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A CHARACTERISATION FOR A GROUPOID GALOIS EXTENSION USING PARTIAL ISOMORPHISMS

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Abstract

Let $S|_R$ be a groupoid Galois extension with Galois groupoid G such that $E_g^{G_{r(g)}} \subseteq C1_g$, for all $g \in G$, where C is the centre of S, $G_{r(g)}$ is the principal group associated to $r(g)$ and $\{E_g\}_{g\in G}$ are the ideals of S. We give a complete characterisation in terms of a partial isomorphism groupoid for such extensions, showing that $G = \dot{\bigcup}_{g\in G}\mathrm{Isom}_R(E_{g^{-1}}, E_g)$ if and only if E_g is a connected commutative algebra or $E_g = E_g^{G_{r(g)}} \oplus E_g^{G_{r(g)}}$, where $E_g^{G_{r(g)}}$ is connected, for all $g \in G$.

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1. Introduction

Several branches of groupoid theory have aroused curiosity as objects of study for their applications and generalisations in many areas, such as algebraic topology, noncommutative geometry, Lie groupoids and theoretical physics (see [2–4, 6–8]). In this paper, we make a breakthrough in groupoid Galois theory by giving a characterisation for a groupoid Galois extension with Galois groupoid given by the disjoint union of the partial isomorphisms between the unitary ideals of that extension.

Our starting point is the work of Chase *et al.* [5]. They proved that the Galois group of a commutative Galois extension with no idempotents other than 0 and 1 is the automorphism group of the ring. Later, Szeto and Xue [9] proved the converse proposition, showing that given a Galois algebra S over a commutative ring R with Galois group G, then $G = \mathrm{Aut}_R(S)$ if and only if either S is commutative with no idempotents other than 0 and 1, or $S \cong R \oplus R$ where R contains no idempotents other than 0 and 1. In this work, we study such characterisations for groupoid Galois extensions in terms of the groupoid of partial isomorphisms and the ideals of the groupoid Galois extension.

The paper is organised as follows. In Section 2 we give some preliminary results about groupoids, groupoid actions and groupoid Galois theory. In Section 3, we consider S, a commutative groupoid Galois extension of R with Galois groupoid G, where G acts by partial isomorphisms over S. We show that each unitary ideal E_g of S, for $g \in G$, is connected if and only if G is the disjoint union of the R-isomorphisms between the ideals E_g^{-1} and E_g.

In Section 4 we take S to be a not necessarily commutative groupoid Galois extension of R with Galois groupoid G. We prove that if the fixed subalgebra of E_g by the action of the principal group associated to $r(g)$ is contained in the projection of the centre of S over the ideal E_g, then G is the disjoint union of the R-isomorphisms between the ideals E_g^{-1} and E_g with $g \in G$ if and only if each ideal E_g is a connected commutative algebra. Moreover, we study the particular case where G is the disjoint union of the principal groups and present some examples concerning the converse in that case.

2. Prerequisites

The algebraic version of groupoid that we adopt in this paper is taken from [7]. Although we do not work with the categorical version of groupoids (see [4] for more details), it should be noted that the algebraic and the categorical definitions of groupoids are equivalent.

A groupoid is a nonempty set G, equipped with a partially defined binary operation (which will be denoted by concatenation), such that the following axioms hold:

(i) for all $g, h, l \in G$, $g(hl)$ exists if and only if $(gh)l$ exists and in this case the two are equal;
(ii) for all $g, h, l \in G$, $g(hl)$ exists if and only if gh and hl exist;
(iii) for each $g \in G$, there exist (unique) elements $d(g), r(g) \in G$ such that $gd(g)$ and $r(g)g$ exist and $gd(g) = g = r(g)g$;
(iv) for each $g \in G$ there exists $g^{-1} \in G$ such that $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$.

For every $g, h \in G$, we say that there exists gh whenever the product gh is defined. It follows by definition that for every $g, h \in G$, we have that there exists gh if and only if $d(g) = r(h)$ and, in this case, $d(gh) = d(h)$ and $r(gh) = r(g)$. We denote by G^2 the subset of the pairs $(g, h) \in G \times G$ such that $d(g) = r(h)$. An element $e \in G$ is called an identity of G if $e = d(g) = r(g^{-1})$, for some $g \in G$. We denote by G_0 the set of all identities of G and by G_e the set of all $g \in G$ such that $d(g) = r(g) = e$. It is easy to see that, for all $e \in G_0$, G_e is a group, called the principal group associated to e. Given G a groupoid and H a nonempty subset of G, we say that H is a subgroupoid of G if it satisfies the following conditions:

(i) for all $g, h \in H$, if there exists gh then $gh \in H$;
(ii) if $g \in H$ then $g^{-1} \in H$, for every $g \in H$.

