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ON SEMIGENERIC TAMENESS AND BASE FIELD EXTENSION

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Abstract. The notions of central endolength and semigeneric tameness are introduced, and their behaviour under base field extension for finite-dimensional algebras over perfect fields are analysed. For k a perfect field, K an algebraic closure and Λ a finite-dimensional k-algebra, here there is a proof that Λ is semigenerically tame if and only if $\Lambda \otimes_k K$ is tame.

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1. Introduction. In this note, k denotes a perfect field, perhaps finite, K an algebraic closure of k, and Λ a finite-dimensional k-algebra.

For an object V with structure of k-vector space and F, a field extension of k, we denote by V^F the object $V \otimes_k F$.

In [**6**] and [**7**], the notion of generic module was introduced in order to generalize the concept of tameness, providing a deeper understanding of representation type problems of finite-dimensional algebras over an arbitrary field.

DEFINITION 1.1. For $M \in \Lambda - \mathrm{Mod}$, we denote $E_M = \mathrm{End}_\Lambda(M)^{op}$. The *endolength* of M is its length as right E_M-module and it is denoted as endol(M). We say that M is *endofinite* if endol(M) $< \infty$. We say that M is *generic* if it is endofinite, indecomposable and it has infinite dimension over k.

DEFINITION 1.2. Λ is *generically trivial* if there are no generic modules in $\Lambda - \mathrm{Mod}$ and Λ is *generically tame* if for each natural number d there is only a finite number of isomorphism classes of generic modules of endolength d in $\Lambda - \mathrm{Mod}$.

THEOREM 1.3 (Theorems 4.4 and 4.5 of [**6**]). *Let us assume that k is algebraically closed, i.e. $k = K$. Then, Λ is of finite representation type if and only if Λ is generically trivial, and Λ is tame if and only if Λ is generically tame.*

A nice way to study the case when the base field is not algebraically closed is to use base field extension, as was proposed in [**12**] and [**13**].

THEOREM 1.4 (Theorem 5.2 of [**13**] and Theorem 2.1 of [**14**]). *If Λ is generically tame, then Λ^K is generically tame.*

Whether the converse of the precedent theorem holds seems to be a quite hard problem, so here I suggest to consider a more tractable type of generic modules:

DEFINITION 1.5. Let G be a generic Λ-module. We say that G is *algebraically rigid* if the Λ^K-module G^K is generic. We say that the generic Λ-module G is *algebraically bounded* if there exists a finite field extension L/k and a natural number n such that $G^L \cong G_1 \oplus \cdots \oplus G_n$, where G_i is an algebraically rigid Λ^L-module for $i \in \{1, \ldots, n\}$.

Also it seems convenient to consider a slightly different way to measure some generic modules.

DEFINITION 1.6. For $M \in \Lambda - \mathrm{Mod}$, we denote $D_M = E_M/\mathrm{rad}(E_M)$ and by Z_M the centre of D_M. Let G be an indecomposable endofinite Λ-module, then it is known that D_G is a division ring (see Proposition 2.2(a)). If D_G is finite-dimensional over the field Z_G, then $\dim_{Z_G}(D_G) = c_G^2$ for a natural number c_G : in this case, we say that G is *centrally finite* and we define its *central endolength* as $c - \mathrm{endol}(G) = c_G \times \mathrm{endol}(G)$. Otherwise, we define $c - \mathrm{endol}(G) = \dim_{Z_G}(D_G) \times \mathrm{endol}(G)$.

DEFINITION 1.7. We say that Λ is *semigenerically tame* if for each $d \in \mathbb{N}$ there is only a finite number of isomorphism classes of algebraically bounded and centrally finite generic modules of central endolength equal to d.

The main result in this paper is the following:

THEOREM 1.8. *Let k be a perfect field, K an algebraic closure of k, and Λ a finite-dimensional k-algebra. Then, Λ is semigenerically tame if and only if Λ^K is generically tame. Moreover, if Λ is semigenerically tame, then each algebraically bounded generic Λ-module is centrally finite.*

The problem of whether semigeneric tameness is equivalent to generic tameness remains open (see 2.20).

2. Some facts about generic modules and base field extension.

It is convenient to recall some important facts.

LEMMA 2.1 (Lemma 1.1 of [**6**]). *Let $M, N \in \Lambda - \mathrm{Mod}$, then*

$$\max\{\mathrm{endol}(M), \mathrm{endol}(N)\} \le \mathrm{endol}(M \oplus N) \le \mathrm{endol}(M) + \mathrm{endol}(N).$$

If $I \neq \emptyset$, then $\mathrm{endol}(\oplus_{i\in I} M) = \mathrm{endol}(M)$.

PROPOSITION 2.2 ([**7**] and [**13**]).

- *(a) The endomorphism ring of an endofinite indecomposable Λ-module G is a local ring with nilpotent radical.*
- *(b) If $M \cong \bigoplus_{i\in I} M_i$ and G is an endofinite indecomposable module such that G is a direct summand of M, then G is a direct summand of M_i for some $i \in I$.*
- *(c) A Λ-module G is endofinite if and only if G is isomorphic to a direct sum $\bigoplus_{j=1}^{s}\left(\bigoplus_{I_j} G_j\right)$, for some natural number s, endofinite indecomposable modules G_j and sets I_j, for $j \in \{1, \ldots, s\}$. Applying Lemma 2.1 and the previous item, we get that if H is an indecomposable direct summand of G, then there exists j such that $H \cong G_j$. Moreover, by Azumaya's decomposition Theorem (12.6 of [**1**]), if $G \cong \oplus_{u\in U} N_u$, where N_u is indecomposable for each u, then (assuming $j \neq j'$ implies $I_j \cap I_{j'} = \emptyset$ and $G_j \ncong G_{j'}$) there exists a bijection $\sigma : U \to I_1 \cup \cdots \cup I_s$ such that $N_u \cong G_j$ if and only if $\sigma(u) \in I_j$.*

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LEMMA 2.3 (Lemma 2.5 of [**12**] and Lemmas 3.2 and 3.3 of [**13**]). *Let M and N be Λ-modules and let L/k be a field extension.*

(a) The natural map $\alpha : \mathrm{Hom}_\Lambda(M,N)^L \to \mathrm{Hom}_{\Lambda^L}(M^L,N^L)$ *is a monomorphism. If $[L:k] < +\infty$ then α is an isomorphism. If $M = N$, then α is a morphism of L-algebras.*

(b) If $[L:k] < +\infty$ and M is endofinite, then M^L is endofinite and

$$\mathrm{endol}(M) \le \mathrm{endol}(M^L) \le [L:k]\,\mathrm{endol}(M).$$

(c) If M is endofinite and indecomposable, and M^L and N^L have a common direct summand, then M is a direct summand of N.

REMARK 2.4. The injectivity in Lemma 2.3(a) can be obtained through the proof of Lemma 3.2 of [**13**]. The proof of 2.3(c) for M generic is the same of Lemma 3.3 (b) of [**13**].

REMARK 2.5. Let L/k be a field extension. Let $\xi : \Lambda^L - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ be the restriction functor of [**13**] and $(\_)^L : \Lambda - \mathrm{Mod} \to \Lambda^L - \mathrm{Mod}$ the scalar extension functor. By Lemma 3.1 of [**13**], the functor ξ is right adjoint to $(\_)^L$. Let us observe that $(\_)^L$ is naturally equivalent to the functor $\Lambda^L \otimes_\Lambda \_$, when we consider the canonical structure of $\Lambda^L - \Lambda$-bimodule of Λ^L, and ξ is naturally equivalent to the functor $\Lambda^L \otimes_{\Lambda^L} \_$, when we consider the canonical structure of $\Lambda - \Lambda^L$-bimodule of Λ^L.

