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**PAPER**

# Generalised symmetries of remarkable (1+2)-dimensional Fokker–Planck equation

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**Abstract**

Using an original method, we find the algebra of generalised symmetries of a remarkable (1+2)-dimensional ultraparabolic Fokker–Planck equation, which is also called the Kolmogorov equation and is singled out within the entire class of ultraparabolic linear second-order partial differential equations with three independent variables by its wonderful symmetry properties. It turns out that the essential subalgebra of this algebra, which consists of linear generalised symmetries, is generated by the recursion operators associated with the nilradical of the essential Lie invariance algebra of the Kolmogorov equation, and the Casimir operator of the Levi factor of the latter algebra unexpectedly arises in the consideration. We also establish an isomorphism between this algebra and the Lie algebra associated with the second Weyl algebra, which provides a dual perspective for studying their properties. After developing the theoretical background of finding exact solutions of homogeneous linear systems of differential equations using their linear generalised symmetries, we efficiently apply it to the Kolmogorov equation.

## 1. Introduction

Generalised (or higher) symmetries of differential equations first appeared in the literature in their present form in Noether's seminal paper [31] in 1918. Since then, they have found various applications in symmetry analysis of differential equations, integrability theory, differential geometry and calculus of variations. See [33, pp. 374–379] for an excellent exposition on the history and development of the theory of generalised symmetries and their applications as well as other monographs on the subject [3–5, 13, 22]. At the same time, despite being under study for over a century, the exhaustive descriptions of generalised-symmetry algebras with complete proofs have only been presented for a small number of specific systems of differential equations. The main reason for this is the computational complexity inherent in all the problems on finding objects that are related to systems of differential equations and defined in the corresponding infinite-order jet spaces. Notably, the generalised-symmetry algebras even of such fundamental and simple models of mathematical physics as the linear $(1 + 1)$-dimensional heat equation [18], the Burgers equation [37], the linear Korteweg–de Vries equation [38] and the $(1 + 1)$-dimensional Klein–Gordon equation [35] were fully described only recently. See also [35] for a review of advances in this field, [34] for constructing the generalised-symmetry algebra of an isothermal no-slip drift flux model, and [7, 41], [8] and [9, 23, 29] for considering generalised symmetries of the Laplace, biharmonic and polyharmonic equations, respectively.

In the present paper, we comprehensively describe the algebra of generalised symmetries of the Kolmogorov equation [15]

$$u_t + xu_y = u_{xx}, \tag{1}$$

which is an ultraparabolic Fokker–Planck equation. This equation is singled out within the entire class $\mathcal{U}$ of ultraparabolic linear second-order partial differential equations with three independent variables by its remarkable symmetry properties. More specifically, it is the unique equation, modulo the point equivalence, whose essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$ is eight-dimensional, which is the maximum such dimension in the class $\mathcal{U}$. This is why we refer to (1) as the *remarkable Fokker–Planck equation*. The above distinguishing properties of the equation (1) within the class $\mathcal{U}$ are analogous to those of the heat equation within the class of linear second-order parabolic partial differential equation with two independent variables, see a discussion in [18]. This is why these two equations are counterparts of each other in their respective classes. As we will show, this relation also manifests on the level of generalised symmetries, see Remark 19.

The extended classical symmetry analysis of the remarkable Fokker–Planck equation was carried out in [16], featuring its numerous interesting symmetry properties. In particular, the point-symmetry pseudogroup $G$ of (1) was computed using the advanced version of the direct method. One- and two-dimensional subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ were classified modulo the action of the essential subgroup $G^{\rm ess}$ of $G$, followed with the exhaustive classification of Lie reductions of the equation (1) and the construction of wide families of its exact solutions.

The algebra $\mathfrak{g}^{\rm ess}$ is wide and has a compound structure. This provides knowledge of many generalised symmetries of (1) for free on the one hand and complicates the computations and analysis within both the classical and the generalised frameworks on the other hand. More specifically, the algebra $\mathfrak{g}^{\rm ess}$ is isomorphic to a semidirect sum ${\rm sl}(2,\mathbb{R}) \ni {\rm h}(2,\mathbb{R})$ of the real order-two special linear Lie algebra ${\rm sl}(2,\mathbb{R})$ and the real rank-two Heisenberg algebra ${\rm h}(2,\mathbb{R})$, where the action of ${\rm sl}(2,\mathbb{R})$ on ${\rm h}(2,\mathbb{R})$ is given by the direct sum of the one- and four-dimensional irreducible representations of ${\rm sl}(2,\mathbb{R})$. Despite the fact that such a structure is similar to those of the essential Lie invariance algebra of the linear (1 + 1)-dimensional heat equation, which is isomorphic to ${\rm sl}(2,\mathbb{R}) \ni {\rm h}(1,\mathbb{R})$, the corresponding computations are of a higher complexity level.

A preliminary analysis of the generalised-symmetry algebra $\Sigma$ of the remarkable Fokker–Planck equation (1) was carried out in [16, 17]. According to [33, Proposition 5.22], any Lie-symmetry operator[1] of the equation (1) is its recursion operator. It was shown in [17] that a complete list of independent operators among such operators is exhausted by those associated with the canonical basis elements of the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$. This is why the associative algebra generated by Lie-symmetry operators of (1) is denoted by $\Upsilon_\mathfrak{r}$. We considered the subalgebra $\Lambda_\mathfrak{r}$ of $\Sigma$ that consists of the generalised-symmetry vector fields obtained by the action of the operators from $\Upsilon_\mathfrak{r}$ on the elementary symmetry vector field $u\partial_u$ of (1). We related this subalgebra to generating solutions of the equation (1) via the iterative action by its Lie-symmetry operators. In this way, taking the group-invariant solutions of the equation (1) as seeds, many more solution families were constructed for it. Nevertheless, the description of the generalised symmetry algebra was left in [16, 17] as an open problem, which we solve in the present paper.

The algebra $\Sigma$ splits over its infinite-dimensional ideal $\Sigma^{-\infty}$ associated with the linear superposition of the solutions and constituted by the vector fields $f(t,x,y)\partial_u$, where the parameter function $f$ runs through the solution set of the equation (1). Thus, $\Sigma = \Sigma^{\rm ess} \in \Sigma^{-\infty}$, where $\Sigma^{\rm ess}$ is a complementary subalgebra to the ideal $\Sigma^{-\infty}$ in $\Sigma$. We show that the subalgebra $\Sigma^{\rm ess}$ coincides with $\Lambda_\mathfrak{r}$. The proof of this assertion is surprisingly unusual. The core of the proof is to show that the entire subalgebra $\Lambda$ of $\Sigma$ constituted by the linear generalised symmetries of the equation (1) coincides with the algebra $\Lambda_\mathfrak{r}$. The latter straightforwardly implies that any subspace consisting of the linear generalised symmetries of order bounded by a fixed $n \in \mathbb{N}$ is finite-dimensional, which allows us to apply the Shapovalov–Shirokov

[1] A *Lie-symmetry operator* of a homogeneous linear system of differential equations $\mathcal{L}$: ${\rm L}u = 0$ is a first-order linear differential operator Q in total derivatives such that the tuple of differential functions Q$u$ is the characteristic of an (essential) Lie symmetry of $\mathcal{L}$.

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theorem [41] and state that $\Sigma^{\rm ess}=\Lambda$. Moreover, this approach requires a preliminary study of the algebra $\Upsilon_\mathfrak{r}$ using methods from ring theory and algebraic geometry, which is uncommon for group analysis of differential equations. The biggest challenge was to analyse how the Casimir operator of the Levi factor $\mathfrak{f}\simeq{\rm sl}(2,\mathbb{R})$ of $\mathfrak{g}^{\rm ess}$ and its multiples are related to the algebra $\Upsilon_\mathfrak{r}$. More specifically, the counterpart C of this operator in $\Upsilon_\mathfrak{r}$ is of degree four as a polynomial, while having order three as a differential operator. This property impacted constructing a basis and, therefore, computing the dimension of the subspace $\Lambda^n_\mathfrak{r}$ of $\Lambda_\mathfrak{r}$, $n\in\mathbb{N}_0$, that is constituted by the elements of $\Lambda_\mathfrak{r}$ whose order is bounded by $n$.

We were impressed to find out that the algebra $\Upsilon_\mathfrak{r}$ is isomorphic to the second Weyl algebra ${\rm W}(2,\mathbb{R})$, which gives rise to the isomorphism between the algebra $\Lambda$ and the Lie algebra ${\rm W}(2,\mathbb{R})^{(-)}$ associated with ${\rm W}(2,\mathbb{R})$. Due to this, we straightforwardly obtain a number of properties of the algebra $\Lambda$. In particular, it is two-generated and $\mathbb{Z}$-graded, its centre is one-dimensional, and its quotient algebra by the centre is simple. It was proved in Corollary 21 of the fourth arXiv version of [18] that the algebra of the linear generalised symmetries of the linear (1 + 1)-dimensional heat equation is isomorphic to the Lie algebra ${\rm W}(1,\mathbb{R})^{(-)}$ associated with the first Weyl algebra ${\rm W}(1,\mathbb{R})$, see Remark 19 below for more details. The above facts strengthen the connection between the heat equation and the remarkable Fokker–Planck equation from the point of view of generalised symmetries, which we also fit into the wider framework of (ultra)parabolic linear second-order partial differential equations with an arbitrary number of independent variables in the conclusion.

The paper is organised as follows. In Section 2, we present the maximal Lie invariance algebra of the remarkable Fokker–Planck equation (1) and describe its key properties. This is followed by the study of the associative algebra $\Upsilon_\mathfrak{r}$ of differential operators generated by the Lie-symmetry operators associated with the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$. We construct, in an explicit form, a basis of the algebra $\Upsilon_\mathfrak{r}$ and, for each $n\in\mathbb{N}_0$, a basis of its subspace of differential operators of order less than or equal to $n$. We also show that the algebra $\Upsilon_\mathfrak{r}$ is isomorphic to the second Weyl algebra ${\rm W}(2,\mathbb{R})$. Section 3 is devoted to the study of the polynomial solutions of (1). The results of this section are used in Section 4 in the course of proving the assertion that the algebra $\Lambda$ coincides with $\Lambda_\mathfrak{r}$. The latter straightforwardly leads to the description of the algebra $\Sigma$ of the generalised symmetries of the equation (1). Proving that the algebra $\Lambda$ is isomorphic to the Lie algebra ${\rm W}(2,\mathbb{R})^{(-)}$ allows us to transfer the known results about ${\rm W}(2,\mathbb{R})^{(-)}$ to $\Lambda$ and, conversely, to study ${\rm W}(2,\mathbb{R})^{(-)}$ from the perspective of its explicit faithful realisation $\Lambda$. Section 5 begins with a review of symmetry approaches for the construction of exact solutions of systems of differential equations. We also essentially develop the theoretical background of finding solutions of homogeneous linear systems of differential equations using their linear generalised symmetries. Then, the developed tools are efficiently applied to the equation (1). The results of the paper and possible avenues for future research are discussed in Section 6.

## 2. Lie-symmetry operators

The maximal Lie invariance algebra of the equation (1) is (see, e.g., [21])

$$\mathfrak{g}:=\langle\mathcal{P}^t,\ \mathcal{D},\ \mathcal{K},\ \mathcal{P}^3,\ \mathcal{P}^2,\ \mathcal{P}^1,\ \mathcal{P}^0,\ \mathcal{I},\mathcal{Z}(f)\rangle,$$

where

$$\begin{aligned}
&\mathcal{P}^t=\partial_t,\ \ \mathcal{D}=2t\partial_t+x\partial_x+3y\partial_y-2u\partial_u,\ \ \mathcal{K}=t^2\partial_t+(tx+3y)\partial_x+3ty\partial_y-(x^2+2t)u\partial_u,\\
&\mathcal{P}^3=3t^2\partial_x+t^3\partial_y+3(y-tx)u\partial_u,\ \ \mathcal{P}^2=2t\partial_x+t^2\partial_y-xu\partial_u,\ \ \mathcal{P}^1=\partial_x+t\partial_y,\ \ \mathcal{P}^0=\partial_y,\\
&\mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x,y)\partial_u.
\end{aligned}$$

Here, the parameter function $f$ of $(t,x,y)$ runs through the solution set of the equation (1).

The vector fields $\mathcal{Z}(f)$ constitute the infinite-dimensional abelian ideal $\mathfrak{g}^{\rm lin}$ of $\mathfrak{g}$ associated with the linear superposition of solutions of (1), $\mathfrak{g}^{\rm lin}:=\{\mathcal{Z}(f)\}$. Thus, the algebra $\mathfrak{g}$ can be represented as a semidirect

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sum, $\mathfrak{g}=\mathfrak{g}^{\rm ess}\Subset\mathfrak{g}^{\rm lin}$, where

$$\mathfrak{g}^{\rm ess}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K},\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I}\rangle$$

is an (eight-dimensional) subalgebra of $\mathfrak{g}$, called the essential Lie invariance algebra of (1).

Up to the skew-symmetry of the Lie bracket, the nonzero commutation relations between the basis vector fields of $\mathfrak{g}^{\rm ess}$ are the following:

$$\begin{gathered}
[\mathcal{P}^t,\mathcal{D}]=2\mathcal{P}^t,\quad [\mathcal{P}^t,\mathcal{K}]=\mathcal{D},\quad [\mathcal{D},\mathcal{K}]=2\mathcal{K},\\
[\mathcal{P}^t,\mathcal{P}^3]=3\mathcal{P}^2,\quad [\mathcal{P}^t,\mathcal{P}^2]=2\mathcal{P}^1,\quad [\mathcal{P}^t,\mathcal{P}^1]=\mathcal{P}^0,\\
[\mathcal{D},\mathcal{P}^3]=3\mathcal{P}^3,\quad [\mathcal{D},\mathcal{P}^2]=\mathcal{P}^2,\quad [\mathcal{D},\mathcal{P}^1]=-\mathcal{P}^1,\quad [\mathcal{D},\mathcal{P}^0]=-3\mathcal{P}^0,\\
[\mathcal{K},\mathcal{P}^2]=-\mathcal{P}^3,\quad [\mathcal{K},\mathcal{P}^1]=-2\mathcal{P}^2,\quad [\mathcal{K},\mathcal{P}^0]=-3\mathcal{P}^1,\\
[\mathcal{P}^1,\mathcal{P}^2]=-\mathcal{I},\quad [\mathcal{P}^0,\mathcal{P}^3]=3\mathcal{I}.
\end{gathered}$$

The algebra $\mathfrak{g}^{\rm ess}$ is nonsolvable. Its Levi decomposition is given by $\mathfrak{g}^{\rm ess}=\mathfrak{f}\Subset\mathfrak{r}$, where the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ coincides with the nilradical of $\mathfrak{g}^{\rm ess}$ and is spanned by the vector fields $\mathcal{P}^3$, $\mathcal{P}^2$, $\mathcal{P}^1$, $\mathcal{P}^0$ and $\mathcal{I}$,

$$\mathfrak{r}=\langle\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I}\rangle.$$

The Levi factor $\mathfrak{f}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K}\rangle$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to ${\rm sl}(2,\mathbb{R})$, the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to the rank-two Heisenberg algebra ${\rm h}(2,\mathbb{R})$ and the real representation of the Levi factor $\mathfrak{f}$ on the radical $\mathfrak{r}$ coincides, in the basis $(\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I})$, with the real representation $\rho_3\oplus\rho_0$ of ${\rm sl}(2,\mathbb{R})$. Here, $\rho_n$ is the standard real irreducible representation of ${\rm sl}(2,\mathbb{R})$ in the $(n+1)$-dimensional vector space. More specifically,

$$\rho_n(\mathcal{P}^t)_{ij}=(n-j)\delta_{i,j+1},\quad \rho_n(\mathcal{D})_{ij}=(n-2j)\delta_{ij},\quad \rho_n(-\mathcal{K})_{ij}=j\delta_{i+1,j},$$

where $i,j\in\{1,2,\dots,n+1\}$, $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and $\delta_{kl}$ is the Kronecker delta, that is, $\delta_{kl}=1$ if $k=l$ and $\delta_{kl}=0$ otherwise, $k,l\in\mathbb{N}_0$. Thus, the entire algebra $\mathfrak{g}^{\rm ess}$ is isomorphic to the algebra $L_{8,19}\simeq{\rm sl}(2,\mathbb{R})\Subset_{\rho_2\oplus\rho_0}{\rm h}(1,\mathbb{R})$ from the classification of indecomposable Lie algebras of dimensions up to eight with nontrivial Levi decompositions, which was carried out in [45].

Lie algebras whose Levi factors are isomorphic to the algebra ${\rm sl}(2,\mathbb{R})$ often arise within the field of group analysis of differential equations as Lie invariance algebras of parabolic partial differential equations. At the same time, the action of Levi factors on the corresponding radicals is usually described in terms of the representations $\rho_0$, $\rho_1$, $\rho_2$ or their direct sums, cf. the essential Lie invariance algebra of the linear (1 + 1)-dimensional heat equation, which is isomorphic to the so-called special Galilei algebra ${\rm sl}(2,\mathbb{R})\Subset_{\rho_1\oplus\rho_0}{\rm h}(1,\mathbb{R})$ [18, Section 3]. To the best of our knowledge, algebras analogous to $\mathfrak{g}^{\rm ess}$ had not been studied in group analysis from the point of view of their subalgebra structure before [16]. See the conclusion for the discussion on (ultra)parabolic second-order multidimensional linear partial differential equations with essential Lie invariance algebras isomorphic to ${\rm sl}(2,\mathbb{R})\Subset_{\rho_{2n-1}\oplus\rho_0}{\rm h}(n,\mathbb{R})$, where ${\rm h}(n,\mathbb{R})$ denotes the rank-$n$ Heisenberg algebra, $n\in\mathbb{N}$.

Consider the Lie-symmetry operators of (1) that are associated with the Lie-symmetry vector fields $-\mathcal{P}^3$, $-\mathcal{P}^2$, $-\mathcal{P}^1$, $-\mathcal{P}^0$ and $-\mathcal{P}^t$, $-\mathcal{D}$, $-\mathcal{K}$ (here, we take minuses for a nicer representation of differential operators),

$$\begin{gathered}
{\rm P}^3:=3t^2{\rm D}_x+t^3{\rm D}_y-3(y-tx),\quad {\rm P}^2:=2t{\rm D}_x+t^2{\rm D}_y+x,\quad {\rm P}^1:={\rm D}_x+t{\rm D}_y,\quad {\rm P}^0:={\rm D}_y,\\
{\rm P}^t:={\rm D}_t,\quad {\rm D}:=2t{\rm D}_t+x{\rm D}_x+3y{\rm D}_y+2,\quad {\rm K}:=t^2{\rm D}_t+(tx+3y){\rm D}_x+3ty{\rm D}_y+x^2+2t.
\end{gathered}$$

The associative operator algebra $\Upsilon_\mathfrak{r}$ generated by the operators ${\rm P}^3$, ${\rm P}^2$, ${\rm P}^1$ and ${\rm P}^0$ admits the following presentation:

$$\begin{aligned}\Upsilon_\mathfrak{r}=\big\langle&{\rm P}^3,{\rm P}^2,{\rm P}^1,{\rm P}^0\mid\\
&[{\rm P}^3,{\rm P}^0]=3,\ [{\rm P}^1,{\rm P}^2]=1,\ [{\rm P}^3,{\rm P}^2]=[{\rm P}^3,{\rm P}^1]=[{\rm P}^2,{\rm P}^0]=[{\rm P}^1,{\rm P}^0]=0\big\rangle.\end{aligned}\tag{2}$$

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We begin describing the properties of the algebra $\Upsilon_{\mathfrak{r}}$ with finding its explicit basis.