Consider S an algebra over a commutative ring K. Following [1], an action of G over S is a pair

$$\beta = (\{E_g\}_{g\in G}, \{\beta_g\}_{g\in G}),$$

where for each $g \in G$, $E_g = E_{r(g)}$ is an ideal of S and $\beta_g : E_{g^{-1}} \to E_g$ is an isomorphism of K-algebras satisfying the following conditions:

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(i) β_e is the identity map Id_{E_e} of E_e for every $e \in G_0$;
(ii) $\beta_g(\beta_h(r)) = \beta_{gh}(r)$ for every $(g,h) \in G^2$ and for every $r \in E_{h^{-1}} = E_{(gh)^{-1}}$.

We say that S is a groupoid Galois extension of a ring $R \subseteq S$ with Galois groupoid G if:

(i) $R = S^G = \{s \in S \mid \beta_g(s1_{g^{-1}}) = s1_g, \text{ for all } g \in G\}$;
(ii) there exist elements $x_i, y_i \in S$, $1 \leq i \leq m$, such that $\sum_{1\leq i\leq m} x_i\beta_g(y_i 1_{g^{-1}}) = \delta_{e,g}1_e$ for all $e \in G_0$ and $g \in G$.

The set $\{x_i, y_i\}_{1\leq i\leq m}$ is called a Galois coordinate system of S over R. A ring S is called a groupoid Galois algebra over R if S is a Galois extension of R and R is contained in the centre of S.

Let S be an algebra over a commutative ring R. Define

$$I_R(S) = \{f_{IJ} : I \to J \mid f_{IJ} \text{ is an } R\text{-isomorphism and } I, J \text{ are ideals of } S \text{ generated by central idempotents}\}.$$

It is easy to see that $I_R(S)$ is a groupoid. If $G \subseteq I_R(S)$ as a subgroupoid, we say that the groupoid G acts over S by partial isomorphisms and, in this case, $\beta_g(s1_{g^{-1}}) = g(s1_{g^{-1}})$, for all $s \in S$. This is the case that we consider in this paper.

Throughout, unless otherwise specified, rings and algebras are associative and unital.

3. Commutative ideals

In this section we assume that S is a commutative groupoid Galois extension of R with Galois groupoid G such that G acts over S by partial isomorphisms and $S = \bigoplus_{e\in G_0} E_e$, where each E_e, $e \in G_0$, is a unitary commutative ideal of S with identity element 1_e.

We say that E_g is connected if it contains no idempotents other than 0 and 1_g. Let ${}_g\mathcal{G}_g = \{h \in G \mid d(h) = d(g) \text{ and } r(h) = r(g)\}$. The following proposition generalises [5, Theorem 3.1].

Proposition 3.1. *Let A be a commutative R-algebra and let $\psi_g, \varphi_g : E_g \to A$ be R-algebra homomorphisms. Then there exists a set $\{e_h^g \mid h \in {}_g\mathcal{G}_g\}$ of pairwise orthogonal idempotents of A such that $\sum_{h\in {}_g\mathcal{G}_g} e_h^g = 1_A$ and $\varphi_g(s_g) = \sum_{h\in {}_g\mathcal{G}_g} \psi_g(h(s_{g^{-1}}1_{h^{-1}}))e_h^g$, for all $g \in G$, where $\sum_{g\in G} s_g = s \in S$.*

Proof. By [1, Theorem 5.3], the map $\rho : S \otimes_R S \to \prod_{h\in G} E_h$ given by $\rho(x \otimes y) = (xh(y1_{h^{-1}}))_{h\in G}$ is an isomorphism of left R-modules. Since E_g is commutative for all $g \in G$, it follows that ρ is an isomorphism of R-algebras.

Let $\rho_g := \rho|_{E_g\otimes E_{g^{-1}}}$. Then

$$\rho_g(x_g \otimes y_{g^{-1}}) = (x_g h(y_{g^{-1}}1_{h^{-1}}))_{h\in G} = (x_g h(y_{g^{-1}}1_{h^{-1}}))_{h\in {}_g\mathcal{G}_g}.$$

Thus, $E_g \otimes E_{g^{-1}} \simeq \prod_{h\in {}_g\mathcal{G}_g} E_h$ as R-algebras. Let θ denote the composition of maps

$$\prod_{h\in {}_g\mathcal{G}_g} E_h \xrightarrow{\rho_g^{-1}} E_g \otimes_R E_{g^{-1}} \xrightarrow{r(g)\otimes g} E_g \otimes_R E_g \xrightarrow{\psi_g\otimes\varphi_g} A \otimes_R A \xrightarrow{k} A,$$

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where k is the contraction homomorphism defined by $k(a_1 \otimes a_2) = a_1 a_2$.

Consider $\overline{1_h} = (0, \ldots, 0, 1_h, 0, \ldots, 0)$ and take $e_h^g = \theta(\overline{1_h})$, $h \in {}_g\mathcal{G}_g$. Then $\{e_h^g\}_{h \in_g \mathcal{G}_g}$ are pairwise orthogonal idempotents of A such that $\sum_{h \in_g \mathcal{G}_g} e_g^h = 1_A$.