LEMMA 2.6 (Lemma 31.4 of [**5**]). *Let Δ_1 and Δ_2 be k-algebras and let B be a $\Delta_1 - \Delta_2$-bimodule such that it is free of finite rank m as right Δ_2-module. Then, for any $M \in \Delta_2 - \mathrm{Mod}$ we have that*

$$\mathrm{endol}(B \otimes_{\Delta_2} M) \le m \times \mathrm{endol}(M).$$

If the functor $B \otimes_{\Delta_2} \_ : \Delta_2 - \mathrm{Mod} \to \Delta_1 - \mathrm{Mod}$ is full, then the equality holds.

Proof. The first part of the statement is Lemma 31.4 of [**5**], and the second part of the statement follows easily from the proof given in [**5**].

□

LEMMA 2.7 (Lemma 3.4 of [**13**]). *Let L/k be an arbitrary field extension. For any Λ^L-module M the endolength of $\xi(M)$ is less than or equal to the endolength of M.*

COROLLARY 2.8. *Let L be an intermediate field of an arbitrary field extension F/k. Then for any Λ-module M we have that* $\mathrm{endol}(M^L) \le \mathrm{endol}(M^F)$.

Proof. Let $\xi : \Lambda^F - \mathrm{Mod} \to \Lambda^L - \mathrm{Mod}$ be the restriction functor. By Lemma 2.7, we get $\mathrm{endol}(M^F) \ge \mathrm{endol}(\xi(M^F))$. It is easy to see that $\xi(M^F) \cong \oplus_{i\in I} M^L$, where the cardinality of I is $[F:L]$, and by Lemma 2.1 we have $\mathrm{endol}(\oplus_{i\in I} M^L) = \mathrm{endol}(M^L)$, so $\mathrm{endol}(M^F) \ge \mathrm{endol}(M^L)$.

□

REMARK 2.9. Let G be an algebraically rigid Λ-module and let L be an intermediate field of K/k. Since $(G^L)^K \cong G^K$ we get that G^L is an indecomposable Λ^L-module. Also it is known that $\dim_k(G) = \dim_L(G^L)$. By Corollary 2.8, we have that $\mathrm{endol}(G) \le \mathrm{endol}(G^L) \le \mathrm{endol}(G^K)$. It follows that G^L is a generic Λ^L-module.

REMARK 2.10. Let $\eta_1 : L \to L'$ be an isomorphism of k-algebras, and K/L and K'/L' algebraic field extensions such that K and K' are algebraically closed. Recall that there exists an isomorphism of k-algebras $\eta_2 : K \to K'$ such that $(\eta_2)_{|L} = \eta_1$. Then, there are induced isomorphisms of k-categories $\mathcal{F}_{1\otimes\eta_1} : \Lambda^L - \text{Mod} \to \Lambda^{L'} - \text{Mod}$ and $\mathcal{F}_{1\otimes\eta_2} : \Lambda^K - \text{Mod} \to \Lambda^{K'} - \text{Mod}$. Thus, we have, by Remark 2.9, the following equivalent definition: a Λ-module G is algebraically rigid if G^L is generic for any algebraic field extension L/k. The argument of Remark 2.9 also exhibits that we can drop out the assumption of genericity for G in Definition 1.5, for algebraic rigidness, and substitute it for indecomposability, in the case of algebraic boundedness.

In this note, a Galois field extension F/k means a normal separable finite field extension F of k.

LEMMA 2.11. *Let F/k be a Galois field extension and L an intermediate field with $n = [L : k]$.*

(a) There is an isomorphism of F-algebras $h_1 : L \otimes_k F \to \times_{i=1}^n F$.
(b) There is an isomorphism of L-algebras $h_2 : L \otimes_k F \to \times_{i=1}^n F$.

Proof. L is separable over k, then it is a simple field extension over k, i.e. $L = k(a)$. Let p be the irreducible monic polynomial of a over k, and $p = \prod_{i=1}^n (x - r_i)$ its factorization on $F[x]$. It is known that we can choose elements of the Galois group $\text{Gal}(F/k)$, namely $\sigma_1, \sigma_2, \ldots, \sigma_n$, such that $\sigma_i(a) = r_i$. Then, there is a k-linear transformation $h_1 : L \otimes_k F \longrightarrow F \times \cdots \times F$, determined by $h_1(l \otimes f) = (\sigma_1(l)f, \ldots, \sigma_n(l)f)$, where $l \in L$ and $f \in F$, and it is easy to verify that h_1 satisfies the first item.

The composition $h_2 = \left(\times_{i=1}^n \sigma_i^{-1}\right) h_1 = L \otimes_k F \to F \times \cdots \times F$, given in homogeneous elements by $h_2(l \otimes f) = \left(l\sigma_1^{-1}(f), \ldots, l\sigma_n^{-1}(f)\right)$, fulfils the second item. □

REMARK 2.12. In the context of Lemma 2.11 and its proof, notice that there are precisely n isomorphism classes of indecomposable $L - F$-bimodules, being $\{F^{\sigma_1}, \ldots, F^{\sigma_n}\}$ a complete set of representatives, where F^{σ_i} is F with its natural structure of right F-module and with the structure of left L-module given by the composition $L \overset{j}{\to} F \overset{\sigma_i}{\to} F$, where j is the inclusion.

If we set $a = r_1$, we get $F = F^{\sigma_1}$ as $L - F$-bimodules.

Also we observe that we can define the $L - F$-bimodule F_{σ_i}, which is F as left L-module and with the structure of right F-module determined by $1 \cdot f = \sigma_i^{-1}(f)$: then $(\sigma_i)^{-1}$ induces an isomorphism of $L - F$-bimodules between F^{σ_i} and F_{σ_i}.

It is easy to see that $h_1 : L \otimes_k F \to \times_{i=1}^n F^{\sigma_i}$ and $h_2 : L \otimes_k F \to \times_{i=1}^n F_{\sigma_i}$ are isomorphisms of $L - F$-bimodules.

LEMMA 2.13. *Let F/k be a Galois field extension, $\xi : \Lambda^F - \text{Mod} \to \Lambda - \text{Mod}$ the restriction functor, $F^{\sigma_1}, \ldots, F^{\sigma_n}$ as in Remark 2.12, and $M \in \Lambda^F - \text{Mod}$. Then, $\xi(M)^F \cong \oplus_{i=1}^n (\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M$ and M is a direct summand of $\xi(M)^F$. Moreover, M is (indecomposable, generic, centrally finite, algebraically rigid, algebraically bounded) if and only if $(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M$ is (respectively indecomposable, generic, centrally finite, algebraically rigid, algebraically bounded) for each i. Also $\dim_F(M) = \dim_F((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)$ and $\text{endol}(M) = \text{endol}((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)$ for each i. If M is indecomposable and endofinite, then $c - \text{endol}(M) = c - \text{endol}((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)$ for each i.*

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Proof. The first part of the claim follows by Remarks 2.5 and 2.12, i.e. we have isomorphisms of $\Lambda \otimes_k F$-modules $\xi(M)^F \cong \Lambda^F \otimes_\Lambda \Lambda^F \otimes_{\Lambda^F} M \cong ((\Lambda \otimes_\Lambda \Lambda) \otimes_k (F \otimes_k F)) \otimes_{\Lambda^F} M \cong \oplus_{i=1}^n (\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M$.

It is easy to see that for $i \in \{1, \ldots, n\}$ there exists $i' \in \{1, \ldots, n\}$ such that $F^{\sigma_i} \otimes_F F^{\sigma_{i'}} \cong F$ as $F - F$-bimodules, so $(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} (\Lambda \otimes_k F^{\sigma_{i'}}) \cong (\Lambda \otimes_\Lambda \Lambda) \otimes_k (F^{\sigma_i} \otimes_F F^{\sigma_{i'}}) \cong \Lambda^F$ as $\Lambda^F - \Lambda^F$-bimodule.