**Lemma 1.** *Fixed any ordering* $(\mathrm{Q}^0,\mathrm{Q}^1,\mathrm{Q}^2,\mathrm{Q}^3)$ *of* $\{\mathrm{P}^0,\mathrm{P}^1,\mathrm{P}^2,\mathrm{P}^3\}$, $\mathrm{Q}^0<\mathrm{Q}^1<\mathrm{Q}^2<\mathrm{Q}^3$, *the monomials* $\mathbf{Q}^\alpha:=(\mathrm{Q}^0)^{\alpha_0}(\mathrm{Q}^1)^{\alpha_1}(\mathrm{Q}^2)^{\alpha_2}(\mathrm{Q}^3)^{\alpha_3}$ *with* $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}_0^4$ *constitute a basis of the algebra* $\Upsilon_{\mathfrak{r}}$.

**Proof.** The required claim follows from Bergman's diamond lemma [2, Theorem 1.2], see therein for all the related notions. Indeed, under the interpretation of the algebra $\Upsilon_{\mathfrak{r}}$ following [2, Section 3], there are exactly four overlap ambiguities, which are related to the products $\mathrm{Q}^3\mathrm{Q}^2\mathrm{Q}^1$, $\mathrm{Q}^3\mathrm{Q}^2\mathrm{Q}^0$, $\mathrm{Q}^3\mathrm{Q}^1\mathrm{Q}^0$ and $\mathrm{Q}^2\mathrm{Q}^1\mathrm{Q}^0$, and each of them is resolvable. □

By default, we use the ordering $\mathrm{P}^3<\mathrm{P}^2<\mathrm{P}^1<\mathrm{P}^0$.

**Lemma 2.** *In the sense of unital algebras, the algebra* $\Upsilon_{\mathfrak{r}}$ *is isomorphic to the quotient algebra of the universal enveloping algebra* $\mathfrak{U}(\mathfrak{r})$ *of* $\mathfrak{r}$ *by the two-sided ideal* $(\iota(\mathcal{I})+1)$ *generated by* $\iota(\mathcal{I})+1$, $\Upsilon_{\mathfrak{r}}\simeq\mathfrak{U}(\mathfrak{r})/(\iota(\mathcal{I})+1)$, *where* $\iota\colon\mathfrak{r}\hookrightarrow\mathfrak{U}(\mathfrak{r})$ *is the canonical embedding of the Lie algebra* $\mathfrak{r}$ *in its universal enveloping algebra* $\mathfrak{U}(\mathfrak{r})$. *Moreover, this defines an isomorphism between the associated Lie algebras* $\Upsilon_{\mathfrak{r}}^{(-)}$ *and* $\big(\mathfrak{U}(\mathfrak{r})/(\iota(\mathcal{I})+1)\big)^{(-)}$.

**Proof.** The correspondence $\mathcal{P}^j\mapsto\mathrm{P}^j$, $j=0,1,2,3$ and $\mathcal{I}\mapsto-1$ linearly extends to the Lie algebra homomorphism $\varphi$ from $\mathfrak{r}$ to the Lie algebra $\Upsilon_{\mathfrak{r}}^{(-)}$ associated with the associative algebra $\Upsilon_{\mathfrak{r}}$. By the universal property of the universal enveloping algebra $\mathfrak{U}(\mathfrak{r})$, the Lie algebra homomorphism $\varphi$ extends to the (unital) associative algebra homomorphism $\hat\varphi\colon\mathfrak{U}(\mathfrak{r})\to\Upsilon_{\mathfrak{r}}$, that is, $\varphi=\hat\varphi\circ\iota$ as homomorphisms of vector spaces. Since the algebra $\Upsilon_{\mathfrak{r}}$ is generated by $\varphi(\mathfrak{r})$, the homomorphism $\hat\varphi$ is surjective.

For the rest of the proof, we identify $\mathfrak{r}$ with its image under the map $\iota$ in $\mathfrak{U}(\mathfrak{r})$. It is clear that $(\mathcal{I}+1)\subset\ker\hat\varphi$. To show the reverse inclusion, consider an arbitrary polynomial $Q\in\mathfrak{U}(\mathfrak{r})$, which in view of the Poincaré–Birkhoff–Witt theorem takes the form

$$Q=c_{i_3i_2i_1i_0j}(\mathcal{P}^3)^{i_3}(\mathcal{P}^2)^{i_2}(\mathcal{P}^1)^{i_1}(\mathcal{P}^0)^{i_0}\mathcal{I}^j$$

with a finite number of nonzero coefficients $c_{i_3i_2i_1i_0j}$, and assume that $Q\in\ker\hat\varphi$,

$$\hat\varphi(Q)=(-1)^jc_{i_3i_2i_1i_0j}(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}=0.$$

Here and in what follows we assume summation with respect to repeated indices. In view of Lemma 1, we have $(-1)^jc_{i_3i_2i_1i_0j}=0$ for each fixed tuple $(i_3,i_2,i_1,i_0)$. Therefore,

$$\begin{aligned}Q&=c_{i_3i_2i_1i_0j}(\mathcal{P}^3)^{i_3}(\mathcal{P}^2)^{i_2}(\mathcal{P}^1)^{i_1}(\mathcal{P}^0)^{i_0}\mathcal{I}^j-(-1)^jc_{i_3i_2i_1i_0j}(\mathcal{P}^3)^{i_3}(\mathcal{P}^2)^{i_2}(\mathcal{P}^1)^{i_1}(\mathcal{P}^0)^{i_0}\\&=c_{i_3i_2i_1i_0j}(\mathcal{P}^3)^{i_3}(\mathcal{P}^2)^{i_2}(\mathcal{P}^1)^{i_1}(\mathcal{P}^0)^{i_0}(\mathcal{I}^j-(-1)^j).\end{aligned}$$

For each $j$, the factor $\mathcal{I}^j-(-1)^j$ is divisible by $\mathcal{I}+1$. Therefore, $\ker\hat\varphi=(\mathcal{I}+1)$ and the isomorphism $\Upsilon_{\mathfrak{r}}\simeq\mathfrak{U}(\mathfrak{r})/(\mathcal{I}+1)$ follow from the first isomorphism theorem for associative algebras.

The isomorphism between the associated Lie algebras $\Upsilon_{\mathfrak{r}}^{(-)}$ and $\big(\mathfrak{U}(\mathfrak{r})/(\mathcal{I}+1)\big)^{(-)}$ follows from the fact that, by definition, the Lie brackets on these algebras are the ring-theoretic commutators on the corresponding associative algebras. □

**Remark 3.** Recall the definition of the $n$th Weyl algebra $\mathrm{W}(n,\mathbb{R})$. It is the quotient of the free associative $\mathbb{R}$-algebra on the alphabet $\{\hat p_1,\dots,\hat p_n,\hat q_1,\dots,\hat q_n\}$ by the two-side ideal generated by $\hat p_i\hat p_j-\hat p_j\hat p_i$, $\hat q_i\hat q_j-\hat q_j\hat q_i$ and $\hat p_i\hat q_j-\hat q_j\hat p_i-\delta_{ij}$. Here and in the rest of this remark, the indices $i$ and $j$ run from 1 to $n$. Recall that $\delta_{ij}$ denotes the Kronecker delta. Hence, the algebra $\mathrm{W}(n,\mathbb{R})$ admits the presentation

$$\mathrm{W}(n,\mathbb{R})=\langle\hat p_1,\dots,\hat p_n,\hat q_1,\dots,\hat q_n\mid\hat p_i\hat p_j-\hat p_j\hat p_i=\hat q_i\hat q_j-\hat q_j\hat q_i=0,\ \hat p_i\hat q_j-\hat q_j\hat p_i=\delta_{ij}\rangle.$$

This algebra can be related to the quotient of the universal enveloping algebra of the rank-$n$ Heisenberg Lie algebra $\mathrm{h}(n,\mathbb{R})$. More specifically, let the elements $p_i$, $q_i$ and $c$ constitute the canonical basis of the Lie algebra $\mathrm{h}(n,\mathbb{R})$, and thus, they satisfy the commutation relations $[p_i,p_j]=[q_i,q_j]=0$ and $[p_i,q_j]=\delta_{ij}c$. The $n$th Weyl algebra $\mathrm{W}(n,\mathbb{R})$ is the quotient of the universal enveloping algebra $\mathfrak{U}\big(\mathrm{h}(n,\mathbb{R})\big)$ of $\mathrm{h}(n,\mathbb{R})$ by the two-sided ideal $(c-1)$ generated by $c-1$, $\mathrm{W}(n,\mathbb{R}):=\mathfrak{U}\big(\mathrm{h}(n,\mathbb{R})\big)/(c-1)$. The canonical basis

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in $\mathrm{W}(n,\mathbb{R})$ consists of monomials $q^\kappa p^\lambda$, where $q^\kappa := q_1^{\kappa_1}\cdots q_n^{\kappa_n}$, $p^\lambda := p_1^{\lambda_1}\cdots p_n^{\lambda_n}$ and $\kappa := (\kappa_1,\dots,\kappa_n)$, $\lambda = (\lambda_1,\dots,\lambda_n)$ are multi-indices running through $\mathbb{N}_0^n$.

The commutation relations in the above basis of $\mathrm{W}(n,\mathbb{R})$ take the form

$$[q^\kappa p^\lambda, q^{\kappa'}p^{\lambda'}] = \sum_{\nu\in\mathbb{N}_0^n} \nu! \left(\binom{\kappa'}{\nu}\binom{\lambda}{\nu} - \binom{\kappa}{\nu}\binom{\lambda'}{\nu}\right) q^{\kappa+\kappa'-\nu}p^{\lambda+\lambda'-\nu}. \tag{3}$$

Note that the series in the commutator $[q^\kappa p^\lambda, q^{\kappa'}p^{\lambda'}]$ is in fact the finite sum with the multi-index $\nu$ satisfying the inequalities $\nu\leqslant\kappa$ and $\nu\leqslant\lambda'$ or $\nu\leqslant\kappa'$ and $\nu\leqslant\lambda$ since $\binom{\kappa}{\nu}=0$ if $\nu$ does not satisfy the inequality $\nu\leqslant\kappa$.

**Remark 4.** The opposite algebra $\mathrm{W}(n,\mathbb{R})^{\rm op}$ of $\mathrm{W}(n,\mathbb{R})$ admits the presentation

$$\mathrm{W}(n,\mathbb{R})^{\rm op} = \langle \check p_1,\dots,\check p_n,\check q_1,\dots,\check q_n \mid \check p_i\check p_j - \check p_j\check p_i = \check q_i\check q_j - \check q_j\check q_i = 0,\ \check p_i\check q_j - \check q_j\check p_i = -\delta_{ij}\rangle.$$

This results in the isomorphism $\mathrm{W}(n,\mathbb{R})^{\rm op} \simeq \mathfrak{U}\big(\mathrm{h}(n,\mathbb{R})\big)/(c+1)$ defined on the algebra generators by the correspondence $\check p_i\mapsto p_i$, $\check q_i\mapsto q_i$. It is clear that the algebras $\mathrm{W}(n,\mathbb{R})$ and $\mathrm{W}(n,\mathbb{R})^{\rm op}$ are isomorphic, where the simplest isomorphism is given by permuting the $p$- and $q$-tuples, $\hat p_i\leftrightarrow\check q_i$, $\hat q_i\leftrightarrow\check p_i$.

**Corollary 5.** *The algebra $\Upsilon_{\mathfrak r}$ is isomorphic to the opposite of the second Weyl algebra and hence to the second Weyl algebra itself, $\Upsilon_{\mathfrak r}\simeq \mathrm{W}(2,\mathbb{R})^{\rm op}\simeq\mathrm{W}(2,\mathbb{R})$.*

The explicit isomorphisms in Corollary 5 are established, for example, by the correspondence on the level of generators in the following way:

$$\big(\tfrac13\mathrm{P}^3,\mathrm{P}^2,\mathrm{P}^1,\mathrm{P}^0\big)\mapsto(\check q_1,\check p_2,\check q_2,\check p_1)\mapsto(\hat p_1,\hat q_2,\hat p_2,\hat q_1).$$

The algebra $\Upsilon_{\mathfrak r}$ possesses two natural filtrations,

$$F_1:\quad \Upsilon_{\mathfrak r} = \bigcup_{n\in\mathbb{N}_0}\Upsilon_n^{\rm ord},\qquad \Upsilon_n^{\rm ord} := \{\mathrm{Q}\in\Upsilon_{\mathfrak r}\mid \operatorname{ord}\mathrm{Q}\leqslant n\},$$
$$F_2:\quad \Upsilon_{\mathfrak r} = \bigcup_{n\in\mathbb{N}_0}\Upsilon_n^{\rm deg},\qquad \Upsilon_n^{\rm deg} := \{\mathrm{Q}\in\Upsilon_{\mathfrak r}\mid \deg\mathrm{Q}\leqslant n\},$$

where $\operatorname{ord}\mathrm{Q}$ is the order of Q as a differential operator and $\deg\mathrm{Q}$ is the degree of Q as a (noncommutative) polynomial in $\{\mathrm{P}^0,\mathrm{P}^1,\mathrm{P}^2,\mathrm{P}^3\}$. It is clear that $\operatorname{ord}\mathrm{Q}\leqslant\deg\mathrm{Q}$ for any $\mathrm{Q}\in\Upsilon_{\mathfrak r}$. Therefore, for each $n\in\mathbb{N}_0$, we have the inclusion $\Upsilon_n^{\rm deg}\subseteq\Upsilon_n^{\rm ord}$. The (unordered) basis of the space $\Upsilon_n^{\rm deg}$ that corresponds to the ordering $\mathrm{P}^3<\mathrm{P}^2<\mathrm{P}^1<\mathrm{P}^0$ is the set

$$\{(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}\mid i_3+i_2+i_1+i_0\leqslant n\},$$

which is the restriction of the corresponding basis of the algebra $\Upsilon_{\mathfrak r}$ to the subspace $\Upsilon_n^{\rm deg}$.

The description of bases of the subspaces $\Upsilon_n^{\rm ord}$, $n\in\mathbb{N}_0$, is more complicated. To construct such bases, we should consider a distinguish element C of $\Upsilon_{\mathfrak r}$. On solutions of the equation (1), its Lie-symmetry operators $\mathrm{P}^t$, D and K associated with its Lie symmetries $-\mathcal{P}^t$, $-\mathcal{D}$ and $-\mathcal{K}$ are equivalent to the elements

$$\begin{aligned}
\hat{\mathrm{P}}^t &:= (\mathrm{P}^1)^2 - \mathrm{P}^2\mathrm{P}^0 = \mathrm{D}_x^2 - x\mathrm{D}_y,\\
\hat{\mathrm{D}} &:= \mathrm{P}^2\mathrm{P}^1 - \mathrm{P}^3\mathrm{P}^0 + 2 = 2t\mathrm{D}_x^2 + x\mathrm{D}_x + (3y-2tx)\mathrm{D}_y + 2,\\
\hat{\mathrm{K}} &:= (\mathrm{P}^2)^2 - \mathrm{P}^3\mathrm{P}^1 = t^2\mathrm{D}_x^2 + (3y+tx)\mathrm{D}_x + t(3y-tx)\mathrm{D}_y + x^2 + 2t
\end{aligned}\tag{4}$$

of the associative algebra $\Upsilon_{\mathfrak r}$, respectively. The associative algebra $\Upsilon_{\mathfrak f}$ generated by $\hat{\mathrm{P}}^t$, $\hat{\mathrm{D}}$ and $\hat{\mathrm{K}}$ is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathfrak f)$ of the Levi factor $\mathfrak f$. In other words, the algebra $\Upsilon_{\mathfrak r}$ contains an isomorphic copy $\Upsilon_{\mathfrak f}$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak f)$. This allows us to consider the counterpart of the Casimir operator $\mathrm{D}^2 - 2(\mathrm{K}\mathrm{P}^t + \mathrm{P}^t\mathrm{K})$ of the Levi factor $\mathfrak f$ inside the algebra $\Upsilon_{\mathfrak r}$.

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This operator is equivalent on the solutions of (1) to the operator

$$\begin{aligned}\mathrm{C} &:= \hat{\mathrm{D}}^2 - 2(\hat{\mathrm{K}}\hat{\mathrm{P}}^t + \hat{\mathrm{P}}^t\hat{\mathrm{K}})\\ &= (\mathrm{P}^3)^2(\mathrm{P}^0)^2 - 6\mathrm{P}^3\mathrm{P}^2\mathrm{P}^1\mathrm{P}^0 - 3(\mathrm{P}^2)^2(\mathrm{P}^1)^2 + 4(\mathrm{P}^2)^3\mathrm{P}^0 + 4\mathrm{P}^3(\mathrm{P}^1)^3 + 3\mathrm{P}^2\mathrm{P}^1 - 9\mathrm{P}^3\mathrm{P}^0\\ &= -12y\mathrm{D}_x^3 - 3x^2\mathrm{D}_x^2 + 18xy\mathrm{D}_x\mathrm{D}_y + 9y^2\mathrm{D}_y^2 + 3x\mathrm{D}_x + (4x^3 + 27y)\mathrm{D}_y.\end{aligned}$$

We observe an interesting phenomenon in the algebra $\Upsilon_{\mathfrak{t}}$. The element C of $\Upsilon_{\mathfrak{t}}$ is a third-order differential operator. At the same time, it is a linear combination of monomials in $(\mathrm{P}^3, \mathrm{P}^2, \mathrm{P}^1, \mathrm{P}^0)$ up to degree four and, in view of Lemma 1, it cannot be represented as a linear combination of monomials of degrees less than or equal to three. Moreover, it can be proved that modulo linearly recombining with later monomials, it is a unique element with such property within the subspace of third-order differential operators in $\Upsilon_{\mathfrak{t}}$.[2] The operator C has a number of other specific properties. In particular, the only third-order differentiation in it is $\mathrm{D}_x^3$, it contains no zero-order term and its coefficients do not depend on $t$.