Note that $\rho_g(1_g \otimes s_{g^{-1}}) = (h(s_{g^{-1}} 1_{h^{-1}}))_{h \in_g \mathcal{G}_g}$, for all $s_{g^{-1}} \in E_{g^{-1}}$. Applying θ on both sides of the equality, we get

$$k \circ (\psi_g \otimes \varphi_g) \circ (r(g) \otimes g) \circ \rho_g^{-1}(\rho_g(1_g \otimes s_{g^{-1}})) = \theta(h((s_{g^{-1}} 1_{h^{-1}}))_{h \in_g \mathcal{G}_g})$$

$$k \circ (\psi_g \otimes \varphi_g)(1_g \otimes s_g) = \theta\Big(\sum_{h \in_g \mathcal{G}_g} h(s_{g^{-1}} 1_{h^{-1}}) \overline{1_h}\Big).$$

So

$$\begin{aligned}
\varphi_g(s_g) &= \sum_{h \in_g \mathcal{G}_g} \theta((h(s_{g^{-1}} 1_{h^{-1}}) 1_l)_{l \in_g \mathcal{G}_g}) \theta(\overline{1_h}) \\
&= \sum_{h \in_g \mathcal{G}_g} k \circ (\psi_g \otimes \varphi_g) \circ (r(g) \otimes g) \circ \rho_g^{-1}((h(s_{g^{-1}} 1_{h^{-1}}) 1_l)_{l \in_g \mathcal{G}_g}) e_h^g \\
&= \sum_{h \in_g \mathcal{G}_g} k \circ (\psi_g \otimes \varphi_g) \circ (r(g) \otimes g)(h(s_{g^{-1}} 1_{h^{-1}}) \otimes 1_{g^{-1}}) e_h^g \\
&= \sum_{h \in_g \mathcal{G}_g} \psi_g(h(s_{g^{-1}} 1_{h^{-1}})) e_h^g. \qquad \square
\end{aligned}$$

Denote by $j : S \star G \to \mathrm{End}_R(S)$ the natural map given by

$$j\Big(\sum_{g \in G} a_g u_g\Big)(x) = \sum_{g \in G} a_g g(x 1_{g^{-1}}) \quad \text{for all } x \in S.$$

By [1, Theorem 5.3], the map j is an isomorphism of rings and S-modules.

Corollary 3.2. *Let $W_{g^{-1}} = \mathrm{Hom}_R(E_{g^{-1}}, E_g)$ be the set of the R-algebra homomorphisms between $E_{g^{-1}}$ and E_g. Then $j^{-1}(W_{g^{-1}})$ consists of all elements of $S \star G$ of the form $\sum_{h \in_g \mathcal{G}_g} e_h^g u_h$, with e_h^g pairwise orthogonal idempotents of E_g such that $\sum_{h \in_g \mathcal{G}_g} e_h^g = 1_g$.*

Proof. Consider the map $h = j|_{\bigoplus_{h \in_g \mathcal{G}_g} E_h \star {}_g\mathcal{G}_g}$. Then h is an isomorphism of R-algebras between $\bigoplus_{h \in_g \mathcal{G}_g} E_h \star {}_g\mathcal{G}_g$ and the set of R-algebra isomorphisms $\mathrm{Isom}_R(E_{g^{-1}}, E_g)$, where

$$j\Big(\sum_{h \in_g \mathcal{G}_g} a_h u_h\Big)(x_{g^{-1}}) = \sum_{h \in_g \mathcal{G}_g} a_h h(x_{g^{-1}} 1_{g^{-1}}) \quad \text{for all } x_{g^{-1}} \in E_{g^{-1}}.$$

Take $A = E_g$ in Proposition 3.1 and $\psi_{g^{-1}} : E_{g^{-1}} \to E_g$ defined by $\psi_{g^{-1}}(x_{g^{-1}}) = x_g$, where $\sum_{g \in G} x_g = x \in S$. By Proposition 3.1, $\varphi_{g^{-1}}(x_{g^{-1}}) = \sum_{h \in_g \mathcal{G}_g} h(x_{g^{-1}} 1_{h^{-1}}) e_h^g$, with e_h^g pairwise orthogonal idempotents of E_g, for all $\varphi_{g^{-1}} \in W_{g^{-1}}$. Consequently, we have $j^{-1}(W_{g^{-1}}) = \{\sum_{h \in_g \mathcal{G}_g} e_h^g u_h\}$. $\square$

Corollary 3.3. *With the same notation as in Corollary 3.2, if $e_h^g \in R1_g$ for all $h \in {}_g\mathcal{G}_g$, then $W_{g^{-1}} = \mathrm{Isom}_R(E_{g^{-1}}, E_g)$.*