Then, the functor $(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} \_ : \Lambda^F - \mathrm{Mod} \to \Lambda^F - \mathrm{Mod}$ is an equivalence of k-categories, so M is indecomposable if and only if $(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M$ is indecomposable for each i.

Since $\Lambda \otimes_k F^{\sigma_i} \cong \Lambda^F$ as right Λ^F-modules, by Lemma 2.6 we get $\mathrm{endol}(M) = \mathrm{endol}((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)$, for each i.

Since $\dim_F(M) = \dim_F((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)$, we get that M is generic if and only if $(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M$ is generic for each i.

By the previous equivalence of k-categories it follows, when M is indecomposable and endofinite, that $\dim_{Z_M}(D_M) = \dim_{Z_{(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M}} \left(D_{(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M}\right)$ for each i.

Using the canonical isomorphism

$$((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)^K \cong (\Lambda \otimes_k F^{\sigma_i} \otimes_F K) \otimes_{\Lambda^K} M^K$$

we can develop an argument similar to the above one and conclude that $((\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M)^K$ is generic if and only if M^K is generic. Then, M is algebraically rigid if and only if $(\Lambda \otimes_k F^{\sigma_i}) \otimes_{\Lambda^F} M$ is algebraically rigid for each i.

By a similar argument, and the additivity of the tensor product, we can verify the part of the statement about algebraic boundedness.

□

Proposition 2.14. *Let L/k be a finite field extension and G an endofinite indecomposable Λ-module. Then:*

(a) $G^L \cong G_1 \oplus \cdots \oplus G_m$, where G_i is an endofinite indecomposable Λ^L-module for $i \in \{1, \ldots, m\}$. Let F/k be a Galois field extension with L an intermediate field, then $m \leq [F : k]$.

(b) G is a direct summand of $\xi(G_i)$, for each i, where $\xi : \Lambda^L - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ is the restriction functor.

(c) G is (generic, algebraically bounded) if and only if G_i is (respectively generic, algebraically bounded) for each i, if and only if G_i is (respectively generic, algebraically bounded) for some i.

(d) If L/k is a Galois field extension, then $\mathrm{endol}(G_1) = \cdots = \mathrm{endol}(G_m)$, $\mathrm{c} - \mathrm{endol}(G_1) = \cdots = \mathrm{c} - \mathrm{endol}(G_m)$ and $D_{G_1} \cong \cdots \cong D_{G_m}$ as k-algebras.

Proof. Let F/k be as in item (a). By Lemma 2.3(b) and Proposition 2.2(c), there exists an endofinite indecomposable direct summand H of G^F.

Let $\xi' : \Lambda^F - \mathrm{Mod} \to \Lambda^L - \mathrm{Mod}$ and $\xi_1 : \Lambda^F - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ be the respective restriction functors.

By Lemma 2.7 and Proposition 2.2(c), we get $\xi_1(H) \cong \bigoplus_{j=1}^s \left(\bigoplus_{I_j} M_j\right)$, where each M_j is indecomposable and endofinite.

By Lemma 2.13, we get that H is a direct summand of $\xi_1(H)^F$ then, by Lemma 2.3(b) and Proposition 2.2(c), there exists $j_0 \in \{1, \ldots, s\}$ such that H is a direct summand of $M_{j_0}^F$. By Lemma 2.3(c), we get $M_{j_0} \cong G$.

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Then, G^F is a direct summand of $\xi_1(H)^F$ and so, by Lemma 2.13 and Proposition 2.2(c), G^F is a finite direct sum of endofinite indecomposable Λ^F-modules. It follows that $G^L \cong G_1 \oplus \cdots \oplus G_m$, where G_i is indecomposable and endofinite for each i, and $m \leq [F : k]$. Moreover, if H_i and $H_{i'}$ are indecomposable direct summands, respectively, of G_i^F and $G_{i'}^F$, then $\mathrm{endol}(H_i) = \mathrm{endol}(H) = \mathrm{endol}(H_{i'})$, $\mathrm{c-endol}(H_i) = \mathrm{c-endol}(H) = \mathrm{c-endol}(H_{i'})$ and $\dim_F(H_i) = \dim_F(H) = \dim_F(H_{i'})$.

It follows that $\dim_F(H) \leq \dim_k(G) \leq [F : k] \times \dim_F(H)$ and $\dim_F(H) \leq \dim_L(G_i) \leq [F : k] \times \dim_F(H)$ for each i, then G is generic if and only if H is generic, if and only if G_i is generic for each i.

Also, using the equivalence of categories of the proof of Lemma 2.13, we get $D_{H_i} \cong D_{H_{i'}}$ as k-algebras.

Let us fix $i \in \{1, \ldots, m\}$, and assume that H_i is an indecomposable direct summand of G_i^F. By the previous argument, G is a direct summand of $\xi_1(H_i)$ and of $\xi_1\left(G_i^F\right)$, and we have $\xi_1\left(G_i^F\right) = \xi\xi'\left(G_i^F\right) \cong \xi\left(\oplus_{s=1}^{[F:L]} G_i\right) \cong \oplus_{s=1}^{[F:L]} \xi(G_i)$: by Lemmas 2.7 and 2.1 and Proposition 2.2(c), it follows that G is a direct summand of $\xi(G_i)$ for each i.

It is easy to verify that G algebraically bounded implies G_i algebraically bounded for each i.

Now let us assume that G_i is algebraically bounded for some $i \in \{1, \ldots, m\}$, and notice that we can choose F such that the indecomposable direct summand H_i of G_i^F is algebraically rigid. Also we have seen that G^F is a direct summand of $\xi_1(H_i)^F$ and so, by Lemma 2.13 and Proposition 2.2(c), G is algebraically bounded.

□

Proposition 2.15. *Let L/k be a finite field extension, H an endofinite indecomposable Λ^L-module, and $\xi : \Lambda^L - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ the restriction functor.*

- *(a) $\xi(H) \cong G_1 \oplus \cdots \oplus G_m$, where G_i is an endofinite indecomposable $\Lambda-$module for $i \in \{1, \ldots, m\}$. Let F/k be a Galois field extension with L an intermediate field, then $m \leq [F : k]$.*
- *(b) There exists $i_0 \in \{1, \ldots, m\}$ such that H is a direct summand of $(G_{i_0})^L$.*
- *(c) H is (generic, algebraically bounded) if and only if G_i is (respectively generic, algebraically bounded) for each i, if and only if G_i is (respectively generic, algebraically bounded) for some i.*

Proof. Let F/k be as in item (a). By Proposition 2.14, we get $H^F \cong H_1 \oplus \cdots \oplus H_n$, where H_j is indecomposable and endofinite for each j, and $n \leq [F : L]$.

Let $\xi_1 : \Lambda^F - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ and $\xi' : \Lambda^F - \mathrm{Mod} \to \Lambda^L - \mathrm{Mod}$ be the respective restriction functors.

By Lemma 2.13, we get that $\xi_1\left(H^F\right)^F$ is a finite direct sum of $n \times [F : k]$ endofinite indecomposable Λ^F-modules, thus $\xi_1\left(H^F\right) = \xi\xi'\left(H^F\right) \cong \oplus_{s=1}^{[F:L]} \xi(H)$ and $\xi(H)$ are finite direct sums of endofinite indecomposable Λ -modules, then we obtain (a).

By Proposition 2.14 and Lemma 2.13, we get that H is (generic, algebraically bounded) if and only if each direct summand of $\xi_1\left(H^F\right)^F$ is (respectively generic, algebraically bounded): the last item of the claim follows applying this and Proposition 2.14 to the isomorphism $\xi_1\left(H^F\right)^F \cong \oplus_{s=1}^{[F:L]}\left(G_1^F \oplus \cdots \oplus G_m^F\right)$.

