximum is attained. Note that D has finite propagation speed if and only if the function F_D is bounded. Now assume D to be Γ-invariant, then so is F_D, which then, since $\Gamma \backslash Y$ is finite, must be bounded. □

Since Γ acts freely on Y, the quotient $\mathcal{F}_\Gamma := \Gamma \backslash \mathcal{F}$ defines a sheaf on the finite set $Y_\Gamma := \Gamma \backslash Y$. The space of sections $\Gamma(\mathcal{F}_\Gamma)$ can be identified with the space of Γ-invariant sections $\Gamma(\mathcal{F})^\Gamma$ of $\mathcal{F}$. Note that the summation map $S : \Gamma_{\mathrm{c}}(\mathcal{F}) \to \Gamma(\mathcal{F}_\Gamma)$ given by $S\varphi(y) := \sum_{\gamma \in \Gamma} \gamma^{-1} \varphi(\gamma y)$ is surjective.

Lemma 2.2. *Let $D : \Gamma_{\mathrm{c}}(\mathcal{F}) \to \Gamma(\mathcal{F})$ be linear, Γ-invariant and of finite propagation speed, then there is a unique operator D_Γ, the* pushdown *of D on $\Gamma(\mathcal{F}_\Gamma)$, such that the diagram*

$$\begin{array}{ccc} \Gamma_{\mathrm{c}}(\mathcal{F}) & \xrightarrow{D} & \Gamma_{\mathrm{c}}(\mathcal{F}) \\ \downarrow S & & \downarrow S \\ \Gamma(\mathcal{F}_\Gamma) & \xrightarrow{D_\Gamma} & \Gamma(\mathcal{F}_\Gamma) \end{array}$$

commutes.

Proof. Define $\tilde{D_\Gamma} : \Gamma_{\mathrm{c}}(\mathcal{F}) \to \Gamma(\mathcal{F}_\Gamma)$ by $\tilde{D_\Gamma}(\varphi) := SD(\varphi)$. The Γ-invariance of D and the finite propagation speed then implies that $\tilde{D}_\Gamma$ vanishes on the kernel of S, hence induces an operator D_Γ as claimed. □

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Let $D : \Gamma_c(\mathcal{F}) \to \Gamma(\mathcal{F})$ be linear, then D has a kernel, i.e. it can be written as

$$D\varphi(x) = \sum_{y \in Y} \langle x|D|y\rangle \varphi(y),$$

where $\langle x|D|y\rangle \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}(y), \mathcal{F}(x))$ and the sum is finite for each $x \in Y$. It follows that D has propagation speed $\leqslant R$ if and only if $\langle x|D|y\rangle = 0$ for all $x, y \in Y$ with $d(x, y) > R$. Further, D is Γ-invariant if and only if $\langle \gamma x|D|\gamma y\rangle = \gamma\langle x|D|y\rangle\gamma^{-1}$ for all $x, y \in Y$ and all $\gamma \in \Gamma$.

Lemma 2.3. *Let D be Γ-invariant and of finite propagation speed, then the operator D_Γ has the kernel:*

$$\langle x|D_\Gamma|y\rangle = \sum_{\gamma \in \Gamma} \gamma^{-1}\langle \gamma x|D|y\rangle.$$