**Theorem 6.** *A basis of the subspace $\Upsilon_n^{\mathrm{ord}}$ of differential operators of order less than or equal to $n \in \mathbb{N}_0$ in $\Upsilon_{\mathfrak{t}}$ is constituted by the products $(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}$, where $i_0, i_1, i_2, i_3 \in \mathbb{N}_0$ with $i_0 + i_1 + i_2 + i_3 \leqslant n$, and $\mathrm{C}^m(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}$, where $i_0, i_1, i_2, i_3 \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $i_0 + i_1 + i_2 + i_3 + 3m = n$.*

**Proof.** Consider the associated graded algebras $\mathrm{gr}_1\Upsilon_{\mathfrak{t}}$ and $\mathrm{gr}_2\Upsilon_{\mathfrak{t}}$ of the algebra $\Upsilon_{\mathfrak{t}}$ with respect to the filtrations $F_1$ and $F_2$, respectively,

$$\mathrm{gr}_1\Upsilon_{\mathfrak{t}} := \bigoplus_{n=0}^{\infty} \Upsilon_n^{\mathrm{ord}}/\Upsilon_{n-1}^{\mathrm{ord}} \quad \text{and} \quad \mathrm{gr}_2\Upsilon_{\mathfrak{t}} := \bigoplus_{n=0}^{\infty} \Upsilon_n^{\mathrm{deg}}/\Upsilon_{n-1}^{\mathrm{deg}},$$

assuming $\Upsilon_{-1}^{\mathrm{ord}} = \Upsilon_{-1}^{\mathrm{deg}} := \{0\}$. The algebra $\Upsilon_{\mathfrak{t}}$ is related to $\mathrm{gr}_1\Upsilon_{\mathfrak{t}}$ and $\mathrm{gr}_2\Upsilon_{\mathfrak{t}}$ via the corresponding initial form maps $\psi_i\colon \Upsilon_{\mathfrak{t}} \to \mathrm{gr}_i\Upsilon_{\mathfrak{t}}$,

$$\psi_1(\mathrm{Q}) := \pi^1_{\mathrm{ord\,Q}-1}(\mathrm{Q}) \quad \text{and} \quad \psi_2(\mathrm{Q}) := \pi^2_{\mathrm{deg\,Q}-1}(\mathrm{Q}),\ \mathrm{Q} \in \Upsilon_{\mathfrak{t}},$$

where $\pi_n^1\colon \Upsilon_{\mathfrak{t}} \to \Upsilon_{\mathfrak{t}}/\Upsilon_n^{\mathrm{ord}}$ and $\pi_n^2\colon \Upsilon_{\mathfrak{t}} \to \Upsilon_{\mathfrak{t}}/\Upsilon_n^{\mathrm{deg}}$ are the canonical projections. Properties of the commutator of differential operators and the presentation (2) of the algebra $\Upsilon_{\mathfrak{t}}$ straightforwardly imply that the algebras $\mathrm{gr}_1\Upsilon_{\mathfrak{t}}$ and $\mathrm{gr}_2\Upsilon_{\mathfrak{t}}$ are commutative. Moreover, the algebra $\mathrm{gr}_2\Upsilon_{\mathfrak{t}}$ is the polynomial algebra $\mathbb{R}[x_0, x_1, x_2, x_3]$ in the variables $x_j := \psi_2(\mathrm{P}^j)$, $j = 0, 1, 2, 3$. Extending $\psi_1$ to the algebra of differential operators in the total derivatives with respect to $x$ and $y$ with coefficients depending on $(t, x, y)$, we denote $X := \psi_1(\mathrm{D}_x)$ and $Y := \psi_1(\mathrm{D}_y)$. Then,

$$\psi_1(\mathrm{P}^0) := Y, \quad \psi_1(\mathrm{P}^1) := X + tY, \quad \psi_1(\mathrm{P}^2) := 2tX + t^2Y, \quad \psi_1(\mathrm{P}^3) := 3t^2X + t^3Y,$$

and the algebra $\mathrm{gr}_1\Upsilon_{\mathfrak{t}}$ can be identified with the polynomial algebra

$$\mathbb{R}[Y,\ X + tY,\ 2tX + t^2Y,\ 3t^2X + t^3Y].$$

The subspace inclusions $i_n\colon \Upsilon_n^{\mathrm{deg}} \hookrightarrow \Upsilon_n^{\mathrm{ord}}$, $n \in \mathbb{N}_0 \cup \{-1\}$, jointly give rise to an algebra homomorphism $f\colon \mathrm{gr}_2\Upsilon_{\mathfrak{t}} \to \mathrm{gr}_1\Upsilon_{\mathfrak{t}}$ that makes the following diagram commutative for each $n \in \mathbb{N}_0 \cup \{-1\}$:

$$\begin{array}{ccc}\Upsilon_n^{\mathrm{deg}} & \xhookrightarrow{\ i_n\ } & \Upsilon_n^{\mathrm{ord}}\\ \Big\downarrow{\scriptstyle \psi_2} & & \Big\downarrow{\scriptstyle \psi_1}\\ \mathrm{gr}_2\Upsilon_{\mathfrak{t}} & \xrightarrow{\ f\ } & \mathrm{gr}_1\Upsilon_{\mathfrak{t}}\end{array}$$

The map $f$ is defined elementwise via the correspondence

$$\mathrm{Q} + \Upsilon^{\mathrm{deg}}_{\mathrm{deg\,Q}-1} \mapsto \psi_1(\mathrm{Q}) + \Upsilon^{\mathrm{ord}}_{\mathrm{deg\,Q}-1}.$$

[2]We first derived this claim after computing the space $\Sigma^3$ of generalised symmetries of the equation (1), see the notation in Section 4 below. It is also an obvious consequence of Theorem 6.

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It is straightforward to verify that it is a well-defined unital homomorphism of associative algebras, and $f(x_j)=\psi_1(\mathrm{P}^j)$, $j=0,1,2,3$. In other words, the image of a differential operator $\mathrm{Q}\in\Upsilon_\mathfrak{r}$ under the composition $f\circ\psi_2$ is its formal symbol if $\operatorname{ord}\mathrm{Q}=\deg\mathrm{Q}$, and it is zero otherwise.

The property $f\circ\psi_2(\mathrm{C})=0$ of the Casimir element $\mathrm{C}\in\Upsilon_\mathfrak{r}$ is equivalent to the fact that the solution set of the polynomial equation $\check{\mathrm{C}}=0$, where

$$\check{\mathrm{C}}:=\psi_2(\mathrm{C})=x_3^2x_0^2-6x_3x_2x_1x_0-3x_2^2x_1^2+4x_2^3x_0+4x_3x_1^3,$$

is a hypersurface in $\mathbb{R}^4$ with the parameterisation

$$x_3=3t^2X+t^3Y,\quad x_2=2tX+t^2Y,\quad x_1=X+tY,\quad x_0=Y,$$

where $(t,X,Y)$ is considered as the coordinate tuple of the affine space $\mathbb{R}^3$.

If $\deg\mathrm{Q}>\operatorname{ord}\mathrm{Q}$, then $f\circ\psi_2(\mathrm{Q})=0$, and thus, the zero locus of the polynomial $\check{\mathrm{C}}$ is contained in the zero locus of the polynomial $\check{\mathrm{Q}}:=\psi_2(\mathrm{Q})$. In other words, the vanishing ideal of the hypersurface $\check{\mathrm{Q}}=0$ in the polynomial algebra $\mathbb{R}[x_0,x_1,x_2,x_3]$ is contained in the vanishing ideal of the hypersurface $\check{\mathrm{C}}=0$ in this algebra. Therefore, by Hilbert's Nullstellensatz in the form [48, Chapter VII, Theorem 14], the polynomial $\check{\mathrm{Q}}$ belongs to the radical of the principal ideal $I:=(\check{\mathrm{C}})$ in $\mathbb{R}[x_0,x_1,x_2,x_3]$, that is, there exists $m\in\mathbb{N}$ such that $\check{\mathrm{Q}}^m\in I$.

We show that the polynomial $\check{\mathrm{C}}$ is irreducible. Assume to the contrary that it is reducible. Since the multipliers $x_3^2$ and $x_0^2$ appear only in the monomial $x_3^2x_0^2$ in $\check{\mathrm{C}}$ and both $x_0$ and $x_3$ do not divide $\check{\mathrm{C}}$, the only possible factorisation of $\check{\mathrm{C}}$ is

$$(x_3x_0+p)(x_3x_0+q)$$

for some homogeneous second-degree polynomials $p,q\in\mathbb{R}[x_0,x_1,x_2,x_3]$ that are affine with respect to $(x_0,x_3)$. Hence, $p+q=-6x_1x_2$ and $pq=-3x_2^2x_1^2+4x_2^3x_0+4x_3x_1^3$. Up to the permutation of $p$ and $q$, we can assume that $q$ does not involve $x_0$ and $x_3$. Then $q$ divides both $x_2^3$ and $x_1^3$, which is impossible if $q$ is not a constant.

The irreducibility of $\check{\mathrm{C}}$ implies its primality since the algebra $\mathbb{R}[x_0,x_1,x_2,x_3]$ is a unique factorisation domain. This is why the ideal $I=(\check{\mathrm{C}})$ is prime. Hence it is radical as well, that is, it coincides with its radical $\sqrt{I}:=\{g\in\mathbb{R}[x_0,x_1,x_2,x_3]\mid g^m\in I\text{ for some }m\in\mathbb{N}\}$.

Moreover, if $\deg\mathrm{Q}-\operatorname{ord}\mathrm{Q}=:m\in\mathbb{N}$, then the polynomial $\check{\mathrm{Q}}$ is a linear combination of monomials of the form $\check{\mathrm{C}}^mx_3^{i_3}x_2^{i_2}x_1^{i_1}x_0^{i_0}$, where $i_3+i_2+i_1+i_0=\operatorname{ord}\mathrm{Q}-3m=\deg\mathrm{Q}-4m$. Indeed, in the light of the above arguments, the polynomial $\check{\mathrm{Q}}$ is of the form $\check{\mathrm{C}}^lF$ for some $l\in\mathbb{N}$, where $F$ is a homogeneous polynomial of the degree $\deg\mathrm{Q}-4l$ with $F\notin(\check{\mathrm{C}})$. This implies that $l\leqslant m$. Assuming that $l<m$, by elementary degree counting we have $\deg F>\operatorname{ord}F$, which thus gives us that $F\in(\check{\mathrm{C}})$. This contradiction proves the required claim.

As a result, we prove that the set $\mathcal{B}$ of the products listed in the theorem's statement spans the subspace $\Upsilon_n^{\mathrm{ord}}$.

Consider the linear combination

$$\begin{aligned}\mathrm{Q}:=\sum_{j=1}^{k+1}\sum_{|i|=n-3j}&\lambda_{ji_0i_1i_2i_3}\mathrm{C}^j(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}\\&+\sum_{|i|\leqslant n}\lambda_{0i_0i_1i_2i_3}(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0},\end{aligned}$$

where $|i|:=i_0+i_1+i_2+i_3$ and $\lambda_{ji_0i_1i_2i_3}\in\mathbb{R}$. Suppose that $\mathrm{Q}=0$. Then,

$$\psi_2(\mathrm{Q})=\sum_{j=1}^{k+1}\hat{\mathrm{C}}^j\sum_{|i|=n-3j}\lambda_{ji_0i_1i_2i_3}x_3^{i_3}x_2^{i_2}x_1^{i_1}x_0^{i_0}+\sum_{|i|\leqslant n}\lambda_{0i_0i_1i_2i_3}x_3^{i_3}x_2^{i_2}x_1^{i_1}x_0^{i_0}=0,\tag{5}$$

where we assign $j=0$ for the terms in the last sum. We have $\deg\hat{\mathrm{C}}^jx_3^{i_3}x_2^{i_2}x_1^{i_1}x_0^{i_0}=|i|+4j$. Since all monomials in (5) are different, they are linearly independent, and thus $\lambda_{ji_0i_1i_2i_3}=0$ for all relevant values of $(j,i_0,i_1,i_2,i_3)$. We obtain that the set $\mathcal{B}$ is linearly independent. Therefore, it is a basis of the subspace $\Upsilon_n^{\mathrm{ord}}$. □

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**Corollary 7.** *The dimension of the subspace $\Upsilon_n^{\mathrm{ord}}$ of the algebra $\Upsilon_\mathfrak{r}$, which consists of differential operators of order less than or equal to n, is*

$$\dim \Upsilon_n^{\mathrm{ord}} = \begin{cases} \frac{1}{18}(n+1)(n+3)(n^2+4n+6) & \text{if} \quad n \equiv 0 \text{ or } 2 \pmod 3, \\ \frac{1}{18}(n+2)^2(n^2+4n+5) & \text{if} \quad n \equiv 1 \pmod 3. \end{cases} \tag{6}$$

*The dimension of the quotient space $\Upsilon_n^{\mathrm{ord}}/\Upsilon_{n-1}^{\mathrm{ord}}$ associated with the nth order differential operators in the algebra $\Upsilon_\mathfrak{r}$ is*

$$\dim \Upsilon_n^{\mathrm{ord}}/\Upsilon_{n-1}^{\mathrm{ord}} = \begin{cases} \frac{1}{9}(2n+3)(n^2+3n+3) & \text{if} \quad n \equiv 0 \pmod 3, \\ \frac{1}{9}(n+2)(2n^2+5n+5) & \text{if} \quad n \equiv 1 \pmod 3, \\ \frac{1}{9}(n+1)(2n^2+7n+8) & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

**Proof.** In view of Theorem 6, the dimension of the space $\Upsilon_n^{\mathrm{ord}}$ is

$$\dim \Upsilon_n^{\mathrm{ord}} = \sum_{k=0}^{n} \binom{k+3}{3} + \sum_{k=1}^{\lfloor n/3 \rfloor} \binom{n-3(k-1)}{3},$$

where $\lfloor x \rfloor$ denotes the 'floor' function. By the induction with respect to the parameter $n \in \mathbb{N}_0$, one can show that the above sum coincides with the value given in (6).

Since $\Upsilon_{n-1}^{\mathrm{ord}} \subset \Upsilon_n^{\mathrm{ord}}$, we have $\dim \Upsilon_n^{\mathrm{ord}}/\Upsilon_{n-1}^{\mathrm{ord}} = \dim \Upsilon_n^{\mathrm{ord}} - \dim \Upsilon_{n-1}^{\mathrm{ord}}$. □

## 3. Polynomial solutions

Given a linear system of differential equations $\mathcal{L}$ and its recursion operator Q that is a linear differential operator in total derivatives with coefficients depending only on the system's independent variables, we call Q a *linear differential recursion operator* of $\mathcal{L}$. Further, the function Q$h$ is a solution of $\mathcal{L}$ whenever the function $h$ is [19], see [42, 43] for first examples of generating solutions of linear differential equations using this approach. The operators $\mathrm{P}^2$ and $\mathrm{P}^3$ are recursion operators[3] of the equation (1) and $u = 1$ is its solution. Hence, this equation possesses the solutions $(\mathrm{P}^3)^k(\mathrm{P}^2)^l 1$, $k, l \in \mathbb{N}_0$, which are polynomials of $(t, x, y)$ and are linearly independent. Moreover, as the following lemma states, these solutions exhaust, up to linearly combining them, all solutions of this equation that are polynomial with respect to $x$.

**Lemma 8.** *The space $\mathscr{P}_n$ of solutions of the remarkable Fokker–Planck equation (1) that are polynomials with respect to $x$ of degree less than or equal to $n \in \mathbb{N}_0$ with coefficients depending on $(t, y)$ is of dimension $(n+1)(n+2)/2$. All of its elements are polynomial with respect to the entire tuple of independent variables $(t, x, y)$ and it admits a basis consisting of the polynomials $(\mathrm{P}^3)^k(\mathrm{P}^2)^l 1$, $0 \leqslant k + l \leqslant n$.*

**Proof.** Substituting the general form $u = \sum_{j=0}^{n} f^j(t, y)x^j$ of polynomials with respect to $x$ of degree less than or equal to $n \in \mathbb{N}_0$ into the equation (1) and splitting with respect to $x$, we derive the system

$$\Delta_j: \quad f_t^j + f_y^{j-1} = (j+1)(j+2)f^{j+2}, \quad j = 0, \dots, n+1,$$

where the equation $\Delta_j$ is obtained by collecting coefficients of $x^j$, and we assume that $f^j = 0$ if $j < 0$ or $j > n$. The equations $\Delta_{n+1}$ and $\partial_y \Delta_n$ take the form $f_y^n = 0$ and $f_{yy}^{n-1} = 0$, respectively. Continuing by the induction with respect to $j$ down to $j = 1$ with the differential consequences $\partial_y^{n-j+1}\Delta_j$, we obtain that $\partial_y^{n-j+1} f^j = 0$, $j = 0, \dots, n$, that is, $f^j$ is a polynomial with respect to $y$ of degree less than or equal to $n - j$

[3] Since the equation (1) is a linear partial differential equation, it is not a surprise that its basic recursion operators $\mathrm{P}^0$, $\mathrm{P}^1$, $\mathrm{P}^2$ and $\mathrm{P}^3$ (see Lemma 10 below) are usual linear differential operators in total derivatives, but not pseudodifferential ones as commonly happens for nonlinear differential equations. This is why it suffices to use here the formal interpretation of recursion operators in the sense of [32] and [33, Definition 5.20] and not to involve the more advanced interpretation of them as certain Bäcklund transformations. The later interpretation was suggested in [36], more explicitly formulated in [10], further developed in [25] and intensively applied later [14, 26, 28, 39, 40].

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with coefficients depending on $t$. More specifically, the equations $\Delta_{n+1}$, $\Delta_n$, $\Delta_{n-1}$, $\Delta_j, j=n-2,\dots,2,1$, respectively, take the form

$$f^n_y=0,\quad f^{n-1}_y=-f^n_t,\quad f^{n-2}_y=-f^{n-1}_t,$$
$$f^j_y=-f^{j+1}_t+(j+2)(j+3)f^{j+3},\quad j=n-3,\dots,1,0.$$

Therefore, $f^n=\tilde f^n(t), f^{n-1}=-\tilde f^n_t(t)y+\tilde f^{n-1}(t), f^{n-2}=\frac12\tilde f^n_{tt}(t)y^2-\tilde f^{n-1}_t(t)y+\tilde f^{n-2}(t)$. In general, $\tilde f^j$ denotes the coefficient of $y^0$ in $f^j$. By the induction with respect to $j$ down to $j=0$, we can show that the coefficients of $y^{n-j}$ and $y^{n-j-1}$ in $f^j$ are equal to $(-1)^{n-j}\partial_t^{n-j}\tilde f^n/(n-j)!$ and $(-1)^{n-j-1}\partial_t^{n-j-1}\tilde f^{n-1}/(n-j-1)!$, respectively. Moreover, the other coefficients of $f^j$ as a polynomial in $y$, except the zero-degree coefficient $\tilde f^j$, are expressed in terms of derivatives of $\tilde f^i$, $i>j$, with respect to $t$. Then, the equation $\Delta_0$: $f^0_t=2f^2$ implies $\partial_t^{n+1}\tilde f^n=0$, $\partial_t^n\tilde f^{n-1}=0$ and $\partial_t^{n-j+1}\tilde f^{n-j}=g^{n-j}$, $j=2,\dots,n$, where $g^{n-j}$ is a polynomial in $t$ expressed in terms of derivatives of $\tilde f^{n-i}$, $i<j$, with respect to $t$. The dimension of the solution space of the system for $\tilde f^j$, $j=0,\dots,n$, is $(n+1)(n+2)/2$ and coincides with $\dim\mathscr P_n$.

The polynomial solutions $(\mathrm P^3)^k(\mathrm P^2)^l1$, $0\leqslant k+l\leqslant n$, of the equation (1) are linearly independent. Their number is equal to $(n+1)(n+2)/2$ as well. Therefore, these polynomials constitute a basis of $\mathscr P_n$. □

**Lemma 9.** *A particular solution of the inhomogeneous equation* $\mathrm Fu=t^r(\mathrm P^3)^i(\mathrm P^2)^j1$, *where* $\mathrm F:=\mathrm D_t+x\mathrm D_y-\mathrm D_x^2$ *and* $i,j,r\in\mathbb N_0$, *is* $u=(r+1)^{-1}t^{r+1}(\mathrm P^3)^i(\mathrm P^2)^j1$.