□

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LEMMA 2.16. *Let G be an endofinite indecomposable Λ-module and L a finite field extension of k. The isomorphism $\alpha : (E_G)^L \to E_{G^L}$ of 2.3(a) induces an isomorphism of L-algebras $(D_G)^L \cong D_{G^L}$.*

Proof. It is clear that α induces an isomorphism of L-algebras

$$\overline{\alpha} : (E_G)^L / \mathrm{rad}\left((E_G)^L\right) \to E_{G^L}/\mathrm{rad}\,(E_{G^L}) = D_{G^L}.$$

Since k is perfect, from Theorem 2.5.36 of [**15**], we have that $(\mathrm{rad}\,(E_G))^L = \mathrm{rad}\left((E_G)^L\right)$, then

$$(D_G)^L = (E_G/\mathrm{rad}\,(E_G))^L \cong (E_G)^L / (\mathrm{rad}\,(E_G))^L = (E_G)^L / \mathrm{rad}\left((E_G)^L\right) \cong D_{G^L}.$$

□

LEMMA 2.17. *Let G be an endofinite indecomposable Λ-module. Let L be a finite field extension of k. By Proposition 2.14, we have that $G^L \cong m_1G_1 \oplus \cdots \oplus m_tG_t$, where $m_1, m_2, \ldots, m_t \in \mathbb{N}$ and $G_1, \ldots, G_t$ are pairwise non-isomorphic endofinite indecomposable Λ^L-modules. Then, we have isomorphisms of $L-$algebras*

$$\begin{aligned} D_{G^L} &\cong \mathrm{End}_{\Lambda^L}(m_1G_1 \oplus \cdots \oplus m_tG_t)^{op} / \mathrm{rad}\,(\mathrm{End}_{\Lambda^L}(m_1G_1 \oplus \cdots \oplus m_tG_t)^{op}) \\ &\cong M_{m_1}(D_{G_1}) \times \cdots \times M_{m_t}(D_{G_t}). \end{aligned}$$

Proof. The isomorphisms follow from the usual description of the radical of an endomorphism algebra of a finite direct sum of modules with local endomorphism algebras (use Proposition 2.2(a)). □

PROPOSITION 2.18. *Let G be an endofinite indecomposable Λ-module, L/k a finite field extension and $G^L \cong m_1G_1 \oplus \cdots \oplus m_tG_t$, where $m_1, m_2, \ldots, m_t \in \mathbb{N}$ and $G_1, \ldots, G_t$ are pairwise non-isomorphic endofinite indecomposable Λ^L-modules. Then:*

(*a*) $\mathrm{endol}\left(G_j\right) = \mathrm{endol}\,(G) \times m_j$ *for* $j \in \{1, \ldots, t\}$. *If L/k is a Galois field extension, then $m_1 = \ldots = m_t$.*
(*b*) $\mathrm{c-endol}\left(G_j\right) = \mathrm{c-endol}\,(G)$ *for each j. Moreover, G is centrally finite if and only if there exists $j \in \{1, \ldots, t\}$ such that G_j is centrally finite.*

Proof. For $j \in \{1, \ldots, t\}$ consider the idempotent e_j of E_{G^L} induced by one of the copies of G_j, i.e. given a monomorphism $\sigma_j : G_j \to G^L$ and an epimorphism $\pi_j : G^L \to G_j$ such that $\pi_j\sigma_j$ is the identity on G_j, we set $e_j = \sigma_j\pi_j$. Notice that $\sigma : G_j \to Ge_j$ is an isomorphism of Λ^L-modules, so $\mathrm{endol}\left(G_j\right) = \mathrm{endol}\left(G^Le_j\right)$.

It is immediate that $E_{G^Le_j} = e_jE_{G^L}e_j$, and so $\mathrm{endol}\left(G^Le_j\right)$ is its length as right $e_jE_{G^L}e_j$-module.

Let $\{0\} = W_0 \subset W_1 \subset \ldots \subset W_u = G$ be a composition series for G as right E_G-module, and observe that $W_{q+1}/W_q \cong D_G$ for $q \in \{0, \ldots, u-1\}$.

It is clear that G^L and W_q^L, for each q, are $(E_G)^L$-modules and E_{G^L}-modules. Also, we have $W_{q+1}^L/W_q^L \cong \left(W_{q+1}/W_q\right)^L \cong (D_G)^L$. The above isomorphism composed with $\overline{\alpha} : (D_G)^L \to D_{G^L}$ gives an isomorphism of E_{G^L}-modules $W_{q+1}^L/W_q^L \cong D_{G^L}$. We also have an isomorphism of $e_jE_{G^L}e_j$-modules $W_{q+1}^Le_j/W_q^Le_j \cong D_{G^L}e_j$.

By Lemma 2.17, we get that $\mathrm{length}_{e_jE_{G^L}e_j}(D_{G^L}) = m_j$: it follows that $\mathrm{endol}\left(G^Le_j\right) = m_ju$.

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By Proposition 2.14, we get, when L/k is a Galois field extension, that $\operatorname{endol}(G_1) = \ldots = \operatorname{endol}(G_t)$, and so $m_1 = m_2 = \cdots = m_t$.

For item (b), we recall (Corollary 1.7.24 of [**15**]) that the centre of $(D_G)^L$ is $(Z_G)^L$, and by Lemmas 2.16 and 2.17 we have $(D_G)^L \cong M_{m_1}(D_{G_1}) \times \cdots \times M_{m_t}(D_{G_t})$, so $(Z_G)^L \cong Z_{G_1} \times \cdots \times Z_{G_t}$ as L-algebras.

It follows that $1 \otimes 1 = e'_1 + \ldots + e'_t$, where $\left\{e'_j\right\}_{j \in \{1,\ldots,t\}}$ is a set of primitive orthogonal idempotents contained in $(Z_G)^L$, thus $(D_G)^L e'_j$ is a $(Z_G)^L e'_j$-vector space with the same dimension that the Z_G-vector space D_G, i.e. $\dim_{Z_G}(D_G) = m_j^2 \times \dim_{Z_{G_j}}\left(D_{G_j}\right)$ for each j.

Then, $c - \operatorname{endol}(G) = \operatorname{endol}(G) \times \sqrt{\dim_{Z_G}(D_G)} = \operatorname{endol}(G) \times m_j \times \sqrt{\dim_{Z_{G_j}}\left(D_{G_j}\right)} = \operatorname{endol}\left(G_j\right) \times \sqrt{\dim_{Z_{G_j}}\left(D_{G_j}\right)} = c - \operatorname{endol}\left(G_j\right)$ for each j.

Now the last part of the item (b) is immediate.

□

Corollary 2.19. *Let L/k be a finite field extension. Then, Λ is semigenerically tame if and only if Λ^L is semigenerically tame.*

Proof. Let G and G' be algebraically bounded and centrally finite Λ-modules such that $G \ncong G'$ and $c - \operatorname{endol}(G) = c - \operatorname{endol}(G')$. By Propositions 2.14 and 2.18, there exist algebraically bounded Λ^L-modules H and H' such that H is a direct summand of G^L, H' is a direct summand of $(G')^L$, and $c - \operatorname{endol}(H) = c - \operatorname{endol}(H')$. By Lemma 2.3(c), we get that $H \ncong H'$: it follows that Λ not semigenerically tame implies Λ^L not semigenerically tame.

Now, let H be an algebraically bounded centrally finite Λ^L-module. By Proposition 2.15, there exists an algebraically bounded Λ-module G such that H is a direct summand of G^L. By Proposition 2.18, we get $c - \operatorname{endol}(G) = c - \operatorname{endol}(H)$. By Proposition 2.14, we know that G^L has a finite number of isomorphism classes of indecomposable direct summands: it follows that Λ semigenerically tame implies Λ^L semigenerically tame.

□

The next corollary provides an example of a situation where generic tameness coincides with semigeneric tameness.