Proof. A computation. □

Suppose that the sheaf $\mathcal{F}$ is hermitian and that Γ acts unitarily. Let $L^2(\mathcal{F})$ be the space of square integrable sections of $\mathcal{F}$. Let S be a set of representatives of $\Gamma\backslash Y$, then $L^2(\mathcal{F}) \cong L^2(\Gamma) \otimes L^2(\mathcal{F}|_S) \cong L^2(\Gamma) \otimes L^2(\mathcal{F}_\Gamma)$, where $L^2(\Gamma)$ is taken with respect to the counting measure and the tensor product is the tensor product in the category of Hilbert spaces. Let $\mathrm{VN}(\Gamma)$ denote the von Neumann algebra of Γ, i.e. the von Neumann algebra of all bounded operators on $L^2(\Gamma)$ that commute with, say, the right action of Γ. It is easy to see that this algebra is topologically generated by the left translations L_γ, $\gamma \in \Gamma$. The algebra $\mathrm{VN}(\Gamma)$ carries a natural finite trace τ given by $\tau(\sum_{\gamma\in\Gamma} c_\gamma L_\gamma) = c_e$. By the above we get that the von Neumann algebra $B(L^2(\mathcal{F}))^\Gamma$ of Γ-invariant operators on $L^2(\mathcal{F})$ is isomorphic to $B(L^2(\mathcal{F}))^\Gamma \cong \mathrm{VN}(\Gamma) \otimes B(L^2(\mathcal{F}_\Gamma))$. Let tr_Γ denote the trace on $B(L^2(\mathcal{F}))^\Gamma$ given by tensoring τ with the usual trace on $B(L^2(\mathcal{F}_\Gamma))$.

Lemma 2.4. *Let $T \in B(L^2(\mathcal{F}))^\Gamma$ with kernel $\langle x|T|y\rangle$, then, for any set S of representatives of $\Gamma\backslash Y$, we have*

$$\mathrm{tr}_\Gamma\, T = \sum_{s \in S} \mathrm{tr}\langle x|T|x\rangle.$$

Proof. As an element of $B(L^2(\mathcal{F}))^\Gamma \cong \mathrm{VN}(\Gamma) \otimes B(L^2(\mathcal{F}|_S))$, the operator T writes as $T = \sum_{\gamma\in\Gamma} L_\gamma \otimes \mathrm{PT}\,|_S$, where $P : \mathcal{F} \to \mathcal{F}|_S$ is the projection. Since the operator $\mathrm{PT}\,|_S$ has kernel $\langle\cdot|T|\cdot\rangle|_{S\times S}$, the claim follows. □

Let $\lambda > 0$. For λ large enough, we may assume that the spectrum of the operator $D+\lambda$ lies in the right half plane. Taking the standard branch of the logarithm, the holomorphic functional calculus then allows us to define the operator $(D+\lambda)^{-s} \in B(L^2(\mathcal{F}))^\Gamma$ for any $s \in \mathbb{C}$. Set $\zeta_{D+\lambda,\Gamma}(s) := \mathrm{tr}_\Gamma((D+\lambda)^{-s})$, and define the *$L^2$-determinant* of $D + \lambda$ as

$$\det{}_\Gamma(D+\lambda) := \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \zeta_{D+\lambda,\Gamma}(s)\right).$$

Proposition 2.5. *The map* $\lambda \mapsto \det_\Gamma(D+\lambda)$ *extends to a holomorphic function on the universal covering of* $\mathbb{C} - \mathrm{Spec}(-D)$.

Proof. Let X be the universal covering in question. The map $\lambda \mapsto \log(D+\lambda)$ extends to an operator valued holomorphic map on X and so does $(D+\lambda)^{-s} = \exp(-s\log(D+\lambda))$. The claim follows. □

A *tower* of subgroups of Γ is a sequence $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots$, with $\cap_j \Gamma_j = \{1\}$ and each Γ_j is normal of finite index in Γ.

For any Γ and any natural number N, fix a standard Nth root of $\det(D_\Gamma + \lambda)$ for $\mathrm{Re}(\lambda) \gg 0$ by requiring that

$$\lim_{\lambda\to+\infty} \frac{\det(D_\Gamma+\lambda)^{1/N}}{|\det(D_\Gamma+\lambda)^{1/N}|} = 1.$$

We now come to the main result of this section.

Theorem 2.6. *Let* D *be* Γ*-invariant and of finite propagation speed. Let* (Γ_j) *be a tower in* Γ*. For* $\mathrm{Re}(\lambda) \gg 0$ *we have locally uniform convergence*

$$\det(D_{\Gamma_j}+\lambda)^{1/[\Gamma:\Gamma_j]} \to \det_\Gamma(D+\lambda),$$

as $j \to \infty$.