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Proof. We claim that for each $f \in W_{g^{-1}}$, there exists $f^{-1} \in W_g$ such that $f \circ f^{-1} = r(g)$ and $f^{-1} \circ f = d(g)$. In fact, given $f \in W_{g^{-1}}$, there exists a set of pairwise orthogonal idempotents $\{e_h^g\}_{h \in {}_g\mathcal{G}_g} \subseteq E_g$ such that $j(\sum_{h \in {}_g\mathcal{G}_g} e_h^g u_h) = f$. Let $f^{-1} = j(\sum_{l \in {}_g\mathcal{G}_g} e_l^{g^{-1}} u_{l^{-1}})$. Then

$$\begin{aligned}\Big(\sum_{h \in {}_g\mathcal{G}_g} e_h^g u_h\Big)\Big(\sum_{l \in {}_g\mathcal{G}_g} e_l^{g^{-1}} u_{l^{-1}}\Big) &= \sum_{h,l \in {}_g\mathcal{G}_g} h(h^{-1}(e_h^g) e_l^{g^{-1}}) u_{hl^{-1}} \\ &= \sum_{h,l \in {}_g\mathcal{G}_g} h(e_h^{g^{-1}} e_l^{g^{-1}}) u_{hl^{-1}} \\ &= \sum_{h \in {}_g\mathcal{G}_g} h(e_h^{g^{-1}} e_h^{g^{-1}}) u_{r(g)} \\ &= \sum_{h \in {}_g\mathcal{G}_g} h(e_h^{g^{-1}}) u_{r(g)} \\ &= 1_g u_{r(g)}.\end{aligned}$$

Analogously, we can show that $(\sum_{l \in {}_g\mathcal{G}_g} e_l^{g^{-1}} u_{l^{-1}})(\sum_{h \in {}_g\mathcal{G}_g} e_h^g u_h) = 1_{g^{-1}} u_{d(g)}$. This proves that $W_{g^{-1}} = \mathrm{Isom}_R(E_{g^{-1}}, E_g)$. □

Corollary 3.4. *With the same notation as in Corollary 3.3, E_g is connected if and only if* $\mathrm{Isom}_R(E_{g^{-1}}, E_g) = W_{g^{-1}} = {}_g\mathcal{G}_g$.

Proof. By Corollary 3.3, $\mathrm{Isom}_R(E_{g^{-1}}, E_g) = W_{g^{-1}}$ and it is immediate that ${}_g\mathcal{G}_g \subseteq W_{g^{-1}}$. If E_g is connected, $j^{-1}(W_{g^{-1}}) = \{1_g u_g\}$. Since $j(1_g u_g)(x_{g^{-1}}) = g(x_{g^{-1}} 1_{g^{-1}})$, it follows that $j(1_g u_g) \in {}_g\mathcal{G}_g$. Thus, $W_{g^{-1}} = {}_g\mathcal{G}_g$.

Conversely, assume $W_{g^{-1}} = {}_g\mathcal{G}_g$. Suppose that there exists $e_h^g \neq 1_g, 0$ in E_g. Then $j(e_h^g u_h + (1_g - e_h^g) u_l) \in W_{g^{-1}}$, but it is not in ${}_g\mathcal{G}_g$, a contradiction. □

Corollary 3.5. *Let $S|_R$ be a commutative groupoid Galois extension. For each $g \in G$, the ideal E_g is connected if and only if $G = \dot{\bigcup}_{g \in G} \mathrm{Isom}_R(E_{g^{-1}}, E_g)$.*

Proof. Since $G = \dot{\bigcup}_{g \in G} {}_g\mathcal{G}_g$, the statement is immediate from Corollary 3.4. □

4. The converse problem

In this section, we assume that S is a not necessarily commutative groupoid Galois extension of R with Galois groupoid G such that G acts over S by partial isomorphisms and $S = \bigoplus_{e \in G_0} E_e$, where each E_g, $g \in G$, is a unitary ideal of S with identity element 1_g.

Lemma 4.1. *Let $\lambda_{IJ} \in I_R(S)$ such that $\lambda_{IJ} \neq \mathrm{Id}_I$ (in the case $I = J$). If $a \neq 0$ is a central idempotent in I such that $\lambda_{IJ}(sa) = s\lambda_{IJ}(a)$ for all $s \in S$, then $\lambda_{IJ} \notin G$.*

Proof. Since S is a groupoid Galois extension over R, there exist elements $x_i, y_i \in S$, $1 \le i \le m$, such that $\sum_{i=1}^m x_i g(y_i 1_{g^{-1}}) = \delta_{e,g} 1_e$, for all $e \in G_0$ and $g \in G$.