Corollary 2.20. *Assume that K/k is a finite field extension and Λ^K is tame. Let G be a generic Λ-module, then G is algebraically bounded and centrally finite.*

Proof. The case $k = K$ is immediate from Theorem 4.6 of [**6**].

If $k \subsetneq K$, by Theorem 17 VI Section 11 of [**11**], the field k is real closed and $K = k\left(\sqrt{-1}\right)$, so $[K : k] = 2$.

In this case, G is algebraically bounded by Proposition 2.14.

Let H be an indecomposable direct summand of G^K. Then, H is generic, by Proposition 2.14, and so H is centrally finite by Theorem 4.6 of [**6**]: by Proposition 2.18 (b) we get that G is centrally finite.

□

The next results exhibit special features associated to algebraically bounded and algebraically rigid modules.

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LEMMA 2.21. *Let G be an algebraically bounded Λ-module and A_G the field of the algebraic elements of Z_G. Let L/k be a finite field extension such that $G^L \cong m_1G_1 \oplus \cdots \oplus m_tG_t$, where $m_1, \ldots, m_t \in \mathbb{N}$ and $G_1, \ldots, G_t$ are pairwise non-isomorphic algebraically rigid Λ^L-modules, then $[A_G : k] = t$.*

Proof. Let Z_0 be a subfield of A_G such that $[Z_0 : k] < \infty$. Let F/Z_0 be a field extension such that F/k is a Galois field and L can be identified with an intermediate field of F/k. (Recall Remark 2.10.)

Applying Lemma 2.11, we have $Z_0 \otimes_k F \cong F \times \cdots \times F$, and so $Z_G \otimes_k F \cong Z_G \otimes_{Z_0} Z_0 \otimes_k F \cong Z_G \otimes_{Z_0} (F \times \cdots \times F) \cong \times_{i=1}^{s} Z_G \otimes_{Z_0} F$, where $s = [Z_0 : k]$.

By Lemma 2.16, we can embed $(Z_G)^F$ in Z_{G^F}, and so there are at least s non-trivial central orthogonal idempotents in D_{G^F} : by Lemma 2.17 and Proposition 7.8 of [**1**], we get $s \leq t$. It follows that $[A_G : k] \leq t$.

Now let F/A_G be a field extension such that F/k is a Galois field extension and L can be identified with an intermediate field of F/k, so we get that $(Z_G)^F \cong \times_{i=1}^{s} Z_G \otimes_{A_G} F$, where $s = [A_G : k]$.

By Theorem 21.2 IV Section 10 of [**11**], we have that $Z_G \otimes_{A_G} F$ is a field: by Lemma 2.17 applied to G^F, and Proposition 7.8 of [**1**], it follows that $s = t$.

□

PROPOSITION 2.22. *Let L/k be an algebraic field extension and G an algebraically rigid Λ^L-module. Then, the morphism of K-algebras $\alpha : (E_G)^K \to E_{G^K}$ induces an injection $\overline{\alpha} : (D_G)^K \to D_{G^K}$.*

Proof. By Lemma 2.3(a), there is a canonical monomorphism $\alpha : (E_G)^K \to E_{G^K}$, and by Proposition 2.2(a) we get that $\operatorname{rad}(E_G)^K$ is nilpotent, so $\alpha\left(\operatorname{rad}(E_G)^K\right) \subset \operatorname{rad}(E_{G^K})$ and α induces a morphism of K-algebras $\overline{\alpha} : (D_G)^K \to D_{G^K}$.

Now, let A_G be the subfield of the algebraic elements of Z_G : by Lemma 2.21 we get $A_G = L$.

Then, by Theorem 21.2 IV Section 10 of [**11**], $(Z_G)^K$ is a field.

By Corollary 1.7.24 of [**15**], the centre of $(D_G)^K$ is $(Z_G)^K$. Now, consider the canonical isomorphism $(D_G)^K \cong D_G \otimes_{Z_G} (Z_G)^K$: by Theorem 1.7.27 of [**15**], we get that $(D_G)^K$ is a simple ring. It follows that $\overline{\alpha}$ is injective.

□

3. Tame case. We recall some known facts, in order to have tools for the proof of Theorem 3.2.

A ring morphism $\eta : R \to S$ induces by restriction a faithful functor $\mathcal{F}_\eta : S-$ Mod $\to R-$ Mod. By Silver's Theorem $\mathcal{F}_\eta$ is full if and only if η is an epimorphism (see [**16**]).

Let Δ be an arbitrary k-algebra. Then:

(1) For any morphism of k-algebras $\eta : \Lambda \to M_n(\Delta)$, we can consider a $\Lambda - \Delta$-bimodule ${}_\eta M = \Delta^n$, where Δ acts by the right canonically and Λ acts by the left by $\lambda \cdot v = \eta(\lambda) v$. Clearly, k acts centrally on ${}_\eta M$.

(2) Now assume that M is a $\Lambda - \Delta$-bimodule, where k acts centrally, and $\tau : M \to \Delta^n$ is an isomorphism of right Δ-modules. Then, we can transfer the Λ-module structure of M to Δ^n in the canonical way, i.e. defining $\lambda \cdot v = \tau\left(\lambda\tau^{-1}(v)\right)$. Notice that now τ is an isomorphism of $\Lambda - \Delta$-bimodules.

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Moreover, there is induced a morphism of k-algebras $\psi : \Lambda \to M_n(\Delta)$ such that $\lambda \mapsto L_\lambda$, where $L_\lambda : \Delta^n \to \Delta^n$ denotes the action of λ, induced by τ, on the $\Lambda - \Delta$-bimodule Δ^n.

(3) Given a $\Lambda - \Delta$-bimodule M we have the right multiplication morphism $\mu : \Delta \to E_M$ given by $\delta \mapsto \mu_\delta$, where $\mu_\delta : M \to M$ denotes right multiplication by δ. If M is free by the right, then μ is injective.

(4) If M is a Λ-module such that $E_M = \Delta \oplus \mathrm{rad}(E_M)$, where Δ is a subalgebra of E_M, then the inclusion map $\Delta \to E_M$ coincides with the right multiplication morphism $\mu : \Delta \to E_M$ described above. In particular, $\Delta = \mathrm{Im}\mu$.

With the previous ideas, it is easy to prove the next claim.

LEMMA 3.1. *Let $G \in \Lambda - \mathrm{Mod}$ be a generic module such that its endomorphisms ring is split over its radical, i.e. $E_G = D \oplus \mathrm{rad}(E_G)$ as k-vector spaces, where D is a subalgebra of E_G and a division k-algebra. Thus, G is a $\Lambda - D$-bimodule and there is associated a morphism of k-algebras $\eta : \Lambda \to M_n(D)$, where $n = \mathrm{endol}(G)$. Moreover, for the induced restriction functor $F_\eta : M_n(D) - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ we get $\mathcal{F}_\eta\left(\mathrm{End}_{M_n(D)}(G)\right) = D^{op}$.*

THEOREM 3.2. *Assume that Λ^K is tame. Let G be an algebraically bounded generic Λ-module. Then, there exists a Galois field extension F/k such that $G^F \cong G_1 \oplus \cdots \oplus G_n$ and for any intermediate field Z of K/F and $i \in \{1, \ldots, n\}$, we have:*

(a) G_i^Z is an algebraically rigid Λ^Z-module;
(b) $E_{G_i^Z} = D_i \oplus \mathrm{rad}\left(E_{G_i^Z}\right)$, where $D_i \cong Z(x)$;
(c) G_i is centrally finite and $\mathrm{c} - \mathrm{endol}(G) = \mathrm{c} - \mathrm{endol}\left(G_i^Z\right) = \dim_{Z(x)}\left(G_i^Z\right) = \mathrm{endol}\left(G_i^K\right)$.

It follows that Λ is semigenerically tame.