Proof. Let, for $\mathrm{Re}(\lambda) \gg 0$ and $\mathrm{Re}(s) \gg 0$:

$$\begin{aligned} F_j(\lambda,s) &:= \frac{\mathrm{tr}(D_{\Gamma_j}+\lambda)^{-s}}{[\Gamma:\Gamma_j]} - \mathrm{tr}_\Gamma(D+\lambda)^{-s} \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\frac{\mathrm{tr}\,\mathrm{e}^{-tD_{\Gamma_j}}}{[\Gamma:\Gamma_j]} - \mathrm{tr}_\Gamma\,\mathrm{e}^{-tD}\right)\mathrm{e}^{-t\lambda}\,dt. \end{aligned}$$

Lemma 2.7. *For any* $z \in \mathbb{C}$*, the operator* e^{zD_Γ} *has kernel*

$$\langle x|\mathrm{e}^{zD_\Gamma}|y\rangle = \sum_{\gamma\in\Gamma} \gamma^{-1}\langle \gamma x|\mathrm{e}^{zD}|y\rangle.$$

Proof. The norms on $\mathcal{F}(x)$ and $\mathcal{F}(y)$ give rise to a norm on $\mathrm{Hom}_\mathbb{C}(\mathcal{F}(y),\mathcal{F}(x))$. We have the estimate

$$\begin{aligned} \|\langle x|D_\Gamma^n|y\rangle\| &= \sup_{\|v\|=1} \|\langle x|D_\Gamma^n|y\rangle v\| \\ &\leqslant \sup_{x,y,v} \|\langle x|D_\Gamma^n|y\rangle v\| \\ &\leqslant \|D_\Gamma^n\| \leqslant \|D_\Gamma\|^n, \end{aligned}$$

where $\|D_\Gamma\|$ is the operator norm on $L^2(\mathcal{F}_\Gamma)$. It follows that we have absolute convergence in

$$\begin{aligned}
e^{zD_\Gamma}\varphi(x) &= \sum_{n\geqslant 0}\frac{z^n}{n!}D_\Gamma^n\varphi(x)\\
&= \sum_{n\geqslant 0}\frac{z^n}{n!}\sum_{y\in Y}\langle x|D_\Gamma^n|y\rangle\varphi(y)\\
&= \sum_{y\in Y}\sum_{n\geqslant 0}\frac{z^n}{n!}\langle x|D_\Gamma^n|y\rangle\varphi(y),
\end{aligned}$$

so that

$$\begin{aligned}
\langle x|e^{zD_\Gamma}|y\rangle &= \sum_{n\geqslant 0}\frac{z^n}{n!}\langle x|D_\Gamma^n|y\rangle\\
&= \sum_{n\geqslant 0}\frac{z^n}{n!}\sum_{\gamma\in\Gamma}\gamma^{-1}\langle\gamma x|D^n|y\rangle\\
&= \sum_{\gamma\in\Gamma}\gamma^{-1}\sum_{n\geqslant 0}\frac{z^n}{n!}\langle\gamma x|D^n|y\rangle\\
&= \sum_{\gamma\in\Gamma}\gamma^{-1}\langle\gamma x|e^{zD}|y\rangle.
\end{aligned}$$

The last equation follows from the above considerations, which are clearly also valid for trivial Γ. □

Lemma 2.8. *There is a sequence $c_j > 0$ tending to zero such that*

$$\left|\frac{\operatorname{tr} e^{zD_{\Gamma_j}}}{[\Gamma:\Gamma_j]} - \operatorname{tr}_\Gamma e^{zD}\right| \leqslant c_j|z|$$

for all $z\in\mathbb{C}$, with $|z|\leqslant 1$ and all $j\in\mathbb{N}$.