**Proof.** Since $u=h:=(\mathrm P^3)^i(\mathrm P^2)^j1$ is a solution of the homogeneous counterpart (1) of the equation to be solved, $\mathrm Fh=0$, we obtain $\mathrm F((r+1)^{-1}t^{r+1}h)=t^rh+(r+1)^{-1}t^{r+1}\mathrm Fh=t^rh$. □

## 4. Generalised symmetries

Hereafter, we use the following notation. The jet variable $u_{kl}$ is identified with the derivative $\partial^{k+l}u/\partial x^k\partial y^l$, $k,l\in\mathbb N_0$. In particular, $u_{00}:=u$. The jet variables $(t,x,y,u_{kl},k,l\in\mathbb N_0)$ constitute the standard coordinates on the manifold $\mathscr F$ defined by the equation (1) and its differential consequences in the infinite-order jet space $\mathrm J^\infty(\mathbb R^3_{t,x,y}\times\mathbb R_u)$ with the independent variables $(t,x,y)$ and the dependent variable $u$. We consider differential functions defined on $\mathscr F$, and $\eta[u]$ denotes a differential function $\eta$ of $u$ that depends on $t$, $x$, $y$ and a finite number of $u_{kl}$. Recall that the order $\operatorname{ord}\eta[u]$ of a differential function $\eta[u]$ is the highest order of derivatives of $u$ involved in $\eta[u]$ if there are such derivatives, and $\operatorname{ord}\eta[u]=-\infty$ otherwise. For a generalised vector field $Q=\eta[u]\partial_u$, we define $\operatorname{ord}Q:=\operatorname{ord}\eta[u]$. The restrictions of the operators $\mathrm D_t$, $\mathrm D_x$ and $\mathrm D_y$ of total derivatives in $t$, $x$ and $y$ to such differential functions on $\mathscr F$ are, respectively,

$$\hat{\mathrm D}_t=\partial_t+\sum_{k,l=0}^{\infty}(u_{k+2,l}-xu_{k,l+1}-ku_{k-1,l+1})\partial_{u_{kl}},$$
$$\hat{\mathrm D}_x=\partial_x+\sum_{k,l=0}^{\infty}u_{k+1,l}\partial_{u_{kl}},\quad \hat{\mathrm D}_y=\partial_y+\sum_{k,l=0}^{\infty}u_{k,l+1}\partial_{u_{kl}}.$$

As for any evolution equation, it is natural to identify the quotient algebra of generalised symmetries of (1) with respect to the equivalence of generalised symmetries with the algebra

$$\Sigma:=\big\{\eta[u]\partial_u\mid \mathrm F\eta[u]=0\big\}\quad\text{with}\quad \mathrm F:=\hat{\mathrm D}_t+x\hat{\mathrm D}_y-\hat{\mathrm D}_x^2$$

of canonical representatives of equivalence classes, see [33, Section 5.1]. The subspace

$$\Sigma^n:=\big\{\eta[u]\partial_u\in\Sigma\mid\operatorname{ord}\eta[u]\leqslant n\big\},\quad n\in\mathbb N_0\cup\{-\infty\},$$

of $\Sigma$ is interpreted as the space of generalised symmetries of order less than or equal to $n$. The subspace $\Sigma^{-\infty}$ can be identified with the subalgebra $\mathfrak g^{\mathrm{lin}}$ of Lie symmetries of the equation (1) that are associated with the linear superposition of solutions of this equation,

$$\Sigma^{-\infty}=\{\mathcal Z(h):=h(t,x,y)\partial_u\mid h_t+xh_y=h_{xx}\}\simeq\mathfrak g^{\mathrm{lin}}.$$

The subspace family $\{\Sigma^n \mid n\in\mathbb{N}_0\cup\{-\infty\}\}$ filters the algebra $\Sigma$. Denote $\Sigma^{[n]}:=\Sigma^n/\Sigma^{n-1}$, $n\in\mathbb{N}$, $\Sigma^{[0]}:=\Sigma^0/\Sigma^{-\infty}$ and $\Sigma^{[-\infty]}:=\Sigma^{-\infty}$. The space $\Sigma^{[n]}$ is naturally identified with the space of canonical representatives of cosets of $\Sigma^{n-1}$ in $\Sigma^n$ and thus assumed as the space of $n$th order generalised symmetries of the equation (1), $n\in\mathbb{N}_0\cup\{-\infty\}$.[4]

In view of the linearity of the equation (1), an important subalgebra of its generalised symmetries consists of its linear generalised symmetries,

$$\Lambda:=\left\{\eta[u]\partial_u\in\Sigma \;\middle|\; \exists\, n\in\mathbb{N}_0,\ \exists\,\eta^{kl}=\eta^{kl}(t,x,y),\ k,l\in\mathbb{N}_0,\ k+l\leqslant n\colon\ \eta[u]=\sum_{k+l\leqslant n}\eta^{kl}u_{kl}\right\}.$$

The subspace $\Lambda^n:=\Lambda\cap\Sigma^n$ of $\Lambda$ with $n\in\mathbb{N}_0$ is constituted by the generalised symmetries with characteristics of the form

$$\eta[u]=\sum_{k+l\leqslant n}\eta^{kl}(t,x,y)u_{kl}. \tag{7}$$

A linear generalised symmetry is of order $n$ if and only if there exists a nonvanishing coefficient $\eta^{kl}$ with $k+l=n$. The quotient spaces $\Lambda^{[n]}=\Lambda^n/\Lambda^{n-1}$, $n\in\mathbb{N}$, and the subspace $\Lambda^{[0]}=\Lambda^0$ are naturally embedded in the respective spaces $\Sigma^{[n]}$, $n\in\mathbb{N}_0$, when taking linear canonical representatives for cosets of $\Sigma^{n-1}$ containing linear generalised symmetries. We interpret the space $\Lambda^{[n]}$ as the space of $n$th order linear generalised symmetries of the equation (1), $n\in\mathbb{N}_0$.

**Lemma 10.** *The algebra $\Lambda$ coincides with the algebra $\Lambda_{\rm r}$ of linear generalised symmetries generated by acting with the recursion operators $\mathrm{P}^3$, $\mathrm{P}^2$, $\mathrm{P}^1$ and $\mathrm{P}^0$ on the elementary seed symmetry vector field $u\partial_u$,*

$$\Lambda=\Lambda_{\rm r}:=\big\langle\big((\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}u\big)\partial_u \mid i_0,i_1,i_2,i_3\in\mathbb{N}_0\big\rangle.$$

**Proof.** The condition $\mathrm{F}\eta[u]=0$ of invariance of the equation (1) with respect to linear generalised symmetries with characteristics $\eta$ of the form (7) can represented as

$$(\eta^{kl}_t+x\eta^{kl}_y-\eta^{kl}_{xx})u_{kl}-k\eta^{kl}u_{k-1,l+1}-2\eta^{kl}_xu_{k+1,l}=0.$$

Splitting this condition with respect to the jet variables $u_{kl}$, we derive the system of determining equations for the coefficients $\eta^{kl}$,

$$\Delta_{kl}\colon\ \mathrm{F}\eta^{kl}-(k+1)\eta^{k+1,l-1}-2\eta^{k-1,l}_x=0,\quad k,l\in\mathbb{N}_0,\quad k+l\leqslant n+1,$$

where we denote $n:=\operatorname{ord}\eta$ and assume $\eta^{kl}=0$ if $k<0$ or $l<0$ or $k+l>n$.

For each $(k,l)\in\mathbb{N}_0\times\mathbb{N}_0$ with $k+l\leqslant n$, we rewrite the equation $\Delta_{k+1,l}$ as

$$2\eta^{kl}_x=\mathrm{F}\eta^{k+1,l}-(k+2)\eta^{k+2,l-1}.$$

In other words, the coefficient $\eta^{kl}$ is defined by the coefficients $\eta^{k+1,l}$ and $\eta^{k+2,l-1}$ modulo a summand depending only on $(t,y)$. Associating $\eta^{k'l'}$ with the point $(k',l')$ in the grid $\mathbb{N}_0\times\mathbb{N}_0$, we geometrically depict this relation pattern in Figure 1a. Therefore, for each fixed $j\in\mathbb{N}$, the coefficients $\eta^{kl}$ with $k+l=j$ define the coefficients $\eta^{kl}$ with $k+l=j-1$ up to summands depending only on $(t,y)$, see Figure 1b. Thus, the induction with respect to $m:=k+l$ from $m=n+1$, where $\eta^{kl}=0$, downwards to $m=0$, in the course of which each induction step is realised as the secondary induction with respect to $l$ from $l=m$ downwards to $l=0$, straightforwardly implies that the coefficient $\eta^{kl}$ is a polynomial with respect to $x$ of the degree at most $2n-2(k+l)$ with coefficients depending on $(t,y)$.

Now, we prove that $\eta^{kl}\in\mathscr{T}$ for any $k,l\in\mathbb{N}_0$, where $\mathscr{T}$ is the space of finite linear combinations of terms $t^r(\mathrm{P}^3)^i(\mathrm{P}^2)^j1$, $i,j,r\in\mathbb{N}_0$. Using Lemma 9, we carry out the induction with respect to $m:=k+l$ in the opposite direction, from $m=0$ upwards to $m=n$, as shown in Figure 2b, where each induction step is performed as the secondary induction with respect to $l$ from $l=0$ upwards to $l=m$. The induction base $k=l=0$ follows in view of Lemma 8 from the equation $\Delta_{00}$: $\mathrm{F}\eta^{00}=0$ and the polynomiality of

[4] The filtration $\Sigma=\cup_{n\in\mathbb{N}_0\cup\{-\infty\}}\Sigma^n$ of the algebra $\Sigma$ gives rise to the associated graded algebra $\operatorname{gr}\Sigma=\oplus_{n\in\mathbb{N}_0}\Sigma^{[n]}$, where $\Sigma^{[n]}:=\Sigma^n/\Sigma^{n-1}$ with $\Sigma^{-1}:=\Sigma^{-\infty}$. In this notation, the space $\Sigma^{[n]}$ is the homogeneous component of degree $n$ of the $\mathbb{N}_0$-graded algebra $\operatorname{gr}\Sigma$.

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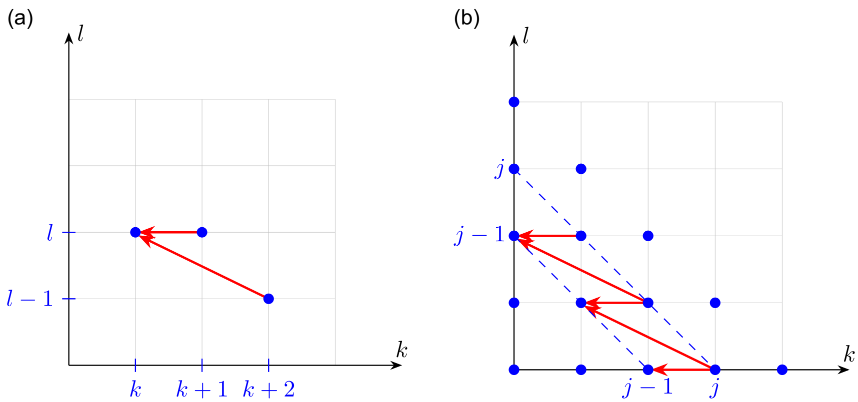


***Figure 1.*** *The first induction (downward). (a) Relation pattern. (b) Induction step.*

$\eta^{00}$ with respect to $x$. On the step $(k,l)$, we have $\eta^{k+1,l-1},\eta^{k-1,l}\in\mathscr{T}$ by the induction supposition. Taking into account $[\mathrm{D}_x,\mathrm{P}^2]=1$ and $[\mathrm{D}_x,\mathrm{P}^3]=3t$, we obtain

$$\big(t^r(\mathrm{P}^3)^i(\mathrm{P}^2)^j1\big)_x=3it^{r+1}(\mathrm{P}^3)^{i-1}(\mathrm{P}^2)^j1+jt^r(\mathrm{P}^3)^i(\mathrm{P}^2)^{j-1}1.$$

Therefore, $\eta^{k-1,l}_x\in\mathscr{T}$ as well. Considering $\Delta_{kl}$ as an inhomogeneous equation with respect to $\eta^{kl}$, we represent $\eta^{kl}$ as the sum of a particular solution $\hat\eta^{kl}$ of this equation according to Lemma 9 and a solution $\check\eta^{kl}$ of the homogeneous counterpart $\mathrm{F}\eta^{kl}=0$ of the equation $\Delta_{kl}$, see Figure 2a for an illustration. Since $\hat\eta^{kl}\in\mathscr{T}$ due to the choice in Lemma 9 and $\eta^{kl}$ is polynomial with respect to $x$ in view of the above arguments, $\check\eta^{kl}$ is also polynomial with respect to $x$ and Lemma 8 implies that $\check\eta^{kl}\in\mathscr{T}$, including only terms with $r=0$. Hence, $\eta^{kl}=\hat\eta^{kl}+\check\eta^{kl}\in\mathscr{T}$.

As a result, we derive the following representation for $\eta$:

$$\eta=\sum_{i,j,k,l\in\mathbb{N}_0}c_{ijkl}W^{ijkl},\quad W^{ijkl}:=\big((\mathrm{P}^3)^i(\mathrm{P}^2)^j1\big)u_{kl}+\sum_{(k',l')\succ(k,l)}V^{ijklk'l'}u_{k'l'},\tag{8}$$

where $(k',l')\succ(k,l)$ means that $k',l'\in\mathbb{N}_0$, $l'\geqslant l$, $k'+l'\geqslant k+l$ and $(k',l')\neq(k,l)$, each $V^{ijklk'l'}$ is an element of $\mathscr{T}$ that is completely defined by $(i,j,k,l,k',l')$, has $r>0$ for each of its summand, and only finite number of $c_{ijkl}$ and of $V^{ijklk'l'}$ are nonzero. In other words, any generalised symmetry $\eta\partial_u$ of the equation (1) is completely defined by the corresponding coefficients $c_{ijkl}$ of $W^{ijkl}$ or, equivalently, of $(\mathrm{P}^3)^i(\mathrm{P}^2)^ju_{kl}$ in its representation (8). At the same time,

$$(\mathrm{P}^3)^i(\mathrm{P}^2)^j(\mathrm{P}^1)^k(\mathrm{P}^0)^lu=\big((\mathrm{P}^3)^i(\mathrm{P}^2)^j1\big)u_{kl}+\sum_{(k',l')\succ(k,l)}\tilde V^{ijklk'l'}u_{k'l'},$$

where the coefficients $\tilde V^{ijklk'l'}$ have the same properties as $V^{ijklk'l'}$. Therefore, $\eta\partial_u\in\Lambda_\mathfrak{r}$, that is, $\Lambda\subseteq\Lambda_\mathfrak{r}$. The inverse inclusion follows from the definitions of $\Lambda$ and $\Lambda_\mathfrak{r}$. Thus, $\Lambda=\Lambda_\mathfrak{r}$. □

**Corollary 11.** *The associative algebra $\Upsilon$ of linear differential recursion operators of the equation* (1) *coincides with the algebra $\Upsilon_\mathfrak{r}$.*

**Corollary 12.** *The algebra $\Lambda=\Lambda_\mathfrak{r}$ is anti-isomorphic to the algebra $\Upsilon_\mathfrak{r}^{(-)}$ and, therefore, to the Lie algebra associated with the quotient of the universal enveloping algebra of the Lie algebra $\mathfrak{r}$ by the principal ideal $(\iota(\mathcal{I})+1)$ generated by $\iota(\mathcal{I})+1$, $\Lambda_\mathfrak{r}\simeq\big(\mathfrak{U}(\mathfrak{r})/(\iota(\mathcal{I})+1)\big)^{(-)}$.*

**Proof.** The correspondence $((\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}u)\partial_u\mapsto(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}$ extended by linearity straightforwardly gives us a vector-space isomorphism $\varphi$ from $\Lambda_\mathfrak{r}$ to $\Upsilon_\mathfrak{r}$. Consider operators $\mathrm{Q},\mathrm{R}\in\Upsilon_\mathfrak{r}$, that is, $\mathrm{Q}=Q^{ij}\mathrm{D}_x^i\mathrm{D}_y^j$ and $\mathrm{R}=R^{ij}\mathrm{D}_x^i\mathrm{D}_y^j$, where only a finite number of the polynomials $Q^{ij}$ and $R^{ij}$ of $(t,x,y)$ are non-zero. Here and in what follows we assume summation with respect two repeated indices

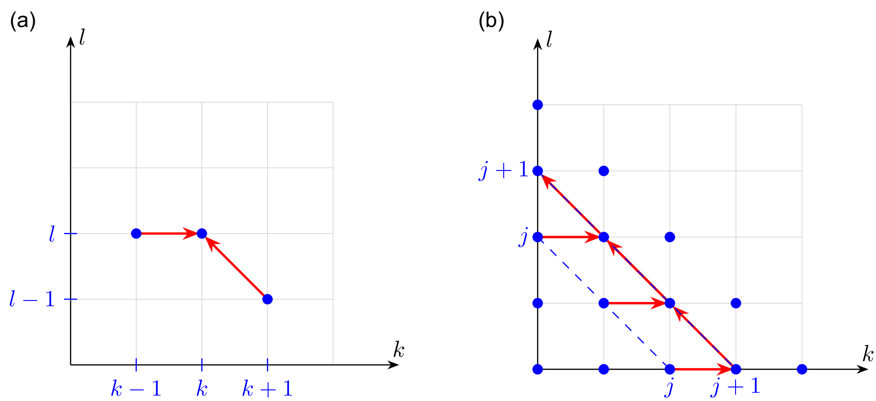


***Figure 2.*** *The second induction (upward). (a) Relation pattern. (b) Induction step.*

$i$ and $j$ through $\mathbb{N}_0$. In view of [33, Proposition 5.15], the commutator $\big[(\mathrm{Q}u)\partial_u, (\mathrm{R}u)\partial_u\big]$ of evolutionary generalised vector fields $(\mathrm{Q}u)\partial_u$ and $(\mathrm{R}u)\partial_u$ from $\Lambda_\mathfrak{r}$ is an evolutionary vector field with characteristic

$$\begin{aligned}\mathrm{pr}\big((\mathrm{Q}u)\partial_u\big)(\mathrm{R}u)-\mathrm{pr}\big((\mathrm{R}u)\partial_u\big)(\mathrm{Q}u)&=\mathrm{D}_x^i\mathrm{D}_y^j(\mathrm{Q}u)\partial_{u_{ij}}(\mathrm{R}u)-\mathrm{D}_x^i\mathrm{D}_y^j(\mathrm{R}u)\partial_{u_{ij}}(\mathrm{Q}u)\\&=R^{ij}\mathrm{D}_x^i\mathrm{D}_y^j(\mathrm{Q}u)-Q^{ij}\mathrm{D}_x^i\mathrm{D}_y^j(\mathrm{R}u)=\mathrm{R}(\mathrm{Q}u)-\mathrm{Q}(\mathrm{R}u)=[\mathrm{R},\mathrm{Q}]u,\end{aligned}$$

where $\mathrm{pr}(\eta\partial_u)$ denotes the prolongation of a generalised vector field $\eta\partial_u$ with respect $x$ and $y$, $\mathrm{pr}(\eta\partial_u)=(\mathrm{D}_x^i\mathrm{D}_y^j\eta)\partial_{u_{ij}}$. Therefore, $\varphi([\mathrm{Q}u\partial_u,\mathrm{R}u\partial_u])=-[\mathrm{Q},\mathrm{R}]$, that is, $\varphi\colon\Lambda_\mathfrak{r}\to\Upsilon_\mathfrak{r}^{(-)}$ is an anti-isomorphism, which combines with Lemma 2 to the second assertion in this theorem. □

We can reformulate Corollary 12, recalling the isomorphism of $\mathfrak{r}$ to the rank-two Heisenberg algebra $\mathrm{h}(2,\mathbb{R})=\langle p_1,p_2,q_1,q_2,c\rangle$, Remark 3 and Corollary 5. In particular,

$$\mathfrak{U}(\mathfrak{r})/(\iota(\mathcal{I})+1)\simeq\mathfrak{U}\big(\mathrm{h}(2,\mathbb{R})\big)/(c+1)\simeq\mathrm{W}(2,\mathbb{R})^{\mathrm{op}}\simeq\mathrm{W}(2,\mathbb{R}).$$

**Corollary 13.** *The algebra $\Lambda=\Lambda_\mathfrak{r}$ of the linear generalised symmetries of the remarkable Fokker–Planck equation (1) is isomorphic to the Lie algebra $\mathrm{W}(2,\mathbb{R})^{(-)}$ associated with the second Weyl algebra $\mathrm{W}(2,\mathbb{R})$, $\Lambda\simeq\mathrm{W}(2,\mathbb{R})^{(-)}$.*

Hence, the centre of the algebra $\Lambda$ is one-dimensional and spanned by $u\partial_u$, and the quotient algebra $\Lambda/\langle u\partial_u\rangle$ is simple.