Proof. Let L/k be a finite field extension such that $G^L \cong H_1 \oplus \cdots \oplus H_n$, where H_i is an algebraically rigid generic Λ^L-module for $i \in \{1, \ldots, n\}$.

Let us fix i for the following argument.

By definition H_i^K is a generic Λ^K-module. By Theorem 4.6 of [**6**] the K-algebra $E_{H_i^K}$ is split over its radical, where $E_{H_i^K} = D \oplus \mathrm{rad}\left(E_{H_i^K}\right)$ and $D \cong K(x)$.

H_i^K has a structure of $\Lambda^K - K(x)$-bimodule and $\mathrm{endol}\left(H_i^K\right) = \dim_{K(x)}\left(H_i^K\right) = d_i$, for some natural number d_i.

By Lemma 3.1, this structure of $\Lambda^K - K(x)$-bimodule determines a morphism of K-algebras $\psi : \Lambda^K \to M_{d_i}(K(x))$.

Then, there exists a finite field extension F_i/k and a morphism of F_i–algebras $\phi : \Lambda^{F_i} \to M_{d_i}(F_i(x))$ such that the following diagram commutes:

$$\begin{array}{ccc}
\Lambda^{F_i} \otimes_{F_i} K & \xrightarrow{\phi \otimes 1_K} & M_{d_i}(F_i(x)) \otimes_{F_i} K \\
 & & \downarrow \eta_1 \\
\downarrow \cong & & M_{d_i}(F_i(x) \otimes_{F_i} K) \\
 & & \downarrow \eta_2 \\
\Lambda^K & \xrightarrow{\psi} & M_{d_i}(K(x)),
\end{array}$$

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where η_1 and η_2 are the canonical isomorphisms: we recall that the canonical morphism of K-algebras $F_i(x)^K \cong K(x)$ is an isomorphism (see Lemma 5.1 of [**13**]).

It follows that associated to ϕ there is a $\Lambda^{F_i} - F_i(x)$-bimodule, denoted by $\underline{G}_i$, such that $\underline{G}_i^K \cong H_i^K$: by Remark 2.10 we get that $\underline{G}_i$ is an algebraically rigid Λ^{F_i}-module.

Observe that $F_i(x) \cong \mathrm{End}_{M_{d_i}(F_i(x))}\left(F_i(x)^{d_i}\right) \cong \mathrm{End}_{M_{d_i}(F_i(x))}\left(\underline{G}_i\right)$, and that the restriction functor $\mathcal{F}_\phi$ identifies $\mathrm{End}_{M_{d_i}(F_i(x))}\left(\underline{G}_i\right)$ with a subalgebra D_i of $E_{\underline{G}_i}$, so $\mathrm{endol}\left(\underline{G}_i\right) \leq d_i$.

Let $\pi : E_{\underline{G}_i} \to D_{\underline{G}_i}$ be the canonical epimorphism, and $\overline{\alpha} : \left(D_{\underline{G}_i}\right)^K \to D_{\underline{G}_i^K}$ the injection of Proposition 2.22. It is not hard to verify that $K(x) \cong \overline{\alpha}\left(\pi\left(D_i\right)^K\right) = D_{\underline{G}_i^K}$, thus $D_{\underline{G}_i} = \pi(D_i) \cong F_i(x)$ and $\underline{G}_i$ is centrally finite.

Moreover, $E_{\underline{G}_i} = D_i \oplus \mathrm{rad}\left(E_{\underline{G}_i}\right)$ and $\mathrm{c-endol}\left(\underline{G}_i\right) = \mathrm{endol}\left(\underline{G}_i\right) = d_i$.

We have a similar argument for $\underline{G}_i^Z$, where Z is an intermediate field of K/F_i, so $\underline{G}_i^Z$ is an algebraically rigid Λ^Z-module, such that $E_{\underline{G}_i^Z} \cong Z(x) \oplus \mathrm{rad}\left(E_{\underline{G}_i^Z}\right)$ and $d_i = \mathrm{endol}\left(\underline{G}_i^Z\right) = \mathrm{c-endol}\left(\underline{G}_i^Z\right) = \dim_{Z(x)}\left(\underline{G}_i^Z\right)$.

Now, we choose a field extension F/L such that F/k is a Galois field extension and we can identify each F_i with an intermediate field of F/k, and let be $G_i = \underline{G}_i^F$ for each i.

By construction $\underline{G}_i^F \cong H_i^F$ for all i, then $G^F \cong H_1^F \oplus \cdots \oplus H_n^F \cong G_1 \oplus \cdots \oplus G_n$. By Proposition 2.18, we get that G is centrally finite and $\mathrm{c-endol}(G) = \mathrm{c-endol}(G_i) = d_i$, for each i.

Now we only need to apply Theorem 1.3 and Lemma 2.3(c) to get that Λ^K tame implies Λ semigenerically tame.

□

REMARK 3.3. Let Λ^K be tame, L/k an algebraic field extension and G an algebraically rigid Λ^L-module. Theorem 3.2, Lemma 2.16 and Corollary 1.7.24 of [**15**] imply that D_G is commutative. For an example let us consider the triangular matrix R-algebra $\Lambda = \begin{pmatrix} R & 0 \\ H & H \end{pmatrix}$, where R is a real closed field and H is the quaternion ring over R. There is a unique, up to isomorphism, generic Λ-module G, and $D_G \cong R(x)[y]/\langle x^2+y^2+1\rangle$ ([**9**]). Also, we have $D_{G^C} \cong (D_G)^C \cong C(x)[y]/\langle x^2+y^2+1\rangle \cong C(t)$, where C is the algebraic closure of R and t can be identified with the element $y/\left(x-\sqrt{-1}\right) \in C(x)[y]/\langle x^2+y^2+1\rangle$ (see [**8**] for a detailed argument). Notice that G is an algebraically rigid generic Λ-module with $D_G \ncong R(w)$ for w a commutative variable.

4. Wild case. Now it is necessary to strength a little bit the definition of wild representation type for a finite-dimensional K-algebra. In order to do so, we are going to use differential tensor algebras and their reduction functors.

DEFINITION 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be additive k-categories and $F : \mathcal{C} \to \mathcal{D}$ a $k-$functor (see page 28 of [**2**]). We say that F is *sharp* if:

1. F preserves indecomposables and isomorphism classes.
2. For any indecomposable $M \in \mathcal{C}$, we have $F(\mathrm{rad}E_M) \subset \mathrm{rad}E_{F(M)}$ and the induced morphism of k-algebras $E_M/\mathrm{rad}E_M \to E_{F(M)}/\mathrm{rad}E_{F(M)}$ is a bijection.

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REMARK 4.2. If $F : \mathcal{C} \to \mathcal{D}$ is full and faithful and idempotents split in $\mathcal{C}$, then F is sharp.

REMARK 4.3. If $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$ are sharp functors, then HF is sharp.

LEMMA 4.4. *Assume that $\mathcal{B}$ is a proper subalgebra (see Definitions 3.2 of [**3**] and 12.1 of [**5**]) of the Roiter ditalgebra $\mathcal{A}$ (see Definition 5.5 of [**5**]). Let $F : \mathcal{B} - \mathrm{Mod} \to \mathcal{A} - \mathrm{Mod}$ be the corresponding extension functor. Then, F is sharp.*

Proof. It is easy to see that F preserves indecomposables and isomorphism classes (see Lemma 2.4 of [**4**]). For sharpness, we can work with the usual characterization of the radical

$$\mathrm{rad}E = \{f \in E \mid 1 - gfh \text{ is invertible for all } g, h \in E\}$$

for a given ring E. Given $M \in \mathcal{B} - \mathrm{Mod}$, $f^0 \in \mathrm{rad}E_M$ and $g, h \in E_{F(M)}$, we have that $g^0, h^0 \in E_M$, then $1_M - g^0 f^0 h^0$ is invertible in E_M. Since $\mathcal{A}$ is a Roiter ditalgebra and $\left(1_{F(M)} - gF\left(f^0\right) h\right)^0 = 1_M - g^0 f^0 h^0$ we get that $1_{F(M)} - gF\left(f^0\right) h$ is invertible in $E_{F(M)}$. Thus, $F\left(\mathrm{rad}E_M\right) \subset \mathrm{rad}E_{F(M)}$.