Proof. Let S_j be a set of representatives of $\Gamma_j\backslash Y$. By Lemma 2.7, we compute

$$\begin{aligned}
\operatorname{tr} e^{zD_{\Gamma_j}} &= \sum_{s\in S_j}\operatorname{tr}\langle s|e^{zD_{\gamma_j}}|s\rangle\\
&= \sum_{s\in S_j}\sum_{\gamma\in\Gamma_j}\operatorname{tr}(\gamma^{-1}\langle\gamma s|e^{zD_{\gamma_j}}|s\rangle).
\end{aligned}$$

Now let $S = S_1$ and assume that $S_j = \cup_{\sigma:\Gamma/\Gamma_j}\sigma S$, where σ runs over a set of representatives of Γ/Γ_j. The above equals

$$\sum_{s\in S}\sum_{\sigma:\Gamma/\Gamma_j}\sum_{\gamma\in\Gamma_j}\operatorname{tr}(\gamma^{-1}\langle\gamma\sigma s|e^{zD_{\gamma_j}}|\sigma s\rangle).$$

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Replacing γ by $\sigma\gamma\sigma^{-1}$, which is possible since Γ_j is normal in Γ, gives

$$\begin{aligned} &= \sum_{s\in S}\sum_{\sigma:\Gamma/\Gamma_j}\sum_{\gamma\in\Gamma_j} \operatorname{tr}(\sigma\gamma^{-1}\sigma^{-1}\langle\sigma\gamma s|\mathrm{e}^{zD_{\gamma_j}}|\sigma s\rangle) \\ &= \sum_{s\in S}\sum_{\sigma:\Gamma/\Gamma_j}\sum_{\gamma\in\Gamma_j} \operatorname{tr}(\sigma\gamma^{-1}\langle\gamma s|\mathrm{e}^{zD_{\gamma_j}}|s\rangle\sigma^{-1}), \end{aligned}$$

since $\mathrm{e}^{zD_{\gamma_j}}$ is Γ-invariant. Now the σ-conjugation vanishes by taking traces, so we get

$$[\Gamma:\Gamma_j]\sum_{s\in S}\sum_{\gamma\in\Gamma_j}\operatorname{tr}(\gamma^{-1}\langle\gamma s|\mathrm{e}^{zD_{\gamma_j}}|s\rangle).$$

It follows that

$$\frac{\operatorname{tr}\mathrm{e}^{zD_{\Gamma_j}}}{[\Gamma:\Gamma_j]} - \operatorname{tr}_\Gamma \mathrm{e}^{zD} = \sum_{s\in S}\sum_{\substack{\gamma\in\Gamma_j\\ \gamma\neq 1}}\operatorname{tr}(\gamma^{-1}\langle\gamma s|\mathrm{e}^{zD_{\gamma_j}}|s\rangle).$$

In this sum we always have $\gamma s \neq s$, so the diagonal of the kernel is never met. Hence, the term of $n=0$ in the sum $\mathrm{e}^{zD} = \sum_{n\geqslant 0}(z^n/n!)D^n$ does not contribute. It follows for $|z|\leqslant 1$

$$\begin{aligned} \left|\frac{\operatorname{tr}\mathrm{e}^{zD_{\Gamma_j}}}{[\Gamma:\Gamma_j]} - \operatorname{tr}_\Gamma \mathrm{e}^{zD}\right| &\leqslant \sum_{s\in S}\sum_{\substack{\gamma\in\Gamma_j\\ \gamma\neq 1}}\sum_{n\geqslant 1}\frac{|z|^n}{n!}|\operatorname{tr}(\gamma^{-1}\langle\gamma s|D^n|s\rangle)| \\ &\leqslant |z|\sum_{s\in S}\sum_{\substack{\gamma\in\Gamma_j\\ \gamma\neq 1}}\sum_{n\geqslant 1}\frac{1}{n!}|\operatorname{tr}(\gamma^{-1}\langle\gamma s|D^n|s\rangle)|, \end{aligned}$$

giving the claim. □

To finish the proof of the theorem we have to show that $F_j(\lambda, s)$ tends to zero locally uniformly in λ when $\operatorname{Re}(\lambda) \gg 0$ and s is in a neighbourhood of zero. To this end, we split the integral into the pieces $\int_0^1$ and $\int_1^\infty$. The second one converges for all s when $\operatorname{Re}(\lambda)$ is large enough, and the first one converges for $\operatorname{Re}(s) > -1$ by the Lemma 2.8. Furthermore, Lemma 2.8 already shows the convergence in question for the $\int_0^1$ part.