Combining Corollaries 7 and 12, we derive that

$$\dim\Lambda^n=\begin{cases}\frac{1}{18}(n+1)(n+3)(n^2+4n+6) & \text{if}\quad n\equiv 0\text{ or }2\pmod 3,\\ \frac{1}{18}(n+2)^2(n^2+4n+5) & \text{if}\quad n\equiv 1\pmod 3,\end{cases}$$

$$\dim\Lambda^{[n]}=\begin{cases}\frac{1}{9}(2n+3)(n^2+3n+3) & \text{if}\quad n\equiv 0\pmod 3,\\ \frac{1}{9}(n+2)(2n^2+5n+5) & \text{if}\quad n\equiv 1\pmod 3,\\ \frac{1}{9}(n+1)(2n^2+7n+8) & \text{if}\quad n\equiv 2\pmod 3.\end{cases}$$

**Theorem 14.** *The algebra of canonical representatives of generalised symmetries of the remarkable Fokker–Planck equation (1) is $\Sigma=\Lambda_\mathfrak{r}\mathbin{\in\!\!\!\!+}\Sigma^{-\infty}$, where*

$$\Lambda_\mathfrak{r}=\big\langle\big((\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}u\big)\partial_u\mid i_0,i_1,i_2,i_3\in\mathbb{N}_0\big\rangle,\qquad\Sigma^{-\infty}:=\big\{\mathfrak{Z}(h)\big\}.$$

*Here, the parameter function $h$ runs through the solution set of (1).*

**Proof.** Lemma 10 obviously implies that $\dim\Lambda^{[n]}<\infty$ for any $n\in\mathbb{N}_0$, and thus $\Sigma^{[n]}=\Lambda^{[n]}$ for any $n\in\mathbb{N}_0$. The last implication is just a particular formulation of the Shapovalov–Shirokov theorem [41, Theorem 4.1] for the equation (1). Therefore, $\Sigma=\Lambda\mathbin{\in\!\!\!\!+}\Sigma^{-\infty}=\Lambda_\mathfrak{r}\mathbin{\in\!\!\!\!+}\Sigma^{-\infty}$. □

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In other words, the algebra $\Sigma$ splits over the infinite-dimensional abelian ideal $\Sigma^{-\infty}$ of trivial generalised symmetries associated with the linear superposition of solutions. The complementary subalgebra to $\Sigma^{-\infty}$ in $\Sigma$, which is naturally called the *essential algebra of generalised symmetries*, is just the algebra $\Lambda=\Lambda_{\mathfrak r}$ of linear generalised symmetries, which is isomorphic to the Lie algebra $\mathrm W(2,\mathbb R)^{(-)}$ associated with the second Weyl algebra $\mathrm W(2,\mathbb R)$.

**Remark 15.** The subspaces $\Lambda^{[1]}$ and $\Lambda^{[2]}$ are in fact subalgebras of $\Lambda$. As Lie algebras, they are isomorphic to the (nil)radical $\mathfrak r$ of the essential Lie invariance algebra $\mathfrak g^{\rm ess}$ of (1) and the algebra $\mathfrak g^{\rm ess}$ itself, respectively. An interesting question is whether the algebra $\Lambda$ contains finite-dimensional noncommutative subalgebras that are $\mathrm{Aut}(\Lambda)$-inequivalent to subalgebras of $\Lambda^{[2]}$.

**Remark 16.** Another concept related to generating generalised symmetries similarly to recursion operators is that of master symmetry. There are various (in general, inequivalent) notions of master symmetries in the literature. According to the definition presented in [33, p. 315], a master symmetry is a generalised (or even nonlocal) vector field $\mathfrak M$ with the property that whenever $Q$ is a generalised symmetry of the system of evolution equations under consideration, so is the Lie bracket $[\mathfrak M, Q]$. Since any generalised symmetry of the system satisfies this property, it is then a master symmetry. Mimicking techniques from [41], it is possible to prove that within the framework of this definition and in the setting of the Shapovalov–Shirokov theorem, *a generalised vector field is a master symmetry of a linear system of differential equations if and only if it is a generalised symmetry of this system.* But even in [33, p. 317], a different notion of master symmetry (see, e.g. [47, Definition 2] for a precise formulation of the corresponding definition) is implicitly used when considering the potential Burgers equation as an example, where the commutators of the presented master symmetry only with the elements of a (proper) abelian subalgebra of the entire algebra of generalised symmetries of this equation are such symmetries. The analogous master symmetry for the Burgers equation is discussed, for example, in [47, Section 1]. The generalised vector field $yu_{yy}\partial_u$, which is the 'time-independent part' of the generalised symmetry $-\frac13(\mathrm P^3(\mathrm P^0)^2u)\partial_u$ of the equation (1), satisfies all the properties of master symmetries according to [47, Definition 2], except that the right-hand side of (1) does not arise as the characteristic of a generalised symmetry from the generated hierarchy.

**Remark 17.** The algebra $\Lambda$ has another remarkable generation property following from the analogous property of $\mathrm W(2,\mathbb R)^{(-)}$ [20]. It is two-generated as a Lie algebra. More specifically, any of its elements can be represented as a linear combination of successive commutators (aka nonassociative monomials) of its two fixed elements, for example, those associated with the operators

$$\mathrm P^3\mathrm P^0+5\mathrm P^2\mathrm P^1\quad\text{and}\quad\big((\mathrm P^3)^3+(\mathrm P^0)^2+1\big)\big((\mathrm P^2)^3+(\mathrm P^1)^2+1\big).$$

Similarly, both (isomorphic) algebras $\Upsilon_{\mathfrak r}$ and $\mathrm W(2,\mathbb R)$ are two-generated as associative algebras.

**Remark 18.** The Lie algebra $\mathrm W(2,\mathbb R)^{(-)}$ has a natural $\mathbb Z$-grading associated with the specific inner semisimple derivation given by the adjoint action of the element $s:=\hat p_1\hat q_1+\hat p_2\hat q_2$,

$$\mathrm{ad}_s(\hat q^\kappa\hat p^\lambda)=(\kappa_1-\lambda_1+\kappa_2-\lambda_2)\hat q^\kappa\hat p^\lambda,$$

where the elements $\hat q^\kappa\hat p^\lambda:=\hat q_1^{\kappa_1}\hat q_2^{\kappa_2}\hat p_1^{\lambda_1}\hat p_2^{\lambda_2}$, $\kappa:=(\kappa_1,\kappa_2)$, $\lambda:=(\lambda_1,\lambda_2)\in\mathbb N_0^2$, constitute the canonical basis of $\mathrm W(2,\mathbb R)$, see Remark 3 and the commutation relation (3). Denote by $\Gamma_m$ the eigenspace of $\mathrm{ad}_s$ associated with the eigenvalue $m\in\mathbb Z$,

$$\Gamma_m:=\big\langle\hat q^\kappa\hat p^\lambda\mid\kappa,\lambda\in\mathbb N_0^2,\ \kappa_1-\lambda_1+\kappa_2-\lambda_2=m\big\rangle.$$

Then, the algebra $\mathrm W(2,\mathbb R)$ is the direct sum of $\Gamma_m$, $m\in\mathbb Z$. Since $\mathrm{ad}_s$ is a derivation of the Lie algebra $\mathrm W(2,\mathbb R)^{(-)}$, this direct sum decomposition is in fact its $\mathbb Z$-grading,

$$\mathrm W(2,\mathbb R)^{(-)}=\bigoplus_{m\in\mathbb Z}\Gamma_m\quad\text{with}\quad[\Gamma_m,\Gamma_{m'}]\subseteq\Gamma_{m+m'}.$$

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The isomorphism between $\Lambda$ and $\mathrm{W}(2,\mathbb{R})^{(-)}$ transfers this grading to the algebra $\Lambda$, where the role of $s$ is played by $\mathfrak{R}:=\big(\frac13\mathrm{P}^3\mathrm{P}^0u+\mathrm{P}^2\mathrm{P}^1u\big)\partial_u$ and the grading components are given by

$$\big\langle\big((\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}u\big)\partial_u\mid i_3,i_2,i_1,i_0\in\mathbb{N}_0,\ i_3-i_0+i_2-i_1=m\big\rangle,\quad m\in\mathbb{Z}.$$

**Remark 19.** The structure of the algebra $\Sigma_{\rm h}$ of generalised symmetries of the linear (1 + 1)-dimensional heat equation

$$u_t=u_{xx} \tag{9}$$

is similar to that of the algebra $\Sigma$. Indeed, the algebra $\Sigma_{\rm h}$ splits over its infinite-dimensional ideal $\Sigma_{\rm h}^{-\infty}$ associated with the linear superposition of solutions of (9), $\Sigma_{\rm h}=\Sigma_{\rm h}^{\rm ess}\ltimes\Sigma_{\rm h}^{-\infty}$. The complementary subalgebra $\Sigma_{\rm h}^{\rm ess}$ to the ideal $\Sigma_{\rm h}^{-\infty}$ in the algebra $\Sigma_{\rm h}$ coincides with the algebra $\Lambda_{\rm h}$ of linear generalised symmetries of (9), see [18]. In view of [18, Corollary 21], it is anti-isomorphic to the Lie algebra arising from the quotient of the universal enveloping algebra $\mathfrak{U}\big(\mathrm{h}(1,\mathbb{R})\big)$ of the rank-one Heisenberg algebra $\mathrm{h}(1,\mathbb{R})$ by the principal two-sided ideal $(c+1)$ generated by $c+1$,

$$\Lambda_{\rm h}\simeq\big(\mathfrak{U}\big(\mathrm{h}(1,\mathbb{R})\big)/(c+1)\big)^{(-)},$$

see Remark 3. Hence, it is isomorphic to the Lie algebra $\mathrm{W}(1,\mathbb{R})^{(-)}$ associated with the first Weyl algebra $\mathrm{W}(1,\mathbb{R})$.

Since any ultraparabolic linear second-order partial differential equation with three independent variables whose essential Lie invariance algebra is eight-dimensional is similar to the remarkable Fokker–Planck equation (1) modulo point transformations, we obtain the following assertion.

**Corollary 20.** *The algebra of canonical representatives of generalised symmetries of any ultraparabolic linear second-order partial differential equation $\mathcal{L}$ with the three independent variables $(t,x,y)$ and the dependent variable $u$ whose essential Lie invariance algebra is eight-dimensional is $\tilde\Sigma=\tilde\Lambda\ltimes\tilde\Sigma^{-\infty}$, where*

$$\tilde\Lambda=\big\langle\big((\tilde{\mathrm{P}}^3)^{i_3}(\tilde{\mathrm{P}}^2)^{i_2}(\tilde{\mathrm{P}}^1)^{i_1}(\tilde{\mathrm{P}}^0)^{i_0}u\big)\partial_u\mid i_0,i_1,i_2,i_3\in\mathbb{N}_0\big\rangle,\quad \tilde\Sigma^{-\infty}:=\big\{h(t,x,y)u\partial_u\big\},$$

$\tilde{\mathrm{P}}^0,\dots,\tilde{\mathrm{P}}^3$ *are Lie-symmetry operators of $\mathcal{L}$ that are associated with (additional to $u\partial_u$) basis elements of the radical of the essential Lie invariance algebra of $\mathcal{L}$, and the parameter function $h$ runs through the solution set of $\mathcal{L}$.*

Thus, the algebra $\tilde\Sigma$ splits over the algebra $\tilde\Sigma^{-\infty}$ of Lie symmetries of the equation $\mathcal{L}$ related to the linear superposition of solutions of this equation, which is the maximal abelian ideal in $\tilde\Sigma$. Its subalgebra complement in $\tilde\Sigma$ is the algebra $\tilde\Lambda$ of canonical representatives of linear generalised symmetries of $\mathcal{L}$, which is isomorphic to the Lie algebra $\mathrm{W}(2,\mathbb{R})^{(-)}$ associated with the second Weyl algebra $\mathrm{W}(2,\mathbb{R})$.

Specific ultraparabolic linear second-order partial differential equation with three independent variables whose essential Lie invariance algebra is eight-dimensional appeared in the literature independently of the equation (1). In particular, these are the Kolmogorov backward equation [49]

$$u_t+xu_y=x^5u_{xx}$$

and the Kramers equations [44]

$$u_t+xu_y=\gamma u_{xx}+\gamma(x-\tfrac34\gamma y)u_x+\gamma u,$$
$$u_t+xu_y=\gamma u_{xx}+\gamma(x+\tfrac{3}{16}\gamma y)u_x+\gamma u.$$

Point transformations mapping these equations to the equation (1) were found in [17, Eq. (6)] and [16, Section 9], respectively. The pushforwards $\tilde{\mathrm{P}}^0,\dots,\tilde{\mathrm{P}}^3$ of $\mathrm{P}^0,\dots,\mathrm{P}^3$ by these transformations take the form

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$$\tilde{\mathrm{P}}^0 = \mathrm{D}_t, \quad \tilde{\mathrm{P}}^1 = y\mathrm{D}_t - x^2\mathrm{D}_x + x,$$
$$\tilde{\mathrm{P}}^2 = y^2\mathrm{D}_t - 2x^2y\mathrm{D}_x + 2yx + x^{-1}, \quad \tilde{\mathrm{P}}^3 = y^3\mathrm{D}_t - 3x^2y^2\mathrm{D}_x + 3(xy^2 - t + x^{-1}y),$$

$$\tilde{\mathrm{P}}^0 = \mathrm{e}^{-\frac{3}{2}\gamma t}\left(\tfrac{1}{\gamma}\mathrm{D}_y - \tfrac{3}{2}\mathrm{D}_x\right), \quad \tilde{\mathrm{P}}^1 = \mathrm{e}^{-\frac{1}{2}\gamma t}\left(\tfrac{1}{\gamma}\mathrm{D}_y - \tfrac{1}{2}\mathrm{D}_x - \tfrac{1}{2}(x + \tfrac{3}{2}\gamma y)\right),$$
$$\tilde{\mathrm{P}}^2 = \mathrm{e}^{\frac{1}{2}\gamma t}\left(\tfrac{1}{\gamma}\mathrm{D}_y + \tfrac{1}{2}\mathrm{D}_x\right), \quad \tilde{\mathrm{P}}^3 = \mathrm{e}^{\frac{3}{2}\gamma t}\left(\tfrac{1}{\gamma}\mathrm{D}_y + \tfrac{3}{2}\mathrm{D}_x + \tfrac{3}{2}(x - \tfrac{1}{2}\gamma y)\right),$$

$$\tilde{\mathrm{P}}^0 = \mathrm{e}^{-\frac{3}{4}\gamma t}\left(\tfrac{1}{\gamma}\mathrm{D}_y - \tfrac{3}{4}\mathrm{D}_x\right), \quad \tilde{\mathrm{P}}^1 = \mathrm{e}^{-\frac{1}{4}\gamma t}\left(\tfrac{2}{\gamma}\mathrm{D}_y - \tfrac{1}{2}\mathrm{D}_x\right),$$
$$\tilde{\mathrm{P}}^2 = \mathrm{e}^{\frac{1}{4}\gamma t}\left(\tfrac{4}{\gamma}\mathrm{D}_y + \mathrm{D}_x + x + \tfrac{3}{4}\gamma y\right), \quad \tilde{\mathrm{P}}^3 = \mathrm{e}^{\frac{3}{4}\gamma t}\left(\tfrac{8}{\gamma}\mathrm{D}_y + 6\mathrm{D}_x + 6(x + \tfrac{1}{4}\gamma y)\right).$$

## 5. Generalised symmetries and finding exact solutions

Generalised symmetries can be used for finding exact solutions of systems of differential equations, advancing the well-established methods that are based on Lie symmetries. However, the computations involving generalised symmetries are much more complicated than their counterparts involving Lie symmetries. This is why such usage of generalised symmetries is not quite common in the literature. Furthermore, in the case of linear systems of differential equations, it has a number of specific features, which we would like to discuss in general before the particular consideration of the equation (1).

### *5.1. Theoretical background*

Given a system $\mathcal{L}$ of differential equations for unknown functions $u = (u^1, \dots, u^m)$ of independent variables $x = (x_1, \dots, x_n)$, $m, n \in \mathbb{N}$, with point-symmetry (pseudo)group $G$, maximal Lie invariance algebra $\mathfrak{g}$ and generalised-symmetry algebra $\Sigma$, one can use these objects or their parts to construct particular exact solutions of $\mathcal{L}$. The range of possibilities includes the following [3–5, 13, 22, 32].

*Generation of solutions via acting by point symmetries*. If $u = f(x)$ is a solution of $\mathcal{L}$ and $\Phi$ is an arbitrary element of $G$, then, $u = \Phi_* f(x)$ (resp. $u = \Phi^* f(x)$) is a solution of $\mathcal{L}$ as well. Here, $\Phi_*$ and $\Phi^*$ denote the pushforward and pullback of functions by the local diffeomorphism $\Phi$, respectively. This induces an equivalence relation of the solution set of $\mathcal{L}$, which we call the $G$-equivalence. All the constructions of exact solutions of $\mathcal{L}$ can be carried out up to this equivalence.

*Lie reductions*. For any subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ that satisfies the transversality condition [33, Eq. (3.34)], the system $\mathcal{S}$ of differential constraints $Q_1[u] = 0, \dots, Q_r[u] = 0$, where $Q_1, \dots, Q_r$ constitute a basis of $\mathfrak{s}$ and $Q_i[u]$ denotes the characteristic of $Q_i$, $i = 1, \dots, r$, is formally compatible with $\mathcal{L}$. The solution set of the extended system $\mathcal{L} \cup \mathcal{S}$ coincides with the set of $\mathfrak{s}$-invariant solutions of $\mathcal{L}$. Since the solution of $\mathcal{S}$ reduces to the solution of a system of $r$ first-order quasilinear partial differential equations with respect to a single function of $n + m$ independent variables, the system $\mathcal{S}$ can be integrated, which gives an ansatz for $u$ in terms of a tuple $\phi$ of new unknown functions of $n - r$ arguments. Substituting this ansatz to $\mathcal{L}$, one obtain the so-called reduced system $\mathcal{L}/\mathfrak{s}$ for $\phi$ with a less number of independent variables than that in the original system $\mathcal{L}$. Any solution of $\mathcal{L}/\mathfrak{s}$ gives, via the ansatz, a solution of $\mathcal{L}$. $G$-equivalent subalgebras of $\mathfrak{g}$ result in $G$-equivalent families of invariant solutions. Hence, in view of the previous points, only $G$-inequivalent subalgebras of $\mathfrak{g}$ of should be used for Lie reductions.

*Solutions invariant with respect to generalised symmetries*. Similarly to Lie reductions, consider a (finite-dimensional) subalgebra $\Theta$ of $\Sigma$ with a basis $(\mathfrak{Q}_1, \dots, \mathfrak{Q}_r)$. The system $\mathcal{S}$ of differential constraints $\mathfrak{Q}_1[u] = 0, \dots, \mathfrak{Q}_r[u] = 0$, where $\mathfrak{Q}_i[u]$ denotes the characteristic of $\mathfrak{Q}_i$, $i = 1, \dots, r$, is formally compatible with $\mathcal{L}$. At the same time, in contrast to Lie reductions, there is no unified way to integrate the extended system $\mathcal{L} \cup \mathcal{S}$, and the deep analysis of the structure of the algebra $\Sigma$ becomes

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important in this context. Since the group $G$ acts on generalised symmetries as well, only $G$-inequivalent subalgebras of $\Sigma$ should be used for constructing solutions that are invariant with respect to generalised symmetries.