Now, given $f \in E_{F(M)}$, there is a decomposition $f = \left(f^0, f^1\right) = \left(f^0, 0\right) + \left(0, f^1\right)$ as a sum of morphisms in $E_{F(M)}$. Here, $\left(0, f^1\right) \in \mathrm{rad}E_{F(M)}$, because $\left(1_{F(M)} - g\left(0, f^1\right) h\right)^0 = 1_M$ is invertible for all $g, h \in E_{F(M)}$. Thus, the induced morphism $\overline{F} : E_M/\mathrm{rad}E_M \to E_{F(M)}/\mathrm{rad}E_{F(M)}$ is surjective. Similarly, $F\left(f^0\right) = \left(f^0, 0\right) \in \mathrm{rad}E_{F(M)}$ implies that $f^0 \in \mathrm{rad}E_M$, so $\overline{F}$ is bijective.

□

LEMMA 4.5. *The restriction of the cokernel functor* $\mathrm{Cok}^2 : \mathcal{P}^2(\Lambda) \to \Lambda - \mathrm{Mod}$ *is sharp.*

Proof. Use Lemma 18.10 and Remark 31.6 of [**5**].

□

In the following, we are going to use results from [**6**, **7**] and [**10**], all of them carefully studied in [**5**].

THEOREM 4.6. *Let Λ^K be of wild representation type, then there exists a $\Lambda^K - K\langle x, y\rangle$-bimodule B, finitely generated as right module, such that the functor $B \otimes_{K\langle x,y\rangle} - : K\langle x, y\rangle - \mathrm{Mod} \to \Lambda^K - \mathrm{Mod}$ is sharp.*

Proof. Let us recall that associated to Λ^K there is a basic finite-dimensional K-algebra Γ, called the *reduced form* of Λ^K (see page 35 of [**2**]), and an equivalence of categories $P\otimes_\Gamma : \Gamma - \mathrm{Mod} \to \Lambda^K - \mathrm{Mod}$ where P is the bimodule of proposition Section 2.2.5 of [**2**]: it is an easy exercise in Morita equivalence to extend the functor of that proposition from finitely generated modules to arbitrary modules.

Γ is wild by Corollary 22.15 of [**5**].

Associated to Γ there is the ditalgebra of Drozd, denoted by $\mathcal{D}^\Gamma$ (see chapter 19 of [**5**]). By Theorems 27.10(1) and 27.14 of [**5**], we have that $\mathcal{D}^\Gamma$ is wild.

By Theorem 27.10(2) of [**5**], there is a reduction functor $F : \mathcal{C} - \mathrm{Mod} \to \mathcal{D}^\Gamma - \mathrm{Mod}$ such that $\mathcal{C}$ is a critical ditalgebra. Reviewing the argument in the proof of the mentioned proposition, we see that F is full and faithful.

Let A (resp. C) be the k-algebra of the degree zero elements of the underlying graded algebra of $\mathcal{D}^{\Gamma}$ (resp. $\mathcal{C}$) (see Definition 2.2 of [**5**]) and consider the canonical embedding $L_{\mathcal{D}^{\Gamma}} : A - \mathrm{Mod} \to \mathcal{D}^{\Gamma} - \mathrm{Mod}$ (resp. $L_{\mathcal{C}} : C - \mathrm{Mod} \to \mathcal{C} - \mathrm{Mod}$).

Reviewing carefully the development of chapter 24 of [**5**], we observe in that reference the proof of the existence of a $C - K\langle x, y\rangle$-bimodule B_0 such that the functor $L_{\mathcal{C}}\left(B_0 \otimes_{K\langle x,y\rangle} \_\right) : K\langle x, y\rangle - \mathrm{Mod} \to \mathcal{C} - \mathrm{Mod}$ is sharp: the functor that produces the wildness of the star algebra of 30.2 of [**5**] is full and faithful, and for the extension functor involved we apply the Lemma 4.4. Also, B_0 is free of finite rank as right module (see Lemma 22.7 of [**5**]).

By the properties of the reduction functors involved and Lemma 22.7 of [**5**], we get that $B_1 = F(B_0)$ is an $A - K\langle x, y\rangle$-bimodule, free of finite rank, and the functor $L_{\mathcal{D}^{\Gamma}}\left(B_1 \otimes_{K\langle x,y\rangle} \_\right)$ is sharp.

Consider the equivalence functor $\Xi : \mathcal{D}^{\Gamma} - \mathrm{Mod} \to \mathcal{P}^1(\Gamma)$ (see Proposition 19.8 of [**5**]). By Lemma 22.20(1) of [**5**], the image of any indecomposable under the composition functor $\Xi L_{\mathcal{D}^{\Gamma}}\left(B_1 \otimes_{K\langle x,y\rangle} \_\right)$ is contained in $\mathcal{P}^2(\Gamma)$. Then, by Lemma 4.5 and the previous arguments, $\mathrm{Cok}^2\,\Xi L_{\mathcal{D}^{\Gamma}}\left(B_1 \otimes_{K\langle x,y\rangle} \_\right)$ is sharp.

By Lemma 22.18(2) of [**5**], there exists a *transitional bimodule*, the $\Gamma - A$-bimodule Z. By construction $Z \otimes_A \_$ is naturally isomorphic to the composition $\mathrm{Cok}\,\Xi L_{\mathcal{D}^{\Gamma}}$. Then, the $\Gamma - K\langle x, y\rangle$-bimodule $Z \otimes_A B_1$ is finitely generated as right module and the functor $Z \otimes_A B_1 \otimes_{K\langle x,y\rangle} \_$ is sharp.

Then, the statement is true for the $\Lambda^K - K\langle x, y\rangle$-bimodule $B = P \otimes_{\Gamma} Z \otimes_A B_1$ and the functor $B \otimes_{K\langle x,y\rangle} \_$.

□

THEOREM 4.7. *If Λ^K is wild, then Λ is not semigenerically tame.*

Proof. This proof contains an adaptation of the argument of [**14**].

By Theorem 4.6, there is a $\Lambda^K - K\langle x, y\rangle$-bimodule B_0, finitely generated as right module, such that $B_0 \otimes_{K\langle x,y\rangle} \_$ is a sharp functor.

As in Section 3, the composition of the epimorphisms of algebras $K\langle x, y\rangle \to K[x, y] \to K(x)[y]$ has an associated restriction functor $\mathcal{F}_{\eta} : K(x)[y] - \mathrm{Mod} \to K\langle x, y\rangle - \mathrm{Mod}$ which is full and faithful. Notice that $\mathcal{F}_{\eta}$ is equivalent to the functor $K(x)[y] \otimes_{K(x)[y]} \_$ when we consider the canonical structure of $K\langle x, y\rangle - K(x)[y]$-bimodule of $K(x)[y]$.

Then, $B_0 \otimes_{K\langle x,y\rangle} K(x)[y] \otimes_{K(x)[y]} \_ : K(x)[y] - \mathrm{Mod} \to \Lambda^K - \mathrm{Mod}$ is sharp.

We also have that $B_0 \otimes_{K\langle x,y\rangle} K(x)[y]$ is finitely generated as right module. Since $K(x)[y]$ is a principal ideal domain, there exists a polynomial $h \in K(x)[y]$ such that $B = B_0 \otimes_{K\langle x,y\rangle} K(x)[y]_h$ is free of finite rank as right module, where $K(x)[y]_h$ denotes the localization of $K(x)[y]$ over h.