We now show that the second integral tends to zero. For $\varphi \in \Gamma(\mathcal{F}_{\Gamma_j})$, let $\|\varphi\|_\infty := \sup_{y\in Y}\|\varphi(y)\|$ the sup-norm. Let $\|D_{\Gamma_j}\|_\infty$ denote the corresponding operator norm.

Lemma 2.9. *There is a constant $c > 0$ such that $\|D_{\Gamma_j}\|_\infty \leqslant c$ for all $j\in\mathbb{N}$.*

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Proof. We estimate

$$\begin{aligned}\|D_{\Gamma_j}\varphi\|_\infty &= \sup_{x\in Y}\|D_{\Gamma_j}\varphi(x)\| \\ &\leqslant \sup_{x\in Y}\sum_{y\in Y}\|\langle x|D|y\rangle\varphi(y)\| \\ &\leqslant \sup_{x\in Y}\sum_{y\in Y}\|\langle x|D|y\rangle\|\,\|\varphi(y)\|.\end{aligned}$$

Now suppose $\|\varphi\|_\infty \leqslant 1$, then we get

$$\|D_{\Gamma_j}\varphi\| \leqslant \sup_{x\in Y}\sum_{y\in Y}\|\langle x|D|y\rangle\| < \infty,$$

independent of j. □

Lemma 2.10. *There are $C_1, C_2 > 0$, such that for all $j \in \mathbb{N}$:*

$$|\operatorname{tr}_\Gamma \mathrm{e}^{zD}|, \left|\frac{\operatorname{tr}\mathrm{e}^{zD_{\Gamma_j}}}{[\Gamma:\Gamma_j]}\right| \leqslant C_1 \mathrm{e}^{|z|C_2}.$$

Proof. We only show the assertion for $|(\operatorname{tr}\mathrm{e}^{zD_{\Gamma_j}})/[\Gamma:\Gamma_j]|$ since the other one is proven analogously. Since for any operator A on a finite dimensional Euclidean space V we have $|\operatorname{tr}A| \leqslant (\dim V)\|A\|_\infty$, it follows that

$$\begin{aligned}|\operatorname{tr}\mathrm{e}^{zD_{\Gamma_j}}| &\leqslant (\#S)r[\Gamma:\Gamma_j]\|\mathrm{e}^{zD_{\Gamma_j}}\|_\infty \\ &\leqslant (\#S)r[\Gamma:\Gamma_j]\mathrm{e}^{\|zD_{\Gamma_j}\|_\infty},\end{aligned}$$

which, by Lemma 2.10, implies the claim. □

Now choose λ with $\operatorname{Re}(\lambda) > C_2$, then it follows by the Lemma 2.10 that the second integral of $F_j(\lambda, s)$ converges dominatedly independent of j. Hence, it suffices that the integrand tends to zero pointwise. This, however, is clear. The theorem is proven. □

3. Laplacians

Let $X_\Gamma = \Gamma\backslash X$ be a finite connected CW-complex with fundamental group Γ and universal covering X. Let $\mathcal{F}_\Gamma$ be a hermitian locally constant sheaf of finite-dimensional complex vector spaces and let $\mathcal{F}$ be its pullback to X. Choosing a basepoint x_0 gives a representation ρ of Γ on the stalk over x_0. For the cohomology, we have $H^p(X_\Gamma, \mathcal{F}_\Gamma) = H^p(\Gamma, \rho)$ for all $p \in \mathbb{Z}$. Let $q \geqslant 0$ and let X_q be the set of q-dimensional cells of X. We construct a discrete proper metric space (Y, d) as follows. The set Y is given by X_q and the distance $d(a, b)$ between two cells equals the minimal number of cells hit by a path joining a given point in a to a given point in b minus one. For any cell c of X, let