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Take λ_{IJ} as in the hypothesis for $a \neq 0$ a central idempotent in I, and assume $\lambda_{IJ} \in G$. Then there exists $g \in G \setminus G_0$ such that $\lambda_{IJ} = g$, which implies that $E_{g^{-1}} = I$ and $E_g = J$. Thus, $\sum_{i=1}^{m} x_i g(y_i 1_{g^{-1}}) = 0$, since $g \notin G_0$. Hence,

$$0 = 0 \cdot g(a) = \sum_{i=1}^{m} x_i g(y_i 1_{g^{-1}}) \cdot g(a) = \sum_{i=1}^{m} x_i g(y_i a) = \sum_{i=1}^{m} x_i y_i g(a) = g(a),$$

which is a contradiction. So $\lambda_{IJ} \notin G$. □

Lemma 4.2. *If* $G = \dot{\bigcup}_{g\in G} \mathrm{Isom}_R(E_{g^{-1}}, E_g)$*, then* $R_g := R1_g$ *is connected for each* $g \in G$.

Proof. Suppose that $G = \dot{\bigcup}_{g\in G} \mathrm{Isom}_R(E_{g^{-1}}, E_g)$. Then ${}_g\mathcal{G}_g = \mathrm{Isom}_R(E_{g^{-1}}, E_g)$. Assume that R_g is not connected. Then there exists $a_g \neq 0, 1_g$, $a_g \in R_g$ such that $a_g^2 = a_g$ and $E_g = E_g a_g \oplus E_g(1_g - a_g)$. Since $a_g \in R_g$, there exists $r \in R$ such that $a_g = r1_g$. Thus, $g^{-1}(a_g) = g^{-1}(r1_g) = r1_{g^{-1}} = a_{g^{-1}}$. Hence, $E_{g^{-1}} = E_{g^{-1}} a_{g^{-1}} \oplus E_{g^{-1}}(1_{g^{-1}} - a_{g^{-1}})$. We have two cases.

Case 1. $|{}_g\mathcal{G}_g| = 1$. In this case, there exists only one partial isomorphism from E_g to $E_{g^{-1}}$. But $\alpha_{g^{-1}} : E_g \to E_{g^{-1}}$ defined by $\alpha_{g^{-1}}(xa_g + y(1_g - a_g)) = xa_{g^{-1}} + y(1_{g^{-1}} - a_{g^{-1}})$ and $\lambda_{g^{-1}} : E_g \to E_{g^{-1}}$ defined by $\lambda_{g^{-1}}(xa_g + y(1_g - a_g)) = ya_{g^{-1}} + x(1_{g^{-1}} - a_{g^{-1}})$ are both partial isomorphisms, which is a contradiction.

Case 2. $|{}_g\mathcal{G}_g| \geq 2$. Choose $h \in {}_g\mathcal{G}_g$ and $h \notin G_0$ (which means that $h \neq \mathrm{Id}_{E_g}$ in the case $d(g) = r(g)$). Thus, $h^{-1} \notin G_0$ and $h^{-1}|_{E_g a_g} \neq \mathrm{Id}_{E_g a_g}$ or $h^{-1}|_{E_g(1_g - a_g)} \neq \mathrm{Id}_{E_g(1_g - a_g)}$. Assume, without loss of generality, that $h^{-1}|_{E_g(1_g - a_g)} \neq \mathrm{Id}_{E_g(1_g - a_g)}$. Define the map $\lambda_{g^{-1}} : E_g \to E_{g^{-1}}$ by $\lambda_{g^{-1}}(xa_g + y(1_g - a_g)) = xa_{g^{-1}} + h^{-1}(y(1_g - a_g))$, for all $x, y \in S$. Then, by Lemma 4.1, $\lambda_{g^{-1}} \notin G$, which is a contradiction. So R_g is connected. □

Lemma 4.3. *If* $G = \dot{\bigcup}_{g\in G} \mathrm{Isom}_R(E_{g^{-1}}, E_g)$ *and* $|{}_g\mathcal{G}_g| > 2$ *for all* $g \in G$*, then* E_g *is a connected* R*-algebra, for all* $g \in G$.

Proof. By Lemma 4.2, R_g is connected for all $g \in G$. We claim that E_g is connected for all $g \in G$. In fact, since S is a groupoid Galois extension over R with groupoid G, then, by [1, Theorem 5.3], S is a finitely generated R-module. Thus, S contains only finitely many minimal idempotents. Since $S = \bigoplus_{e\in G_0} E_e$, E_e contains only finitely many minimal idempotents, for all $e \in G_0$.

Let $g \in G$ and $\{a_{g,i} \mid i = 1, 2, \ldots, q\}$ be the minimal central idempotents in E_g. As well as $E_g \simeq E_{g^{-1}}$, the ideals E_g and $E_{g^{-1}}$ have the same number of minimal central idempotents. Let $\{a_{g^{-1},i} \mid i = 1, 2, \ldots, q\}$ be the minimal central idempotents in $E_{g^{-1}}$. If $q = 1$, we are done.