All the above techniques also work if $\mathcal{L}$ is a homogeneous linear system of differential equations, but their application has specific features in this case. In particular, the generation of solutions via acting by point symmetries can be extended using the intermediate complexification. More specifically, if the system $\mathcal{L}$ has real-analytical coefficients, we can complexify it, assuming both independent and dependent variables to be complex. After finding families of exact solutions of the complexified system $\mathcal{L}$, we extend these families by acting with the complexified version of the group $G$. Then, assuming the independent variables to be real, we take the real and imaginary parts of the obtained solutions, thus constructing new families of (real) exact solutions of $\mathcal{L}$.

Moreover, denote by $\Lambda$ the algebra of linear generalised symmetries of $\mathcal{L}$, which is a subalgebra of $\Sigma$, and by $\Upsilon$ the associative algebra of linear differential recursion operators of $\mathcal{L}$. As discussed in the beginning of Section 3, $\mathrm{Q}\in\Upsilon$ if and only if $(\mathrm{Q}u)\partial_u\in\Lambda$ [33, Proposition 5.22], and then, the function $\mathrm{Q}h$ is a solution of $\mathcal{L}$ whenever the function $h$ is. This gives one more specific way for generating solutions of linear systems of differential equations. One should select only $G^{\rm ess}$-inequivalent elements[5] of $\Upsilon$, but even this selection does not guarantee nontrivial results. In addition, it is necessary to carefully analyse the action of particular elements of $\Upsilon$ on known families of solutions of $\mathcal{L}$ [19]. For convenience, we call $\langle(\mathrm{Q}u)\partial_u\rangle$-invariant solutions of $\mathcal{L}$ merely Q-*invariant solutions* of $\mathcal{L}$. Thus, a solution $h$ of $\mathcal{L}$ is Q-invariant if and only if $\mathrm{Q}h=0$. It is clear that given an element Q of $\Upsilon$, the action by any element from the principal left ideal $\Upsilon\mathrm{Q}$ on any Q-invariant solution of $\mathcal{L}$ results in the zero solution of $\mathcal{L}$. More generally, any element R of $\Upsilon$ with $\mathrm{QR}\in\Upsilon\mathrm{Q}$ maps the space of Q-invariant solutions of $\mathcal{L}$ to itself. In particular, this is the case for any element of $\Upsilon\mathrm{Q}+\mathrm{C}_\Upsilon(\mathrm{Q})$, that is, for the sum of any elements of the principal left ideal $\Upsilon\mathrm{Q}$ and of the centraliser $\mathrm{C}_\Upsilon(\mathrm{Q})$ of Q in $\Upsilon$. This is why the study of algebraic structure of $\Upsilon$ is relevant in the context of generating solutions of $\mathcal{L}$ via acting by elements of $\Upsilon$.

The relation of linear generalised symmetries of linear partial differential equations with separation of variables in these equations is well known for a long time [30].

For the further consideration, we need some auxiliary statements.

**Lemma 21.** *Suppose that $\mathbb{F}$ is an algebraically closed field, $V$ is a vector space over $\mathbb{F}$, $A\in\mathrm{End}(V)$, $P\in\mathbb{F}[x]$, $\lambda_1,\dots,\lambda_r$ are all the distinct roots of $P$ with multiplicities $k_1,\dots,k_r$, respectively. Denote by* id *the identity endomorphism on $V$. Then,*

$$\ker P(A)=\bigoplus_{i=1}^r\ker\,(A-\lambda_i\,\mathrm{id})^{k_i}.$$

**Proof.** Let $\lambda$ be a root of the polynomial $P$ with multiplicity $k$. Then, this polynomial can be factored in the form $P(x)=R(x)(x-\lambda)^k$, where $R\in\mathbb{F}[x]$ with $R(\lambda)\neq0$. For the proof of the lemma, it suffices to show that

$$\ker P(A)=\ker R(A)\oplus\ker\,(A-\lambda\,\mathrm{id})^k.$$

Shifting $x$, $x-\lambda\mapsto x$, or, equivalently, replacing $A$ by $A+\lambda\,\mathrm{id}$, we can assume without loss of generality that $\lambda=0$, and thus $P(x)=R(x)x^k$ with $R(0)\neq0$. We represent $R$ as $R(x)=R(0)(1-Q(x))$, where $Q\in x\mathbb{F}[x]$. Let $v\in\ker P(A)$. Then, $R(A)v=w$, where $w\in\ker A^k$. Hence, $v=\hat v+\check v$, where $\hat v\in\ker R(A)$ and

$$\check v=\frac{1}{R(0)}\sum_{j=0}^k\big(Q(A)\big)^jw\in\ker A^k.$$

[5] It is a quite common situation for a linear system $\mathcal{L}$ of differential equations that the corresponding point-symmetry (pseudo)group $G$ consists of fibre-preserving transformations that are affine with respect to the dependent variables. Let $G^{\rm ess}$ denote the (pseudo)subgroup of all elements of $G$ that induce homogeneous linear transformations of the unknown functions of $\mathcal{L}$. The natural action of group $G^{\rm ess}$ preserves the algebra $\Lambda$ of linear generalised symmetries of $\mathcal{L}$. Hence, the $G^{\rm ess}$-equivalence of these symmetries is well-defined and induces the $G^{\rm ess}$-equivalence of linear differential recursion operators of $\mathcal{L}$.

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Therefore, $\ker P(A) = \ker R(A) + \ker A^k$. Suppose that $v \in \ker R(A) \cap \ker A^k$. Since the polynomial $R$ and the monomial $x^k$ are coprime, according to Bézout's identity, there exist polynomials $N$ and $M$ such that $N(x)R(x) + M(x)x^k = 1$. Then, $v = \big(N(A)R(A) + M(A)A^k\big)v = 0$. This proves that the above sum of kernels is direct. □

**Corollary 22.** *Suppose that $V$ is a vector space over a field $\mathbb{F}$, $A \in \mathrm{End}(V)$, $P \in \mathbb{F}[x]$ and $P = P_1 \cdots P_s$ is a factorisation with coprime polynomials $P_1, \ldots, P_s$. Then,*

$$\ker P(A) = \ker P_1(A) \oplus \cdots \oplus \ker P_s(A).$$

**Proof.** If the field $\mathbb{F}$ is algebraically closed, the corollary's statement directly follows from Lemma 21.

Otherwise, let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, and $\bar V := \bar{\mathbb{F}} \otimes_{\mathbb{F}} V$ and $\bar A := 1_{\bar{\mathbb{F}}} \otimes_{\mathbb{F}} A \colon \bar V \to \bar V$ be the counterparts of $V$ and $A$ under this closure. The vector space is naturally embedded in $\bar V$ via identifying it with $1_{\bar{\mathbb{F}}} \otimes_{\mathbb{F}} V$. In view of Lemma 21, we have $\ker P(\bar A) = \ker P_1(\bar A) \oplus \cdots \oplus \ker P_s(\bar A)$. Since $\ker P(A) = \ker P(\bar A) \cap V$ and $\ker P_j(A) = \ker P_j(\bar A) \cap V$, $j = 1, \ldots, s$, we obtain the required equality for $A$. □

**Lemma 23.** *Suppose that $V$ is a vector space over a field $\mathbb{F}$, $A, B \in \mathrm{End}(V)$, $C := [A, B]$ commutes with $A$ and $C \ker A = \ker A$. Then, for any $r \in \mathbb{N}$,*

$$\ker A^r = \sum_{i=0}^{r-1} B^i \ker A. \tag{10}$$

The condition $[A, C] = 0$ implies $C \ker A \subseteq \ker A$, but we need the stronger condition $C \ker A = \ker A$.

**Proof.** Denote the direct sum in the right-hand side of the last equality by $U_r$.

We can show by induction with respect to $i$ that for any $r > i$,

$$A^r B^i = \big(rC + BA\big)\big((r-1)C + BA\big) \cdots \big((r-i+1)C + BA\big)A^{r-i}.$$

Therefore, $B^i \ker A \subseteq \ker A^r$ for any $r > i$, $i \in \mathbb{N}$, which obviously implies $U_r \subseteq \ker A^r$.

The inverse inclusion $\ker A^r \subseteq U_r$ is proved by the induction with respect to $r$. The induction base $r = 1$ is obvious since $U_1 = \ker A$. Let $r > 1$. The induction hypothesis is $\ker A^{r-1} \subseteq U_{r-1}$. Suppose that $v \in \ker A^r$. Then, $A^{r-1}v = w$, where $w \in \ker A$. Since $C \ker A = \ker A$, then $C^k \ker A = \ker A$ for any $k \in \mathbb{N}$. For any $\hat v \in \ker A$, we have $A^{r-1}B^{r-1}\hat v = (r-1)!C^{r-1}\hat v$. Therefore, there exists $\hat v \in \ker A$ such that $A^{r-1}B^{r-1}\hat v = w$, and then $\check v := v - B^{r-1}\hat v \in \ker A^{r-1}$. As a result, $v = B^{r-1}\hat v + \check v \in B^{r-1} \ker A + \ker A^{r-1} \subseteq B^{r-1} \ker A + U_{r-1} = U_r$. □

**Remark 24.** If in addition to the condition of Lemma 23, we have $\ker A \cap \ker C = \{0\}$, then the sum in (10) is direct. Indeed, then the operator $C$ and, therefore, all its powers are injective on $\ker A$. Let $v \in B^i \ker A \cap B^j \ker A$, $i < j$. Then, $0 = A^j v = A^j B^i \hat v = A^j B^j \hat v = (j-1)!C^{j-1}\hat v$, which implies that $v = 0$.

**Corollary 25.** *Suppose that $V$ is a vector space over a field $\mathbb{F}$, $A, B \in \mathrm{End}(V)$ such that $[A, B] = c\,\mathrm{id}$ with some nonzero $c \in \mathbb{F}$. Then, for any $r \in \mathbb{N}$,*

$$\ker A^r = \bigoplus_{i=0}^{r-1} B^i \ker A.$$

We use the above assertions in the context of finding solutions that are invariant with respect to generalised symmetries. If generalised vector field $(\mathrm{Q}u)\partial_u$ is a (linear) generalised symmetry of $\mathcal{L}$, then for any polynomial $P$ of Q, the generalised vector field $(P(\mathrm{Q})u)\partial_u$ is such as well. If $P = P_1 \cdots P_s$ is a factorisation with coprime polynomials $P_1, \ldots, P_s$, then the space of $P(\mathrm{Q})$-invariant solutions of $\mathcal{L}$ is the direct sum of the spaces of $P_j(\mathrm{Q})$-invariant solutions of $\mathcal{L}$, $j = 1, \ldots, s$. Therefore, when finding $P(\mathrm{Q})$-invariant solutions of $\mathcal{L}$ for fixed Q and various $P$, it suffices to consider only the polynomials $P(x) = (x - \lambda)^k$, $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ in the real case, realify the corresponding space of invariant solutions. In addition, one should select only $G^{\mathrm{ess}}$-inequivalent recursion operators Q, where the action of $G^{\mathrm{ess}}$ on such operators, Q, is induced by the action on the corresponding linear generalised symmetries, $(\mathrm{Q}u)\partial_u$. Moreover, finding $(\mathrm{Q} - \lambda E)^k$-invariant solutions of $\mathcal{L}$, where $E$ is the $m \times m$ identity

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matrix, is simplified when the operator $\mathrm{Q}-\lambda E$ fits into the framework of Lemma 23. More specifically, we assume that $V$ is the solution space of $\mathcal{L}$. If there exists $\mathrm{S}\in\Upsilon$ such that the operator $[\mathrm{Q},\mathrm{S}]$ commutes with Q and its restriction to the kernel of $\mathrm{Q}-\lambda E$ is a surjection, then the space of $(\mathrm{Q}-\lambda E)^k$-invariant solutions of $\mathcal{L}$ can be constructed as the sum of the images of the space of $(\mathrm{Q}-\lambda E)$-invariant solutions of $\mathcal{L}$ under successive action of S up to $k-1$ times.

In fact, the generalised reduction procedure can be systematically realised only for linear generalised symmetries of the form $(P(\mathrm{Q})u)\partial_u$, where Q is in addition to a first-order Lie-symmetry operator, $\mathrm{Q}=(\xi^i(x)\mathrm{D}_i)E-H(x)$ with an $m\times m$ matrix function $H(x)=\big(\eta^{ab}(x)\big)$ of $x$, and $\xi^i\neq 0$ for some $i$. Here and in what follows the index $i$ runs from 1 to $n$, the indices $a$ and $b$ run from 1 to $m$, and we assume summation with respect to repeated indices. As stated above for the case of general Q, it suffices to consider very particular polynomials $P$. More specifically, let $\lambda_1, \ldots, \lambda_r$ be all the distinct roots of $P$ over $\mathbb{C}$, and let $k_j$ be the multiplicity of $\lambda_j$, $j=1,\ldots,r$. Then, the space of $P(\mathrm{Q})$-invariant solutions of $\mathcal{L}$ decomposes into the direct sum of the spaces of $(\mathrm{Q}-\lambda_jE)^{k_j}$-invariant solutions of $\mathcal{L}$ for each $\lambda_j\in\mathbb{R}$ and $(\mathrm{Q}-\lambda_jE)^{k_j}(\mathrm{Q}-\bar\lambda_jE)^{k_j}$-invariant solutions of $\mathcal{L}$ for each unordered pair $\{\lambda_j,\bar\lambda_j\}$ with $\lambda_j\in\mathbb{C}\setminus\mathbb{R}$, where $\bar\lambda_j$ denotes the complex conjugate of $\lambda_j$. In other words, the construction of solutions of $\mathcal{L}$ that are invariant with respect to generalised symmetries associated with polynomials of a single Lie-symmetry operators Q reduces to the case when polynomials are powers of binomials $\mathrm{Q}-\lambda E$ and, if $\lambda\in\mathbb{C}\setminus\mathbb{R}$, the separation of the real and imaginary parts of the obtained solutions. For each binomial power $(\mathrm{Q}-\lambda E)^k$, the corresponding ansatz is just a representation of the general solution of the system $(\mathrm{Q}-\lambda E)^k u=0$, and thus it takes the form

$$u^a=\mathrm{e}^{\lambda\zeta}\sum_{j=0}^{k-1}\sum_{b=1}^{m}f^{ab}(x)\varphi^{bj}(\omega_1,\ldots,\omega_{n-1})\zeta^j,\quad a=1,\ldots,m. \tag{11}$$

Here, $\omega_1=\omega_1(x)$, ..., $\omega_{n-1}=\omega_{n-1}(x)$ are functionally independent solutions of the equation $\xi^i(x)\partial_i\omega=0$, $\zeta=\zeta(x)$ is a particular solution of the equation $\xi^i(x)\partial_i\zeta=1$, and $(f^{1b},\ldots,f^{mb})$ are linearly independent solutions of the system $\xi^i(x)\partial_i f^a=\eta^{ab}(x)f^b$. The selection of $G^{\mathrm{ess}}$-inequivalent Lie-symmetry operators reduces to the classification of one-dimensional subalgebras in the essential Lie invariance algebra of the system $\mathcal{L}$. Substituting the ansatz (11) into this system results in a homogeneous linear system of differential equations with respect to the new unknown functions $\varphi^{bj}$ of less number of independent variables $\omega_1, \ldots, \omega_{n-1}$.

### *5.2. Point symmetries and Lie-invariant solutions*

Now, we present the results on point symmetries and Lie reductions of the equation (1) that were obtained in [16, 17].

**Theorem 26** *([16]). The complete point-symmetry pseudogroup G of the remarkable Fokker–Planck equation* (1) *consists of the transformations of the form*

$$\tilde t=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x=\frac{\hat x}{\gamma t+\delta}-\frac{3\gamma\hat y}{(\gamma t+\delta)^2},\quad \tilde y=\frac{\hat y}{(\gamma t+\delta)^3},$$
$$\tilde u=\sigma(\gamma t+\delta)^2\exp\left(\frac{\gamma\hat x^2}{\gamma t+\delta}-\frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2}+\frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)$$
$$\times\exp\big(3\lambda_3(y-tx)-\lambda_2x-(3\lambda_3^2t^3+3\lambda_3\lambda_2t^2+\lambda_2^2t)\big)\big(u+f(t,x,y)\big), \tag{12}$$

*where* $\hat x:=x+3\lambda_3t^2+2\lambda_2t+\lambda_1$, $\hat y:=y+\lambda_3t^3+\lambda_2t^2+\lambda_1t+\lambda_0$; $\alpha$, $\beta$, $\gamma$ *and* $\delta$ *are arbitrary constants with* $\alpha\delta-\beta\gamma=1$; $\lambda_0, \ldots, \lambda_3$ *and* $\sigma$ *are arbitrary constants with* $\sigma\neq 0$, *and* $f$ *is an arbitrary solution of* (1).

Pulling back an arbitrary solution $u=h(t,x,y)$ of (1) by an arbitrary point-symmetry transformation of the form (12), we obtain, in the notation of Theorem 26, the formula of generating new solutions of (1) from known ones under the action of elements of $G$,

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$$u=\frac{\mathrm{e}^{\lambda_2x-3\lambda_3(y-tx)+3\lambda_3^2t^3+3\lambda_3\lambda_2t^2+\lambda_2^2t}}{\sigma(\gamma t+\delta)^2}\exp\left(-\frac{\gamma\hat x^2}{\gamma t+\delta}+\frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2}-\frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)$$
$$\times h\left(\frac{\alpha t+\beta}{\gamma t+\delta},\ \frac{\hat x}{\gamma t+\delta}-\frac{3\gamma\hat y}{(\gamma t+\delta)^2},\ \frac{\hat y}{(\gamma t+\delta)^3}\right)-f(t,x,y). \tag{13}$$

Due to the complexification trick, applying the formula (13) for generating solutions to a real-analytical solution of the equation (1), one can assume all the constant parameters in (1) to be complex and then take the real and imaginary parts of the obtained solutions.

We use the modified transformation composition as the group operation in $G$. More specifically, this composition respects the natural domains of transformations of the form (13), see [16, Section 3] for details. The point transformations of the form

$$\mathscr{Z}(f):\quad \tilde t=t,\quad \tilde x=x,\quad \tilde y=y,\quad \tilde u=u+f(t,x,y),$$

where the parameter function $f=f(t,x,y)$ is an arbitrary solution of the equation (1), are associated with the linear superposition of solutions of this equation and thus can be considered as trivial. They constitute the normal pseudosubgroup $G^{\rm lin}$ of the pseudogroup $G$. The pseudogroup $G$ splits over $G^{\rm lin}$, $G=G^{\rm ess}\ltimes G^{\rm lin}$, where $G^{\rm ess}$ is the *subgroup* of $G$ consisting of the transformations of the form (12) with $f=0$ and with their natural domains, and thus it is an eight-dimensional Lie group.

We exhaustively carried out Lie reductions of the equation (1) in [16], beginning with the classification of $G^{\rm ess}$-inequivalent one- and two-dimensional subalgebras of $\mathfrak g^{\rm ess}$.