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$\mathcal{F}(c)$ denote the stalk of $\mathcal{F}$ at a given fixed point of c. (These points should be chosen Γ-invariantly.) This way, $\mathcal{F}$ induces a hermitian sheaf $\mathcal{F}_Y$ on Y on which Γ acts unitarily. The combinatorial Laplacian $\Delta_{q,\mathcal{F}}$ now induces an operator on $\mathcal{F}_Y$ that is Γ-invariant and of finite propagation speed. The theorem of the last section applies.

We will make use of this fact to prove a theorem on zeta functions of graphs. So we restrict to the case $\dim X = 1$, so X is a tree and X_Γ is a finite graph. We give X and X_Γ a metric by giving each edge the metric of the unit interval. It then makes sense to speak of geodesics and especially of closed geodesics in X_Γ. Let γ be a closed geodesic and x a point on its trace. Since the sheaf $\mathcal{F}$ is locally constant, parallel transport along γ gives a monodromy operator m_γ on the stalk $\mathcal{F}(x)$. We define the zeta function $Z_{\Gamma,\mathcal{F}}$ of $\mathcal{F}_\Gamma$ as

$$Z_{\Gamma,\mathcal{F}}(u) := \prod_\gamma \det(1 - u^{l(\gamma)} m_\gamma),$$

where the product runs over all primitive closed geodesics, i.e. those which are not a power of a shorter one. The definition does not depend on the choices of points x on the geodesics. In general the product will be infinite but it is easy to show convergence for $u \in \mathbb{C}$ with $|u|$ sufficiently small. In [**5**] it is shown that $Z_{\Gamma,\mathcal{F}}$ extends to a rational function, indeed a polynomial, on $\mathbb{C}$. Note that Γ, being the fundamental group of a finite graph, contains a tower.

Lemma 3.1. *Let (Γ_j) be a tower in Γ, then $Z_{\Gamma_j,\mathcal{F}}(u)^{1/[\Gamma:\Gamma_j]}$ tends to 1 for any $u \in \mathbb{C}$ with $|u|$ small.*

Proof. Let X_Γ^{b} be the barycentric subdivision of X_Γ, then X_Γ^{b} is bipartite, i.e. the vertex set of X_Γ^{b} can be written as a disjoint union $V = V_0 \cup V_1$, where vertices in V_i are only connected by edges to vertices in V_{1-i} for $i = 0, 1$. Let (Y, d) be the discrete proper metric space attached to the set of edges X_1^{b} of X^{b}. The sheaf $\mathcal{F}$ can also be considered as a locally constant sheaf on X^{b}. We will construct two operators T_0 and T_1 on $\mathcal{F}_Y$. Note first that for $a, b \in Y$, having a common vertex in X^{b}, then parallel transport gives an isomorphism $\varphi_{a,b} : \mathcal{F}(a) \to \mathcal{F}(b)$. For $v \in \mathcal{F}(a)$, let δ_v be the section of $\mathcal{F}_Y$ that maps a to v and is zero elsewhere. Then the T_i are defined by

$$T_i \delta_v := \sum_{\substack{b \in Y \\ a \sim_i b}} \delta_{\varphi_{a,b}(v)},$$

where $a \sim_i b$ means that a and b have a common vertex in V_i for $i = 0, 1$. This prescription defines two Γ-invariant operators of finite propagation speed on $\mathcal{F}_Y$. A computation shows

$$\begin{aligned} Z_{\Gamma,\mathcal{F}}(u) &= \det(1 - u(T_0 T_1)_\Gamma) \\ &= u^{2l} \det((1/u) - (T_0 T_1)_\Gamma), \end{aligned}$$