Assume $q > 1$. Let $h \in {}_g\mathcal{G}_g$ be such that $h \notin G_0$ and $h^{-1}(a_{g,1}) = a_{g^{-1},j}$, for some j. Then we have the following four cases.

Case 1. $j = 1$ and $h^{-1}(sa_{g,1}) = sa_{g^{-1},1}$ for all $s \in S$. Then, by Lemma 4.1, $h^{-1} \notin G$, which is a contradiction.

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Case 2. $j = 1$ and $h^{-1}(sa_{g,1}) \neq sa_{g^{-1},1}$ for some $s \in S$. Then we have the partial isomorphism $\lambda_{g^{-1}} : E_g \to E_{g^{-1}}$ defined by $\lambda_{g^{-1}}|_{S a_{g,1}} = h^{-1}|_{S a_{g,1}}$ and $\lambda_{g^{-1}}(sa_{g,i}) = sa_{g^{-1},i}$ for all $i \neq 1$, $s \in S$. Thus, by Lemma 4.1, $\lambda_{g^{-1}} \notin G$, a contradiction again.

Case 3. $j \neq 1$ and $q > 2$. Consider the partial isomorphism $\lambda_{g^{-1}} : E_g \to E_{g^{-1}}$ such that $\lambda_{g^{-1}}|_{S a_{g,1}} = h^{-1}|_{S a_{g,1}} : S a_{g,1} \to S a_{g,j}$, $\lambda_{g^{-1}}(sa_{g,j}) = sa_{g^{-1},1}$ and $\lambda_{g^{-1}}(sa_{g,k}) = sa_{g^{-1},k}$, for all $k \neq 1, j$. Then, by Lemma 4.1, $\lambda_{g^{-1}} \notin G$, a contradiction.

Case 4. $j \neq 1$ and $q = 2$. Then $h^{-1}(a_{g,1}) = a_{g^{-1},2}$. Since $|{}_g\mathcal{G}_g| > 2$, there exists $l \in {}_g\mathcal{G}_g$ such that $l \neq h$ and $l \notin G_0$. Thus, $l^{-1}(a_{g,1}) = a_{g^{-1},1}$ or $l^{-1}(a_{g,1}) = a_{g^{-1},2}$. If $l^{-1}(a_{g,1}) = a_{g^{-1},1}$, then we have Case 1 or Case 2. Hence, assume $l^{-1}(a_{g,1}) = a_{g^{-1},2}$. Note that hl^{-1} exists, since $d(h) = d(g) = d(l) = r(l^{-1})$. Then we have $hl^{-1} : E_g \to E_g$, $hl^{-1}(a_{g,1}) = a_{g,1}$ and $hl^{-1} \notin G_0$, since $h \neq l$. This is either Case 1 or Case 2, which leads to a contradiction.

So, $q = 1$ and E_g is a connected R-algebra for all $g \in G$. □

Observation 4.4. Lemmas 4.2 and 4.3 are also true if $G = I_R(S)$. But if S is a Galois extension over R with Galois groupoid G such that $G = I_R(S)$ and $|{}_g\mathcal{G}_g| > 2$ for all $g \in G$, all ideals of S generated by central idempotents are connected. In particular, S is an ideal of itself and it is generated by a central idempotent, so S is connected. This implies that $1_e = 1_S$ for all $e \in G_0$, and then $G_0 = \{e\}$ which means that G is a group. This case was worked out in [9].

In the particular case where $G = \dot{\bigcup}_{e \in G_0} \mathrm{Aut}_R(E_e)$, we have the commutativity of the ideals E_g, $g \in G$, as the next result shows.

Theorem 4.5. *Let S be a groupoid Galois algebra of R with Galois groupoid G. If $G = \dot{\bigcup}_{e \in G_0} \mathrm{Aut}_R(E_e)$ and $|{}_g\mathcal{G}_g| > 2$ for all $g \in G$, then E_g is a connected commutative R-algebra, for all $g \in G$.*

Proof. Since $G = \dot{\bigcup}_{e \in G_0} \mathrm{Aut}_R(E_e)$, we have $G = \dot{\bigcup}_{e \in G_0} G_e$. Let $g \in G$. As $S|_R$ is a groupoid Galois extension with Galois groupoid G, $E_g|_{E_g^{G_{r(g)}}}$ is a Galois extension with group $G_{r(g)}$. Note that in this case ${}_g\mathcal{G}_g = G_{r(g)}$ and so $|G_{r(g)}| > 2$. Next we will show that $E_g|_{E_g^{G_{r(g)}}}$ is a Galois algebra, that is, $E_g^{G_{r(g)}} \subseteq C_g$, where C_g is the centre of E_g.