**Lemma 27.** *A complete list of $G^{\rm ess}$-inequivalent one-dimensional subalgebras of $\mathfrak g^{\rm ess}$ is exhausted by the subalgebras*

$$\mathfrak s_{1.1}=\langle\mathcal P^t+\mathcal P^3\rangle,\quad \mathfrak s^\delta_{1.2}=\langle\mathcal P^t+\delta\mathcal I\rangle,\quad \mathfrak s^\nu_{1.3}=\langle\mathcal D+\nu\mathcal I\rangle,\quad \mathfrak s^\mu_{1.4}=\langle\mathcal P^t+\mathcal K+\mu\mathcal I\rangle,$$
$$\mathfrak s^\varepsilon_{1.5}=\langle\mathcal P^2+\varepsilon\mathcal P^0\rangle,\quad \mathfrak s_{1.6}=\langle\mathcal P^1\rangle,\quad \mathfrak s_{1.7}=\langle\mathcal P^0\rangle,\quad \mathfrak s_{1.8}=\langle\mathcal I\rangle,$$

*where $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,0,1\}$, and $\mu$ and $\nu$ are arbitrary real constants with $\nu\geqslant0$.*

The families of solutions of the equation (1) that are invariant with respect to the subalgebras $\mathfrak s^0_{1.2}$, $\mathfrak s^\varepsilon_{1.5}$, $\mathfrak s_{1.6}$ and $\mathfrak s_{1.7}$ are parameterised by the general solutions of the (1 + 1)-dimensional linear heat equations with the zero and the inverse square potentials,

- $u=|x|^{-\frac14}\vartheta^\mu\Big(\frac94\tilde\varepsilon y,|x|^{\frac32}\Big)$ with $\mu=\frac5{36}$, $\tilde\varepsilon:=\operatorname{sgn}x$, (14)
- $u=|t|^{-\frac12}\mathrm e^{-\frac{x^2}{4t}}\vartheta^0\Big(\frac13t^3+2\varepsilon t-t^{-1},2y-(t+\varepsilon t^{-1})x\Big)$ with $\varepsilon\in\{-1,1\}$, (15)
- $u=\vartheta^0\Big(\frac13t^3,y-tx\Big)$, (16)
- $u=\vartheta^0(t,x)$, (17)

where $\vartheta^\mu=\vartheta^\mu(z_1,z_2)$ is an arbitrary solution of the equation $\vartheta^\mu_1=\vartheta^\mu_{22}+\mu z_2^{-2}\vartheta^\mu$. It was shown in [16] that only Lie reductions of codimension one give essentially new solutions. The solutions obtained using Lie reductions of codimensions two and three are superfluous since they give solutions that are $G^{\rm ess}$-equivalent to elements of the families (14)–(17) corresponding to known invariant solutions of the equations $\vartheta^\mu_1=\vartheta^\mu_{22}+\mu z_2^{-2}\vartheta^\mu$ with $\mu\in\{0,\frac5{36}\}$. Recall that a complete collection of inequivalent Lie-invariant solutions of the (1 + 1)-dimensional linear heat equation ($\mu=0$) was presented in Examples 3.3 and 3.17 in [33] and then enhanced in [46, Section A]. An analogous collection for all nonzero values of $\mu$ was constructed in [16, Section A], see also [11, 12].

The sole possibility to extend the Lie reduction procedure using the complexification trick is to assume the real constants appearing in optimal lists of subalgebras, like $\mu$ and $\nu$ in Lemma 27, to be complex.

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### *5.3. Generation of solutions by recursion operators*

The analysis of generating solutions of the remarkable Fokker–Planck equation (1) via acting by elements of the associative algebra $\Upsilon$ of its linear differential recursion operators was initiated in [16, 17]. Therein, only solutions that are invariant with respect to one-dimensional subalgebras of the (nil)radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ were considered as seed ones. In this section, we systematically analyse the use of arbitrary Lie-invariant solutions of (1) within this framework. It suffices to consider the solutions of (1) that are invariant with respect to the $G^{\rm ess}$-inequivalent one-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ that are listed in Lemma 27. Recall that the subalgebra $\mathfrak{s}_{1.8}$ is not appropriate for Lie reduction. Acting on an arbitrary solution $u=h(t,x,y)$ of the equation (1) by an element

$$\mathrm{Q}=\sum_{(i_3,i_2,i_1,i_0)\in\mathbb{N}_0^4} c_{i_3i_2i_1i_0}(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0} \tag{18}$$

of the algebra $\Upsilon=\Upsilon_{\mathfrak{r}}$, where only finitely many real constants $c_{i_3i_2i_1i_0}$ are nonzero, we obtain a solution $\mathrm{Q}h$ of (1). The problem is that applying this procedure to a known solution, one may obtain a solution that is known as well.

For each of fixed subalgebra $\mathfrak{s}^*_{1.k}$ from Lemma 27, where $*$ denotes a value of the tuple of parameters of the $k$th subalgebra family, $k=1,\dots,7$, we denote by $\mathscr{S}^*_k$ the family of $\mathfrak{s}^*_{1.k}$-invariant solutions of (1) and by B the element of $\Upsilon$ that is associated up to the multiplier $-1$ with its canonical basis element. We directly compute low-degree generators[6] of the centraliser $\mathrm{C}_\Upsilon(\mathrm{B})$ of B in $\Upsilon$ and expand an arbitrary $\mathrm{Q}\in\Upsilon$, that is, an arbitrary operator of the form (18) with respect to a basis of $\Upsilon$ in which powers of these generators are right multipliers in the basis elements. Since the action by elements of $\mathrm{C}_\Upsilon(\mathrm{B})$ preserves the family $\mathscr{S}^*_k$, such an expansion allows us to determine the action of which elements of $\Upsilon$ may lead, up to linearly combining solutions, to new solutions of (1). In the course of this analysis, we have to use the explicit representation (4) of the operators $\hat{\mathrm{P}}^t$, $\hat{\mathrm{D}}$ and $\hat{\mathrm{K}}$ in terms of the canonical generators of the algebra $\Upsilon$.

In the expansions of Q below, the indices $j$ run through $\mathbb{N}_0$, we assume summation with respect to repeated indices, and only a finite number of polynomials $R$ indexed by tuples of indices $j$ are nonzero.

$\mathfrak{s}_{1.1}$**.** The centraliser $\mathrm{C}_\Upsilon(\mathrm{B})$ of $\mathrm{B}:=\hat{\mathrm{P}}^t+\mathrm{P}^3$ in the algebra $\Upsilon$ contains the operators

$$\mathrm{H}^1:=\mathrm{P}^1-\frac{1}{6}(\mathrm{P}^0)^2,\quad \mathrm{H}^2:=\mathrm{P}^2+\frac{2}{27}(\mathrm{P}^0)^3-\frac{2}{3}\mathrm{P}^1\mathrm{P}^0,\quad \mathrm{B}=\mathrm{P}^3-\mathrm{P}^2\mathrm{P}^0+(\mathrm{P}^1)^2.$$

We expand the operator Q as $\mathrm{Q}=R^{j_1j_2j_3}(\mathrm{P}^0)\,(\mathrm{H}^1)^{j_1}(\mathrm{H}^2)^{j_2}\mathrm{B}^{j_3}$. It implies that action of an arbitrary element Q of $\Upsilon$ on an arbitrary solution $h$ from the family $\mathscr{S}_1$ is a (finite) linear combinations of solutions of the form $(\mathrm{P}^0)^j\hat{h}$, where $j\in\mathbb{N}_0$ and $\hat{h}\in\mathscr{S}^0_2$. Thus, when using seed solutions from $\mathscr{S}_1$, only acting with powers of $\mathrm{P}^0$ may give solutions out of $\mathscr{S}_1$.

$\mathfrak{s}^\delta_{1.2}$**.** The operator $\mathrm{B}:=\hat{\mathrm{P}}^t+\delta$ commutes with the operators

$$\mathrm{P}^0,\quad \hat{\mathrm{P}}^t=-\mathrm{P}^2\mathrm{P}^0+(\mathrm{P}^1)^2,\quad \mathrm{H}:=\mathrm{P}^3(\mathrm{P}^0)^2-3\mathrm{P}^2\mathrm{P}^1\mathrm{P}^0+2(\mathrm{P}^1)^3.$$

In view of Lemma 7 from [18], any solution $h\in\mathscr{S}^\delta_2$ can be represented as $(\mathrm{P}^0)^k\tilde{h}$ with $k:=\max\big(\{2i_3+i_2-i_0\mid c_{i_3i_2i_1i_0}\neq0\}\cup\{0\}\big)$ for some $\tilde{h}\in\mathscr{S}^\delta_2$, and thus

$$\mathrm{Q}h=\mathrm{Q}(\mathrm{P}^0)^k\tilde{h}=R^{j_1j_2j_3}(\mathrm{P}^1)\,(\mathrm{P}^0)^{j_1}(\hat{\mathrm{P}}^t)^{j_2}\mathrm{H}^{j_3}\tilde{h}.$$

It implies that action of an arbitrary element Q of $\Upsilon$ on an arbitrary solution $h$ from the family $\mathscr{S}^\delta_2$ is a (finite) linear combinations of solutions of the form $(\mathrm{P}^1)^j\hat{h}$, where $j\in\mathbb{N}_0$ and $\hat{h}\in\mathscr{S}^\delta_2$. In other words, the only way for generating new solutions of the equation (1) from its solutions from the family $\mathscr{S}^0_2$ via acting by its linear differential recursion operators is to use powers of $\mathrm{P}^1$.

$\mathfrak{s}^\nu_{1.3}$**.** Some families of Lie-invariant solutions are preserved when acting by all linear differential recursion operators. Consider the span $\mathscr{S}_3$ of all $\mathfrak{s}^\nu_{1.3}$-invariant solutions, where the parameter $\nu$ runs

[6] Finding the entire centralisers $\mathrm{C}_\Upsilon(\mathrm{B})$ is a nontrivial problem for most elements B of $\Upsilon\simeq\mathrm{W}(2,\mathbb{R})$. See [6] and references therein, where the analogous problem is considered for the simpler first Weyl algebra $\mathrm{W}(1,\mathbb{R})$.

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through $\mathbb{R}$, $\mathscr{S}_3:=\sum_{\nu\in\mathbb{R}}\mathscr{S}_3^\nu$. Since $\mathrm{P}^3\hat{\mathrm{D}}=(\hat{\mathrm{D}}-3)\mathrm{P}^3$, $\mathrm{P}^2\hat{\mathrm{D}}=(\hat{\mathrm{D}}-1)\mathrm{P}^2$, $\mathrm{P}^1\hat{\mathrm{D}}=(\hat{\mathrm{D}}+1)\mathrm{P}^1$, $\mathrm{P}^0\hat{\mathrm{D}}=(\hat{\mathrm{D}}+3)\mathrm{P}^0$ and thus

$$(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}(\hat{\mathrm{D}}-\nu)=(\hat{\mathrm{D}}-\nu-3i_3-i_2+i_1+3i_0)(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0},$$

the action by $(\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}$ maps $\mathscr{S}_3^\nu$ into $\mathscr{S}_3^{\nu'}$ with $\nu'=\nu+3i_3+i_2-i_1-3i_0$. Therefore, the action of any element Q of the algebra $\Upsilon$ maps $\mathscr{S}_3$ into itself.

$\mathfrak{s}^\mu_{1.4}$. The centraliser $\mathrm{C}_\Upsilon(\mathrm{B})$ with $\mathrm{B}=\hat{\mathrm{P}}^t+\hat{\mathrm{K}}+\mu$ contains the associative algebra generated by

$$\hat{\mathrm{P}}^t+\hat{\mathrm{K}}=(\mathrm{P}^2)^2-\mathrm{P}^3\mathrm{P}^1-\mathrm{P}^2\mathrm{P}^0+(\mathrm{P}^1)^2,\quad \mathrm{H}:=(\mathrm{P}^3)^2+3\mathrm{P}^3\mathrm{P}^1+3\mathrm{P}^2\mathrm{P}^0+(\mathrm{P}^0)^2,\quad \mathrm{C},$$

$$\begin{aligned}\mathrm{S}:=&(\mathrm{P}^3)^3\mathrm{P}^0-3(\mathrm{P}^3)^2\mathrm{P}^2\mathrm{P}^1+2\mathrm{P}^3(\mathrm{P}^2)^3+3\mathrm{P}^3(\mathrm{P}^2)^2\mathrm{P}^0-6\mathrm{P}^3\mathrm{P}^2(\mathrm{P}^1)^2-\mathrm{P}^3(\mathrm{P}^1)^2\mathrm{P}^0\\&-\mathrm{P}^3(\mathrm{P}^0)^3+3(\mathrm{P}^2)^3\mathrm{P}^1+6(\mathrm{P}^2)^2\mathrm{P}^1\mathrm{P}^0-3\mathrm{P}^2(\mathrm{P}^1)^3+3\mathrm{P}^2\mathrm{P}^1(\mathrm{P}^0)^2-2(\mathrm{P}^1)^3\mathrm{P}^0\\&-4(\mathrm{P}^3)^2+8(\mathrm{P}^0)^2+12\mathrm{P}^2\mathrm{P}^0.\end{aligned}$$

Applying arguments similar to those above to the pair $\hat{\mathrm{P}}^t+\hat{\mathrm{K}}$ and H, we conclude that only the action by polynomials

$$\mathrm{P}^3\mathrm{P}^2(\mathrm{P}^1)^{j_1}(\mathrm{P}^0)^{j_2},\quad \mathrm{P}^3(\mathrm{P}^1)^{j_1}(\mathrm{P}^0)^{j_2},\quad \mathrm{P}^2(\mathrm{P}^1)^{j_1}(\mathrm{P}^0)^{j_2},\quad (\mathrm{P}^1)^{j_1}(\mathrm{P}^0)^{j_2}$$

might result in nontrivial solution generations. It is not clear how this consideration can be modified using higher-degree elements like C and S.

$\mathfrak{s}^\varepsilon_{1.5}$, $\mathfrak{s}_{1.6}$, $\mathfrak{s}_{1.7}$. The centralisers $\mathrm{C}_\Upsilon(\mathrm{B})$ of $\mathrm{B}:=\mathrm{P}^2+\varepsilon\mathrm{P}^0$, $\mathrm{B}:=\mathrm{P}^1$ and $\mathrm{B}:=\mathrm{P}^0$ in the algebra $\Upsilon$ contain the associative algebras generated by the sets $\{\mathrm{P}^3-3\varepsilon\mathrm{P}^1,\mathrm{P}^2,\mathrm{P}^0\}$, $\{\mathrm{P}^3,\mathrm{P}^1,\mathrm{P}^0\}$ and $\{\mathrm{P}^2,\mathrm{P}^1,\mathrm{P}^0\}$, respectively. For each of these associative algebras, we expand an arbitrary element Q of $\Upsilon$, involving its elements as described above. This implies that the only way to construct essentially new solutions via acting by linear differential recursion operators starting with the seeds from the families $\mathscr{S}_5^\varepsilon$, $\mathscr{S}_6$ and $\mathscr{S}_7$ is to use powers of $\mathrm{P}^1$, $\mathrm{P}^2$ and $\mathrm{P}^3$, respectively.

We combine the above results with results of Sections 5.3 to construct wide families of exact solutions of the equation (1). More specifically, we have shown that for the families (14), (15), (16) and (17) of solutions of the equation (1) that are invariant with respect to the subalgebras $\mathfrak{s}^0_{1.2}$, $\mathfrak{s}^\varepsilon_{1.5}$, $\mathfrak{s}_{1.6}$ and $\mathfrak{s}_{1.7}$, only the action by the monomials $(\mathrm{P}^1)^k$, $(\mathrm{P}^1)^k$, $(\mathrm{P}^2)^k$ and $(\mathrm{P}^3)^k$, $k\in\mathbb{N}$, respectively, in general leads to essentially new solutions of (1),

- $u=(\mathrm{P}^1)^k\Big(|x|^{-\frac14}\vartheta^\mu\big(\tfrac94\tilde\varepsilon y,|x|^{\frac32}\big)\Big)$ with $\mu=\frac{5}{36}$, $\tilde\varepsilon:=\operatorname{sgn}x$, (19)
- $u=(\mathrm{P}^1)^k\Big(|t|^{-\frac12}\mathrm{e}^{-\frac{x^2}{4t}}\vartheta^0\big(\tfrac13t^3+2\varepsilon t-t^{-1},2y-(t+\varepsilon t^{-1})x\big)\Big)$ with $\varepsilon\in\{-1,1\}$, (20)
- $u=(\mathrm{P}^2)^k\vartheta^0\Big(\tfrac13t^3,y-tx\Big)$, (21)
- $u=(\mathrm{P}^3)^k\vartheta^0(t,x)$. (22)

Even if the action of an element Q of $\Upsilon$ preserve a solution family, it may result in an interesting solution generation within this family. An example of such a generation is presented in Section 3. The solution $u=1$ is invariant with respect to the four-dimensional subalgebra $\langle\mathcal{P}^t,\mathcal{D}+2\mathcal{I},\mathcal{P}^1,\mathcal{P}^0\rangle$ of the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$ of (1). In other words, this solution belongs to $\mathscr{S}_2^0\cap\mathscr{S}_2^2\cap\mathscr{S}_6\cap\mathscr{S}_7$. Lemma 8 states that any solution of the equation (1) that is polynomial with respect to $x$ is polynomial with respect to the entire tuple of independent variables $(t,x,y)$ and is a (finite) linear combination of the basis polynomials $(\mathrm{P}^3)^k(\mathrm{P}^2)^l1$, $k,l\in\mathbb{N}_0$, thus belonging to the space $\mathscr{S}_3:=\sum_{\nu\in\mathbb{R}}\mathscr{S}_3^\nu$.

### *5.4. Generalised reductions*

Using the theory developed in Section 5.1, we also revisit and extend the results of [16, 17] on particular generalised reductions of the equation (1). According to this theory, only polynomials of Lie-symmetry operators of the equation (1) can be systematically used in the procedure of its generalised reduction.

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In view of Lemma 21, it suffices to just consider the powers of binomials of the form $\mathrm{Q}-\lambda$, where, up to the $G^{\rm ess}$-equivalence, Q runs through the Lie-symmetry operators of (1) associated with the basis elements of the subalgebras $\mathfrak{s}^*_{1.k}$, $k=1,\dots,7$, from Lemma 27. Nevertheless, only some families even of these specific invariant solutions can be described completely.

It turns out that the easiest and the most complete description is achieved when the corresponding subalgebra is contained in the (nil)radical $\mathfrak{r}$ of the algebra $\mathfrak{g}^{\rm ess}$ and thus, modulo the $G^{\rm ess}$-equivalence, it is one of the subalgebras $\mathfrak{s}^\varepsilon_{1.5}$, $\mathfrak{s}_{1.6}$ and $\mathfrak{s}_{1.7}$ or, equivalently, $\mathrm{Q}\in\{\mathrm{P}^2+\varepsilon\mathrm{P}^0,\mathrm{P}^1,\mathrm{P}^0\}$. In this case, up to the $G^{\rm ess}$-equivalence, the parameter $\lambda$ in $\mathrm{Q}-\lambda$ can be set to zero, that is, it suffices to merely describe $\mathrm{Q}^n$-invariant solutions. In view of the commutation relations $[\mathrm{P}^2+\varepsilon\mathrm{P}^0,\mathrm{P}^1]=1$, $[\mathrm{P}^1,\mathrm{P}^2]=1$ and $[\mathrm{P}^0,\mathrm{P}^3]=3$, Corollary 25 implies that the spaces of $(\mathrm{P}^2+\varepsilon\mathrm{P}^0)^n$-, $(\mathrm{P}^1)^n$- and $(\mathrm{P}^0)^n$-invariant solutions, $n\in\mathbb{N}$, are the direct sums of spaces of solutions of the form (20), (21) and (22) with fixed $k$, respectively, where $k$ runs from 0 to $n-1$. Combining the complexification trick, the $G^{\rm ess}$-action and the linear superposition of solutions, we obtain the entire span of solutions that are invariant with respect to polynomials of Lie-symmetry operators associated with elements of $\mathfrak{r}$.