where l is the number of edges of X_Γ. Theorem 2.6 implies that

$$Z_{\Gamma_j,\mathcal{F}}(u)^{1/[\Gamma:\Gamma_j]} \to u^{2l} \det{}_\Gamma((1/u) - (T_0 T_1)_\Gamma),$$

as $j \to \infty$ when $\mathrm{Re}(1/u) \gg 0$. The operator T_0T_1 has the property that the diagonal of the kernel $\langle x|(T_0T_1)^n|x\rangle$ vanishes for any $n > 0$. This implies, by Lemma 2.4, that $\det_\Gamma((1/u) - (T_0T_1)) = \det_\Gamma(1/u) = u^{-2l}$, whence the claim. □

We will say a graph is *regular of valency* $q+1$ for $q \in \mathbb{N}$ if every edge has two distinct endpoints and every vertex is connected to $q+1$ distinct edges. To see which topological information is encoded in $Z_{\Gamma,\mathcal{F}}$ we will connect it to the combinatorial Laplacian $\Delta_{0,\mathcal{F}_\Gamma}$.

Proposition 3.2. *Let $\Delta = \Delta_{0,\mathcal{F}_\Gamma}$ and let n be the number of vertices of X_Γ. Assume X_Γ is regular of valency $q+1$ for $q \geqslant 2$. Then*

$$Z_{\Gamma,\mathcal{F}}(u) = (1-u^2)^{-\chi(\rho)} u^n \det(\Delta - (q+1-qu-(1/u))),$$

where $\chi(\rho) = \dim H^0(\Gamma,\rho) - \dim H^1(\Gamma,\rho)$ is the Euler characteristic.

Proof. In [5], it is shown that $Z_{\Gamma,\mathcal{F}}(u)$ equals

$$Z_{\Gamma,\mathcal{F}}(u) = (1-u^2)^{-\chi(\rho)} \det(1 - A_\rho u + qu^2),$$

where A_ρ is the adjacency operator of $\mathcal{F}_\Gamma$. A calculation shows that $\Delta = q+1-A_\rho$. The claim follows. □

The next theorem is the main result of this paper.

Theorem 3.3. *Let $\Delta = \Delta_{0,\mathcal{F}_\Gamma}$. For $|u|$ small we have*

$$Z_{\Gamma,\mathcal{F}}(u) = \frac{\det(\Delta + qu + (1/u) - q - 1)}{\det_\Gamma(\Delta + qu + (1/u) - q - 1)}.$$

Proof. At first, note that, since the cohomology is computed by a finite-dimensional complex, it follows that

$$\chi(\rho) = \tfrac{1}{2}(1-q)n \dim \rho,$$

where n is the number of edges on X. Next, when $\mathrm{Re}(1/u)$ tends to infinity, then so does $\mathrm{Re}(\lambda_u)$ with $\lambda_u = qu + (1/u) - q - 1$. Fix a tower (Γ_j). For $\mathrm{Re}(1/u) \gg 0$ we have that $Z_{\Gamma_j,\mathcal{F}}(u)^{1/[\Gamma:\Gamma_j]}$ tends to 1 as j tends to infinity. On the other hand, by Theorem 2.6 it follows that with $\Delta_j = \Delta_{0,\mathcal{F}_{\Gamma_j}}$ we have $\det(\Delta_j + \lambda_u)^{1/[\Gamma:\Gamma_j]}$ tends to $\det_\Gamma(\Delta + \lambda_u)$. By Proposition 3.2 we infer that

$$\det_\Gamma(\Delta + \lambda_u) = (1-u^2)^{-[(q-1)/2]\dim(\rho)n} u^{-n},$$

and thus the claim. □

Using the results of [1], the theorem can be extended to arbitrary finite graphs, where the number q has to be replaced by the operator $qf(x) = q(x)f(x)$, where $q(x)$ is the number of edges emanating from the vertex x.

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