The canonical algebra morphism $K(x)[y] \to K(x)[y]_h$ is an epimorphism, so the functor $B \otimes_{K(x)[y]_h} \_ : K(x)[y]_h - \mathrm{Mod} \to \Lambda^K - \mathrm{Mod}$ also is sharp.

Since B is of finite rank as right module and Λ is finite-dimensional, there exist a finite field extension L/k and a $\Lambda^L - L(x)[y]_h$-bimodule $\underline{B}$, such that $\underline{B}$ is free as $L(x)[y]_h$-module with $\mathrm{rank}_{L(x)[y]_h}(\underline{B}) = \mathrm{rank}_{K(x)[y]_h}(B) = n$ and $\underline{B}^K \cong B$ as $\Lambda^K - K(x)[y]_h$-bimodules. Of course, it is assumed that $h \in L(x)[y]$.

Let $\{p_i\}_{i \in I}$ be an infinite set of non-equivalent primes of $L[y]$, each one relatively prime to h in $L(x)[y]$.

There is a canonical isomorphism of $L(x)$-algebras $L(x) \otimes_L L[y] \cong L(x)[y]$ and so, for each $i \in I$, there exists an isomorphism of $L(x)$-algebras $L(x) \otimes_L (L[y]/\langle p_i\rangle) \cong (L(x)[y])/\langle p_i\rangle$. Then, choosing F_i as a normal closure of $L[y]/\langle p_i\rangle$ we get, by the

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previous isomorphisms, Lemma 2.11(a) and Lemma 5.1 of [13], the isomorphisms of $k(x)$-algebras

$$((L(x)[y])/\langle p_i\rangle)^{F_i} \cong L(x)\otimes_L (L[y]/\langle p_i\rangle)\otimes_L F_i \cong L(x)\otimes_L (F_i\times\cdots\times F_i)\cong F_i(x)\times\cdots\times F_i(x),$$

where the number of factors is $\dim_L(L[y]/\langle p_i\rangle)$.

For each $i\in I$, let us consider the left $L(x)[y]_h$-module $G_i = L(x)[y]_h/\langle p_i\rangle$. Let $p_i=\prod_{j=1}^{\mathrm{grad}(p_i)}\left(y-r_{i,j}\right)$ be its factorization on $F_i[y]$. From Remark 2.12, we get that $G_i^{F_i}\cong\oplus_{j=1}^{\dim_L(L[y]/\langle p_i\rangle)}H_{i,j}$, where $H_{i,j}$ is the $F_i(x)[y]_h$-module $F_i(x)$, where the action of y is multiply by $r_{i,j}$.

There is a canonical isomorphism $F_i(x)[y]_h\otimes_{F_i}K\cong K(x)[y]_h$ of $K(x)$-algebras. It is easy to see that we can identify $H_{i,j}^K$ with the $K(x)[y]_h$-module $K(x)$, where the action of y is to multiply by $r_{i,j}$.

Notice that $E_{G_i}\cong(L(x)[y])/\langle p_i\rangle$, $D_{H_{i,j}}=E_{H_{i,j}}\cong F_i(x)$ and $D_{H_{i,j}^K}=E_{H_{i,j}^K}\cong K(x)$, then $1=\mathrm{endol}(G_i)=\mathrm{endol}\left(H_{i,j}\right)=\mathrm{endol}\left(H_{i,j}^K\right)$, for each i and each j.

Moreover, we get that the monomorphism $\alpha:\left(E_{H_{i,j}}\right)^K\to E_{H_{i,j}^K}$ of Lemma 2.3(a) is bijective.

By Lemma 2.6, we have that $\mathrm{endol}\left(\underline{B}\otimes_{L(x)[y]_h}G_i\right)\le n$ and $\mathrm{endol}\left(B\otimes_{K(x)[y]_h}H_{i,j}^K\right)\le n$, for each i and each j.

Then, by Proposition 2.2(c) and Lemma 2.1, we have that $\underline{B}\otimes_{L(x)[y]_h}G_i\cong\oplus_{t=1}^{s_i}\left(\oplus_{I_{i,t}}U_{i,t}\right)$, where $s_i\in\mathbb{N}$ for each $i\in I$, $I_{i,t}$ is a set for $t\in\{1,\dots,s_i\}$, $U_{i,t}$ is an indecomposable Λ^L-module of endolength less or equal to n, for each i and each t, and $t\neq t'$ implies $U_{i,t}\ncong U_{i,t'}$.

Also by sharpness, we get that $B\otimes_{K(x)[y]_h}H_{i,j}^K$ is indecomposable, and clearly it is of infinite dimension over K.

It is easy to verify that for any $i\in I$ there exists a commutative diagram of categories and functors

$$\begin{array}{ccccc}
L(x)[y]_h-\mathrm{Mod} & \xrightarrow{\ -\otimes_L F_i\ } & F_i(x)[y]_h-\mathrm{Mod} & \xrightarrow{\ -\otimes_{F_i}K\ } & K(x)[y]_h-\mathrm{Mod}\\
\downarrow{\scriptstyle \underline{B}\otimes_{L(x)[y]_h}-} & & \downarrow{\scriptstyle \underline{B}^{F_i}\otimes_{F_i(x)[y]_h}-} & & \downarrow{\scriptstyle B\otimes_{K(x)[y]_h}-}\\
\Lambda^L-\mathrm{Mod} & \xrightarrow{\ -\otimes_L F_i\ } & \Lambda^{F_i}-\mathrm{Mod} & \xrightarrow{\ -\otimes_{F_i}K\ } & \Lambda^K-\mathrm{Mod}.
\end{array}$$

From the previous diagram, and the fact that $B\otimes_{K(x)[y]_h}H_{i,j}^K$ is a generic Λ^K-module, we get that $\underline{B}^{F_i}\otimes_{F_i(x)[y]_h}H_{i,j}$ is indecomposable for each i and each j, then $\underline{B}^{F_i}\otimes_{F_i(x)[y]_h}H_{i,j}$ is an algebraically rigid Λ^{F_i}-module for each i and each j.

By Proposition 2.2 and Lemma 2.3(c), we get that $U_{i,t}$ is an algebraically bounded Λ^L-module, for each i and each t.

By sharpness of $B\otimes_{K(x)[y]_h}-$, we get that $i\neq i'$ implies that $B\otimes_{K(x)[y]_h}H_{i,j}^K\ncong B\otimes_{K(x)[y]_h}H_{i',j}^K$ and so, by Lemma 2.3(c), $U_{i,t}\ncong U_{i',t'}$.

Also by sharpness of $B\otimes_{K(x)[y]_h}-$, we get that $\mathrm{endol}\left(B\otimes_{K(x)[y]_h}H_{i,j}^K\right)=\mathrm{c-endol}\left(B\otimes_{K(x)[y]_h}H_{i,j}^K\right)$ for each i and each j.

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By the previous commutative diagram and the mentioned isomorphism of K-algebras $\alpha : \left(E_{H_{i,j}}\right)^K \to E_{H^K_{i,j}}$, it follows that $\left(D_{\underline{B}^{F_i} \otimes_{F_i(x)[y]_h} H_{i,t}}\right)^K \cong D_{B \otimes_{K(x)[y]_h} H^K_{i,t}}$, and so we get

$$\mathrm{c}-\mathrm{endol}\left(\underline{B}^{F_i} \otimes_{F_i(x)[y]_h} H_{i,t}\right) = \mathrm{c}-\mathrm{endol}\left(B \otimes_{K(x)[y]_h} H^K_{i,t}\right).$$

Now by Proposition 2.18 we have $\mathrm{c}-\mathrm{endol}\left(U_{i,j}\right) \leq n$: it follows that Λ^L is not semigenerically tame and, by Corollary 2.19, Λ is not semigenerically tame. □

Keeping in mind Drozd's theorem, it is clear that Theorem 1.8 is a consequence of Theorems 3.2 and 4.7.

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