Considering the subalgebra $\mathfrak{s}_{1.1}$, where $\mathrm{Q}=\hat{\mathrm{P}}^t+\mathrm{P}^3$, we can also set $\lambda=0$ modulo the $G^{\rm ess}$-equivalence and, since $[\hat{\mathrm{P}}^t+\mathrm{P}^3,\mathrm{P}^0]=-3$, apply Corollary 25. As a result, we conclude that the space of $(\hat{\mathrm{P}}^t+\mathrm{P}^3)^n$-invariant solutions of (1), $n\in\mathbb{N}$, is the direct sums of spaces of solutions of the form $(\mathrm{P}^0)^k h$ with fixed $k$ and an arbitrary $h\in\mathscr{S}_1$, where $k$ runs from 0 to $n-1$. These solutions arise as the result of the only essential nontrivial solution generation using linear differential recursion operators and seed solutions from the set $\mathscr{S}_1$. Recall [16, Section 5] that the function $h=h(t,x,y)$ belongs to $\mathscr{S}_1$ if and only if

$$h=\mathrm{e}^{\frac{3}{10}t(t^4-5tx+10y)}w(z_1,z_2),\quad z_1:=y-\tfrac14t^4,\quad z_1:=x-t^3,$$

where $w$ is an arbitrary solution of the reduced equation $z_2w_1=w_{22}-3z_1w$. The problem is that no nonzero solutions of the latter equation and, therefore, no nonzero elements of $\mathscr{S}_1$ are known.

The description of generalised reductions associated with polynomials of $\hat{\mathrm{P}}^t$ is more involved comparing to the previous cases. Up to the $G^{\rm ess}$-equivalence, the parameter $\lambda$ in the operator $\hat{\mathrm{P}}^t+\lambda$ can be gauged at most to $\delta\in\{-1,0,1\}$. In other words, we should consider the operator $\hat{\mathrm{P}}^t+\delta$ associated with the basis element of the subalgebra $\mathfrak{s}^\delta_{1.2}$ for each $\delta\in\{-1,0,1\}$. We have $[\hat{\mathrm{P}}^t+\delta,\mathrm{P}^1]=\mathrm{P}^0$, $[\hat{\mathrm{P}}^t+\delta,\mathrm{P}^0]=0$ and, in view of [18, Lemma 7], $\mathrm{P}^0\ker(\hat{\mathrm{P}}^t+\delta)=\ker(\hat{\mathrm{P}}^t+\delta)$. Lemma 23 in this setting implies that the space of $(\hat{\mathrm{P}}^t+\delta)^n$-invariant solutions of (1), $n\in\mathbb{N}$, is the sum of the spaces of solutions of the form $(\mathrm{P}^0)^kh$ with fixed $k$ and an arbitrary $h\in\mathscr{S}^\delta_2$, where $k$ runs from 0 to $n-1$. The $\mathfrak{s}^\delta_{1.2}$-invariant solutions are of the form $u=\mathrm{e}^{\delta t}w(x,y)$, where $w$ is an arbitrary solution of the equation $xw_y=w_{xx}-\delta w$. For $\delta=0$, the last equation is reduced by a point transformation to a (1 + 1)-dimensional linear heat equation with an inverse square potential, and thus the generated solutions are precisely given by (19). In the case $\delta\neq0$, we were able to construct only those among the $\mathfrak{s}^\delta_{1.2}$-invariant solutions that are in addition $\mathrm{P}^0$-invariant, that is, that are $\mathfrak{s}^\delta_{2.2}$-invariant, where $\mathfrak{s}^\delta_{2.2}=\langle\mathcal{P}^t+\delta\mathcal{I},\mathcal{P}^0\rangle$. The solution generation using linear differential recursion operators and seed solutions from the set $\mathscr{S}^\delta_2\cap\mathscr{S}_7$ results only in certain solutions of the form (22). Hence, it is necessary to find other $\mathfrak{s}^\delta_{1.2}$-invariant solutions with $\delta\neq0$, which is an open nontrivial problem.

The analysis of generalised reductions associated with polynomials of $\hat{\mathrm{D}}$ and of $\hat{\mathrm{P}}^t+\hat{\mathrm{K}}$ is far more complicated. Modulo the $G^{\rm ess}$-equivalence, the parameter $\lambda$ can at most be made nonnegative in the operator $\hat{\mathrm{D}}-\lambda$ and cannot be changed in the operator $\hat{\mathrm{P}}^t+\hat{\mathrm{K}}-\lambda$. This is why the consideration of all the subalgebras $\mathfrak{s}^\nu_{1.3}$ and $\mathfrak{s}^\mu_{1.4}$ from Lemma 27 is relevant here. The space $\mathscr{S}_3:=\sum_{\nu\in\mathbb{R}}\mathscr{S}^\nu_3$ is preserved by the action of any element Q of $\Upsilon$, see Section 5.3. We can show analogously that there is no obvious way to derive $(\hat{\mathrm{D}}-\lambda)^n$-invariant solutions from analogous solutions with lower values of $n$. The same claim holds for $(\hat{\mathrm{P}}^t+\hat{\mathrm{K}}-\lambda)^n$-invariant solutions. Following the consideration in the last paragraph of Section 5.1, we construct generalised ansatzes for such solutions and the corresponding reduced systems, which are

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$$u=|t|^{\frac12\lambda-1}\sum_{j=0}^{n-1}w^j(z_1,z_2)\zeta^j,\quad \omega_1:=|t|^{-\frac32}y,\quad \omega_2:=|t|^{-\frac12}x,\quad \zeta:=\frac12\ln|t|,$$
$$(2\varepsilon'z_2-3z_1)w^j_1=2\varepsilon'w^j_{22}+z_2w^j_2-(\lambda-2)w^j-(j+1)w^{j+1},\quad j=0,\dots,n-1,\quad w^n:=0$$

with $\varepsilon':=\operatorname{sgn}t$ for the operator $(\hat{\mathrm D}-\lambda)^n$ and

$$u=\frac{e^{\theta(t,x,y)+\lambda\zeta}}{t^2+1}\sum_{j=0}^{n-1}w^j(z_1,z_2)\zeta^j,\quad \theta(t,x,y):=-\frac{3t^3y^2+t(2x(t^2+1)-3ty)^2}{4(t^2+1)^3},$$
$$z_1:=\frac{y}{(t^2+1)^{\frac32}},\quad z_2:=\frac{(t^2+1)x-3ty}{(t^2+1)^{\frac32}},\quad \zeta:=\arctan t,$$
$$z_2w^j_1=3z_1w^j_2+w^j_{22}+(z_2^2-\lambda)w^j-(j+1)w^{j+1},\quad j=0,\dots,n-1,\quad w^n:=0$$

for the operator $(\hat{\mathrm P}^t+\hat{\mathrm K}-\lambda)^n$. At the same time, even in the case $n=1$, when the corresponding invariant solutions are in fact Lie-invariant and the corresponding reduced systems are just single equations, the reduction procedure did not result to finding new exact solutions of the equation (1) [16, Section 5].

**Remark 28.** Solutions of the equation (1) that are invariant with respect to linear differential recursion operators that are not polynomials of single Lie-symmetry operators, even the simplest among such operators, for example, $(\mathrm P^0)^2+\mathrm P^1$, need a separate consideration.

## 6. Conclusion

The successful exhaustive classical symmetry analysis of the remarkable Fokker–Planck equation (1) in [16] inspired us to study its generalised symmetries as well. To this end, we began with computing the generalised symmetries of (1) up to order four by using the excellent package Jets by Baran and Marvan [1] for Maple, which is based on results of [27]. Carefully analysing the computation results, we made two interesting observations that allowed us to precisely conjecture the statement of Theorem 14.

The first observation was that all the linear generalised symmetries of order not greater than four are generated by the action of the Lie-symmetry operators of (1) associated with the radical $\mathfrak r$ of the algebra $\mathfrak g^{\rm ess}$ on the elementary Lie symmetry $u\partial_u$. In other words, $\Lambda^4=\Lambda^4_{\mathfrak r}$.

The second observation concerned the unexpected involvement of the Casimir operator of the Levi factor $\mathfrak f$ of $\mathfrak g^{\rm ess}$ in the consideration of the algebra $\Upsilon_{\mathfrak r}$. The counterpart C of this operator in the algebra $\Upsilon_{\mathfrak r}$ has degree four as a polynomial of $(\mathrm P^3,\mathrm P^2,\mathrm P^1,\mathrm P^0)$, while it is of order three as a differential operator. This degree–order inconsistency hinted that straightforwardly computing the dimensions of the subspaces $\Lambda^n$ of the algebra $\Lambda$ via evaluating the dimensions of the corresponding subspaces of the solution space of the system of determining equations $\Delta_{kl}$, $k,l\in\mathbb N_0$, with order restrictions is very difficult, perhaps even impossible.

Recall that the standard approach to finding the algebra of generalised symmetries of a linear system of differential equations includes the following steps:

1. For each $n\in\mathbb N_0$, compute the dimension of the space of canonical representatives of linear generalised symmetries of order less than or equal to $n$.
2. If all the dimensions obtained in the previous step are finite, then apply the Shapovalov–Shirokov theorem [41] to state that the linear generalised symmetries exhaust all generalised symmetries up to their equivalence and linear superposition of solutions.
3. By comparing the dimensions for each fixed order $n$, check whether the algebra of linear generalised symmetries is generated by the action of known linear differential recursion operators on simple seed symmetries, in particular, by the action of Lie-symmetry operators on the elementary Lie symmetry $u\partial_u$.

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For a number of systems of differential equations, their generalised-symmetry algebras were computed via following these steps in the presented order [18, 35, 41].

In contrast, we begin by showing that the entire algebra $\Lambda$ of linear generalised symmetries of the equation (1) coincides with the algebra $\Lambda_{\mathfrak{r}}$ of generalised symmetries generated by the action of the Lie-symmetry operators $\mathrm{P}^3$, $\mathrm{P}^2$, $\mathrm{P}^1$ and $\mathrm{P}^0$ on the vector field $u\partial_u$. In other words, we effectively start with step 3, leaving aside the dimension counting.

From the equality $\Lambda = \Lambda_{\mathfrak{r}}$, we derive $\dim \Lambda^{[n]} = \dim \Lambda_{\mathfrak{r}}^{[n]}$. At the same time, computing the dimension $\dim \Lambda_{\mathfrak{r}}^{[n]}$ is a nontrivial problem, once again due to the above inconsistency between the degree and the order of the operator C. However, we have managed to transfer the problem to the context of ring theory and algebraic geometry, which has allowed us to overcome this issue, prove the inequality $\dim \Lambda_{\mathfrak{r}}^{[n]} < \infty$ for any $n \in \mathbb{N}_0$ and thus apply the Shapovalov–Shirokov theorem. This has resulted in the proof of Theorem 14, thereby completing the description of the algebra $\Sigma$ of the equation (1).

A natural question to be addressed is whether there are more examples of differential equations for which the computation of their generalised-symmetry algebras using the approach developed in this paper is beneficial.

We also intend to extend the study of generalised symmetries to other (1 + 2)-dimensional ultraparabolic Fokker–Planck equations, in particular to prove Conjecture 8 from [19] on the generalised-symmetry algebra of the fine Fokker–Planck equation $u_t + xu_y = x^2u_{xx}$.

There is an important observation that if a homogeneous linear differential equation possesses a sufficiently large number of linearly independent essential Lie symmetries, then all its generalised symmetries are generated by acting with recursion operators related to such Lie symmetries on the simplest seed Lie symmetry $u\partial_u$. Examples of this situation include the linear (1 + 1)-dimensional heat equation, the (1 + 1)-dimensional Klein–Gordon equation and the remarkable Fokker–Planck equation, where sufficient sets of recursion operators are exhausted by selections of Lie-symmetry operators, as well as the linear Korteweg–de Vries equation, where one in addition needs to use the inversion of a Lie-symmetry operator associated with the space translations. It is an open question what are necessary and sufficient conditions for linear systems of differential equations whose algebras of generalised symmetries are exhausted by those generated using Lie symmetries. Examples of the opposite situation can be constructed from the above ones using differential substitutions like Darboux transformations such that the essential Lie invariance algebras of the mapped equations are trivial while their algebras of generalised symmetries are quite large.

We have developed a theoretical framework for using linear generalised symmetries of homogeneous linear systems of differential equations or, equivalently, their linear differential recursion operators for constructing and generating their exact solutions. The procedure of generalised reduction has been shown to properly work in the case of polynomials of single Lie-symmetry operators, Q, which can be reduced to the consideration of powers of elementary binomials, $\mathrm{Q} - \lambda$. The developed techniques have been efficiently applied to the remarkable Fokker–Planck equation (1), which have essentially extended results from [16, 17].

In the context of the classical group analysis, the linear (1 + 1)-dimensional heat equation (9) and the remarkable Fokker–Planck equation (1) are related to each other since they have similar Lie- and point-symmetry properties within the classes of parabolic linear second-order partial differential equations with two independent variables and of ultraparabolic linear second-order partial differential equations with three independent variables, respectively. Surprisingly, this relation manifests on the level of generalised symmetries as well. In particular, both the respective algebras $\Lambda_{\mathrm{h}}$ and $\Lambda$ of linear generalised symmetries are generated by the action of the Lie-symmetry operators associated with the radicals of the corresponding essential Lie invariance algebras on the elementary Lie-symmetry vector fields $u\partial_u$. Therefore, the algebras $\Lambda_{\mathrm{h}}$ and $\Lambda$ are isomorphic to the Lie algebras $\mathrm{W}(1,\mathbb{R})^{(-)}$ and $\mathrm{W}(2,\mathbb{R})^{(-)}$, respectively.

The above relation can be embedded in a much wider framework. For each $n \in \mathbb{N}$, consider the class $\mathcal{U}_n$ of (ultra)parabolic linear second-order partial differential equations with $1 + n$ independent variables $t, x_1, \ldots, x_n$ and dependent variable $u$, where the corresponding (symmetric) matrices of coefficients of

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second-order derivatives of the dependent variable $u$ are of rank one, and the number $n+1$ of independent variables is essential in the sense that none among them plays the role of a parameter even up to their point transformations. The equation

$$\mathcal{F}_n\colon\quad u_t+\sum_{i=1}^{n-1}x_iu_{x_{i+1}}=u_{x_1x_1}$$

belongs to the class $\mathcal{U}_n$, and the equations $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide with the equations (9) and (1), respectively. An in-depth preliminary analysis allows us to conjecture that for each $n\in\mathbb{N}$, the equation $\mathcal{F}_n$ is singular within the class $\mathcal{U}_n$ and has the following properties.

1. The dimension of the essential Lie invariance algebra $\mathfrak{g}_n^{\rm ess}$ of $\mathcal{F}_n$ is equal to $2n+4$, and this algebra is isomorphic to the algebra ${\rm sl}(2,\mathbb{R})\in_{\rho_{2n-1}\oplus\rho_0}{\rm h}(n,\mathbb{R})$, see Section 2 for the notation. The Levi factor $\mathfrak{f}_n$ and the (nil)radical $\mathfrak{r}_n$ of $\mathfrak{g}_n^{\rm ess}$ are isomorphic to the algebras ${\rm sl}(2,\mathbb{R})$ and ${\rm h}(n,\mathbb{R})$, respectively.
2. The dimension of $\mathfrak{g}_n^{\rm ess}$ is maximal among those of the essential Lie invariance algebras of equations from the class $\mathcal{U}_n$, and each equation whose essential Lie invariance algebra is of this maximal dimension is reduced to $\mathcal{F}_n$ by a point transformation in the space $\mathbb{R}^{1+n}_{t,x_1,\dots,x_n}\times\mathbb{R}_u$.
3. The essential point-symmetry group $G_n^{\rm ess}$ of the equation $\mathcal{F}_n$ is isomorphic to the Lie group $\big({\rm SL}(2,\mathbb{R})\ltimes_{\varrho_{2n-1}\oplus\varrho_0}{\rm H}(n,\mathbb{R})\big)\times\mathbb{Z}_2$, where ${\rm H}(n,\mathbb{R})$ denotes the rank-$n$ Heisenberg group and $\varrho_m$ is the irreducible representation of ${\rm SL}(2,\mathbb{R})$ in $\mathbb{R}^{m+1}$.
4. A complete list of discrete point-symmetry transformations of the equation $\mathcal{F}_n$ that are independent up to combining with each other and with continuous point-symmetry transformations of this equation is exhausted by the single involution $\mathscr{I}$ alternating the sign of $u$, $\mathscr{I}\colon(t,x_1,\dots,x_n,u)\mapsto(t,x_1,\dots,x_n,-u)$. Thus, the quotient group of the complete point-symmetry pseudogroup $G_n$ of $\mathcal{F}_n$ with respect to its identity component is isomorphic to $\mathbb{Z}_2$.
5. The algebra of canonical representatives of generalised symmetries of $\mathcal{F}_n$ is $\Sigma_n=\Lambda_n\in\Sigma_n^{-\infty}$. Here, $\Lambda_n$ is the subalgebra of linear generalised symmetries of $\mathcal{F}_n$, which is generated by acting with the Lie-symmetry operators associated with the canonical basis of the complement of the centre $\langle u\partial_u\rangle$ in the (nil)radical $\mathfrak{r}_n$ of $\mathfrak{g}_n^{\rm ess}$ on the elementary seed symmetry vector field $u\partial_u$, and $\Sigma_n^{-\infty}$ is the ideal associated with linear superposition of solutions of $\mathcal{F}_n$.
6. The algebra $\Lambda_n$ is isomorphic to the Lie algebra ${\rm W}(n,\mathbb{R})^{(-)}$ associated with the $n$th Weyl algebra ${\rm W}(n,\mathbb{R})$, $\Lambda_n\simeq{\rm W}(n,\mathbb{R})^{(-)}$. Hence, the algebra $\Lambda_n$ is $\mathbb{Z}$-graded.
7. The algebra $\Lambda_n$ is two-generated as a Lie algebra, that is, there is a pair of its elements such that $\Lambda_n$ coincides with its subalgebra containing all successive commutators (aka nonassociative monomials) of these two elements.

The complete and rigorous proofs of the listed properties constitute the subject of a substantial research programme whose realisation will result in a deeper understanding of symmetry properties of linear second-order partial differential equations. It will essentially extend the results of [18, 24, 33] on the linear (1 + 1)-dimensional heat equation $\mathcal{F}_1$ and of [16] and this paper on the remarkable Fokker–Planck equation $\mathcal{F}_2$ to $\mathcal{F}_n$ with an arbitrary $n\in\mathbb{N}$.

Establishing the isomorphism between the algebras $\Upsilon_{\mathfrak{r}}$ and ${\rm W}(2,\mathbb{R})$ (resp. $\Lambda$ and ${\rm W}(2,\mathbb{R})^{(-)}$) allows us to transfer the results naturally obtained for one of them to the other, and the mapped results can be not as apparent as their counterparts. See Remarks 17 and 18, where the transfers go from the abstract algebras to their realisations. Unexpected examples of opposite transfers are given by filtrations of the above associative algebras. Both the filtration $F_2$ of the algebra $\Upsilon_{\mathfrak{r}}$ with respect to the degree of its elements as (noncommutative) polynomials in $\{{\rm P}^0,{\rm P}^1,{\rm P}^2,{\rm P}^3\}$, which is given in Section 2, and its counterpart for the algebra ${\rm W}(2,\mathbb{R})$ are natural in the context of general noncommutative polynomial algebras. At the same time, we see no way to naturally interpret, for the second Weyl algebra ${\rm W}(2,\mathbb{R})$, the image of the other filtration of $\Upsilon_{\mathfrak{r}}$ presented therein and associated with the order of elements of $\Upsilon_{\mathfrak{r}}$ as differential operators. There are other similar natural filtrations of $\Upsilon_{\mathfrak{r}}$ that are related to various interpretations of

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the order of differential operators. In particular, we can assign to each element of $\Upsilon_\mathfrak{r}$ the order of its counterpart obtained by excluding multiple derivatives with respect to $x$ according to the Kovalevskaya form $u_{xx} = u_t + xu_y$ of the equation (1). The discussion in this paragraph is of significant interest since, as far as we know, the general problems of classifying filtrations and gradings of the Weyl algebras and of the associated Lie algebras have not been solved.

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