Since S is a groupoid Galois algebra over R, $R1_g \subseteq C1_g = C_g \subseteq S1_g$. Note that $R1_g = \{x_{r(g)} \in E_g \mid h(x_{r(g)}) = x_{r(g)}, \forall h \in G_{r(g)}\} = E_g^{G_{r(g)}}$. Thus $E_g|_{E_g^{G_{r(g)}}}$ is a Galois algebra. So, by [9, Theorem 4.4], E_g is a connected commutative R-algebra. □

Theorem 4.6. *Let S be a groupoid Galois algebra of R with Galois groupoid G. If $G = \dot{\bigcup}_{e \in G_0} \mathrm{Aut}_R(E_e)$ and $|{}_g\mathcal{G}_g| = 2$ for all $g \in G$, then either E_g is a connected commutative R-algebra, or $E_g = R1_g \oplus R1_g$, where $R1_g$ is connected, for all $g \in G$.*

Proof. It was shown in the proof of the Theorem 4.5 that $R1_g = E_g^{G_{r(g)}}$ and that $E_g|_{E_g^{G_{r(g)}}}$ is a Galois algebra. Since $\mathrm{Aut}_R(E_e) = G_e$ and ${}_g\mathcal{G}_g = G_{r(g)}$ for all $g \in G$, the result follows from [9, Theorem 4.5]. □

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The converse of each of the two previous results is false. The following examples shows that one can take a G-groupoid Galois algebra with $E_g = R1_g \oplus R1_g$, where $R1_g$ is connected, for all $g \in G$, or each E_g is a connected commutative ideal, but $G \neq \dot{\bigcup}_{e\in G_0} \mathrm{Aut}_R(E_e)$.

Example 4.7. Let K be a field and let S be a commutative K-algebra such that $S = Ke_1 \oplus Ke_2 \oplus Ke_3 \oplus Ke_4$, where e_1, e_2, e_3 and e_4 are minimal central idempotents of S and $R \subsetneq K1_S$ a subring of S such that $2 < |\mathrm{Aut}_R(K)| < \infty$. Define $g_{\sigma ij} : Ke_i \to Ke_j$ by $g_{\sigma ij}(ke_i) = \sigma(k)e_j$, where $\sigma \in \mathrm{Aut}_R(K)$ and $1 \leq i, j \leq 4$. Let G be the groupoid $G = \{g_{\sigma ij} \mid \sigma \in \mathrm{Aut}_R(K) \text{ and } 1 \leq i, j \leq 4\}$. Then $S^G = \{ae_1 + ae_2 + ae_3 + ae_4 \mid a \in K^{\mathrm{Aut}_R(K)}\}$ and $S|_{S^G}$ is a groupoid Galois algebra, with Galois coordinate system given by $\{x_i = y_i = e_i \mid 1 \leq i \leq 4\}$. Then $|_g\mathcal{G}_g| > 2$, for all $g \in G$. Clearly, $G \neq \dot{\bigcup}_{e\in G_0} \mathrm{Aut}_R(E_e)$.

Example 4.8. Let $\mathbb{C}$ be the complex number field and S be a commutative $\mathbb{C}$-algebra such that $S = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$, where e_1, e_2 and e_3 are minimal central idempotents of S. Define $g_{\sigma ij} : \mathbb{C}e_i \to \mathbb{C}e_j$ by $g_{\sigma ij}(xe_i) = \sigma(x)e_j$, where $\sigma \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ and $1 \leq i, j \leq 3$. Let $G = \{g_{\sigma ij} \mid \sigma \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) \text{ and } 1 \leq i, j \leq 3\}$. Then we see that G is a groupoid, $S^G = \{ae_1 + ae_2 + ae_3 \mid a \in \mathbb{R}\}$ and $S|_{S^G}$ is a groupoid Galois algebra, with Galois coordinate system given by $\{x_i = y_i = e_i \mid 1 \leq i \leq 3\}$. Note that, in this case, $|_g\mathcal{G}_g| = 2$, for all $g \in G$. Immediately, $G \neq \dot{\bigcup}_{e\in G_0} \mathrm{Aut}_R(E_e)$.

Example 4.9. Let S be a commutative ring, $R = S^G$ and

$$G = \{d(g), r(g), g, g^{-1}, h, h^{-1} \mid d(h) = d(g) \neq r(g) = r(h)\} \subseteq I_R(S).$$

Suppose that $S = R_{g^{-1}} \oplus R_{g^{-1}} \oplus R_g \oplus R_g$, where $R_g = R1_g$ is connected. Then define $g : R_{g^{-1}} \oplus R_{g^{-1}} \to R_g \oplus R_g$ by $g(r_1 1_{g^{-1}} + r_2 1_{g^{-1}}) = r_1 1_g + r_2 1_g$, $h : R_{g^{-1}} \oplus R_{g^{-1}} \to R_g \oplus R_g$ by $h(r_1 1_{g^{-1}} + r_2 1_{g^{-1}}) = r_2 1_g + r_1 1_g$. Thus S is a groupoid Galois algebra over R, with $E_{h^{-1}} = E_{g^{-1}} = R_{g^{-1}} \oplus R_{g^{-1}}$, $E_h = E_g = R_g \oplus R_g$ and Galois coordinate system given by $\{x_1 = y_1 = 1_{g^{-1}} + 1_{g^{-1}}, x_2 = y_2 = 1_g + 1_g\}$. Note that, in this case, $|_g\mathcal{G}_g| = 2$, for all $g \in G$. We have $|\mathrm{Aut}_R E_g| = |\mathrm{Aut}_R E_{g^{-1}}| = 2$ and $|\mathrm{Aut}_R E_g \cup \mathrm{Aut}_R E_{g^{-1}}| = 4$, which shows that $G \neq \dot{\bigcup}_{e\in G_0} \mathrm{Aut}_R(E_e)$, since $|G| = 6$.

In fact, in all of the previous examples, $G = \dot{\bigcup}_{g\in G} \mathrm{Isom}_R(E_{g^{-1}}, E_g)$. At the end of this section we give a complete characterisation of this case. First, to achieve the commutativity of the ideals we need to add one more condition.

Theorem 4.10. *If $G = \dot{\bigcup}_{g\in G} \mathrm{Isom}_R(E_{g^{-1}}, E_g)$, $|_g\mathcal{G}_g| > 2$ and $E_g^{G_{r(g)}} \subseteq C1_g$, for all $g \in G$, where C is the centre of S, then either E_g is a connected commutative algebra.*

Proof. Note that $C1_g = C_g$ is the centre of the group Galois extension $E_g|_{E_g^{G_{r(g)}}}$ with Galois group $G_{r(g)}$. Thus $E_g|_{E_g^{G_{r(g)}}}$ is a Galois algebra. So the statement follows by [9, Theorem 4.4]. □

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Theorem 4.11. *If* $G = \dot{\bigcup}_{g\in G} \operatorname{Isom}_R(E_{g^{-1}}, E_g)$*,* $|{}_g\mathcal{G}_g| = 2$ *and* $E_g^{G_{r(g)}} \subseteq C1_g$*, for all* $g \in G$*, where* C *is the centre of* S*, then either* E_g *is a connected commutative algebra, or* $E_g = E_g^{G_{r(g)}} \oplus E_g^{G_{r(g)}}$*, where* $E_g^{G_{r(g)}}$ *is connected for all* $g \in G$*.*

Proof. As mentioned in the proof of the previous theorem, C_g is the centre of the group Galois extension $E_g|_{E_g^{G_{r(g)}}}$ with Galois group $G_{r(g)}$. The statement follows by [9, Theorem 4.5]. □

A more general characterisation is given in the next result.

Theorem 4.12. *Let* $S|_R$ *be a groupoid Galois extension with Galois groupoid* G *such that* $E_g^{G_{r(g)}} \subseteq C1_g$*, for all* $g \in G$*, where* C *is the centre of* S*. Then we have* $G = \dot{\bigcup}_{g\in G} \operatorname{Isom}_R(E_{g^{-1}}, E_g)$ *if and only if* E_g *is a connected commutative algebra or* $E_g = E_g^{G_{r(g)}} \oplus E_g^{G_{r(g)}}$*, where* $E_g^{G_{r(g)}}$ *is connected for all* $g \in G$*.*

Proof. (⇒) This implication is a consequence of Theorems 4.10 and 4.11.

(⇐) Suppose that E_g is a connected commutative algebra. Then it is immediate that S is a commutative algebra. So, by Corollary 3.5, $G = \dot{\bigcup}_{g\in G} \operatorname{Isom}_R(E_{g^{-1}}, E_g)$.

Assume now that $E_g = E_g^{G_{r(g)}} \oplus E_g^{G_{r(g)}}$, where $E_g^{G_{r(g)}}$ is connected for all $g \in G$. Then the only isomorphisms in $\operatorname{Isom}_R(E_{g^{-1}}, E_g)$ are $\psi_g(x_{g^{-1}} \oplus y_{g^{-1}}) = x_g \oplus y_g$ and $\phi_g(x_{g^{-1}} \oplus y_{g^{-1}}) = y_g \oplus x_g$, which means that $|\operatorname{Isom}_R(E_{g^{-1}}, E_g)| = 2 = |{}_g\mathcal{G}_g|$. Because ${}_g\mathcal{G}_g \subseteq \operatorname{Isom}_R(E_{g^{-1}}, E_g)$, it follows that ${}_g\mathcal{G}_g = \operatorname{Isom}_R(E_{g^{-1}}, E_g)$. Since $G = \dot{\bigcup}_{g\in G} {}_g\mathcal{G}_g$, we see that $G = \dot{\bigcup}_{g\in G} \operatorname{Isom}_R(E_{g^{-1}}, E_g)$. □

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