Canad. Math. Bull. Vol. **26** (4), 1983

METRIZABILITY CONDITIONS FOR COMPLETELY DISTRIBUTIVE LATTICES

BY

G. GIERZ, J. D. LAWSON AND A. R. STRALKA

ABSTRACT. A lattice is said to be essentially metrizable if it is an essential extension of a countable lattice. The main result of this paper is that for a completely distributive lattice the following conditions are equivalent: (1) the interval topology on L is metrizable, (2) L is essentially metrizable, (3) L has a doubly order-generating sublattice, (4) L is an essential extension of a countable chain.

Discussions of very strong completeness properties for distributive lattices are often, at least implicitly, highly topological in nature. The strength of the completeness properties a lattice possesses can be measured by the quality of the various intrinsic topologies which the order structure of the lattice can create. In this paper we shall discuss metrizability and metrizability-like conditions. For these very restrictive topological properties, it is natural that we should begin with an examination of those lattices with the strongest completeness properties—completely distributive lattices.

For completely distributive lattices, the interval topology is compact, Hausdorff and compatible with the order structure. We are able to provide several purely order-theoretic conditions, largely based upon the idea of essential extension, which are equivalent to the interval topology being metrizable. Our discussion then leads to a set of conditions, again involving essential extensions, which implies complete distributivity.

For a lattice L, a an element of L, and A a subset of L, $\uparrow a = \{x \in L \mid x \geq a\}$ and $\uparrow A = \bigcup \{\uparrow a \mid a \in A\}$. The sets $\downarrow a$ and $\downarrow A$ are defined dually. The interval topology on L, denoted $\mathrm{Int}(L)$, has a subbase for its family of closed sets consisting of all sets of the form $\uparrow x$ and $\downarrow x$ where $x \in L$. Papert-Strauss in [6] showed that the interval topology on a completely distributive lattice L is compact and Hausdorff and when L is equipped with this topology, it becomes a topological lattice; i.e., the meet and join operations, thought of as maps from $L \times L$ into L, are continuous. A map between complete lattices is said to

Received by the editors July 16, 1982 and, in revised form, October 19, 1982 and January 7, 1983.

1980 AMS Subject Classification Numbers: 06D10, 06B30, 54H12

be a complete homomorphism if it preserves arbitrary infima and arbitrary suprema. A homomorphism between two completely distributive lattices is complete if and only if it is continuous relative to the interval topology. Let I be the unit interval of the real line with the usual order and, let I^ω denote the countable product of copies of I. Putting together current information about completely distributive lattices, we have:

THEOREM 1. *Let L be a completely distributive lattice. Then the following conditions are equivalent:*

(i) $\mathrm{Int}(L)$ *is metrizable.*

(ii) *There is an algebraic and topological imbedding of* $(L, \wedge, \vee, \mathrm{Int}(L))$ *into* $(I^\omega, \wedge, \vee, \mathrm{Int}(I^\omega))$.

(iii) *There is a countable, point-separating family of continuous homomorphisms from* $(L, \wedge, \vee, \mathrm{Int}(L))$ *into* $(I, \wedge, \vee, \mathrm{Int}(I))$.

(iv) *There is a complete isomorphism from L into I^ω.*

(v) *There is a countable, point-separating family of complete homomorphisms from L into I.*

In view of Theorem 1 we shall say that any completely distributive lattice satisfying the equivalent conditions listed in Theorem 1 is metrizable.

Recall that a lattice L is said to be an essential extension of a lattice M if M can be imbedded into L in such a way that whenever ψ is a lattice congruence on L such that $\psi \cap M \times M = \Delta_M$, then $\psi = \Delta_L$. The concept of essential extension plays a prominent role in our search for a purely order-theoretic analog of metrizability. In the paper [1] Banaschewski and Bruns discuss essential extensions of distributive lattices in depth.

DEFINITION 2. The lattice L is said to be essentially metrizable if it is an essential extension of a countable lattice.

That essential metrizability has more to do with metrizability than with separability will become clear when we come to Proposition 4. It might be useful to keep in mind the chain J which is formed by applying the lexicographic ordering to the subset $[0, 1] \times \{0, 1\}$ of the plane. Then relative to the interval topology, J is a compact, totally disconnected, separable Hausdorff space—but it is not metrizable. In effect, J can be thought of as a space similar to the Cantor chain except that it has uncountably many gaps.

PROPOSITION 3. *Let L be a distributive lattice and let $a, b \in L$, $a \le b$. If L is essentially metrizable, then so are $\downarrow b$, $\uparrow a$ and the interval $[a, b] = \uparrow a \cap \downarrow b$.*

Proof. First of all, we verify Proposition 3 for $\downarrow b$. Let $D \subseteq L$ be a countable sublattice and assume that L is an essential extension of D. We define $D' \subseteq \downarrow b$ by $D' = \{d \wedge b \mid d \in D\}$. The distributive law implies that D' is a (countable) sublattice of $\downarrow b$. Let Θ be a congruence relation on $\downarrow b$. We may extend Θ to a

https://doi.org/10.4153/CMB-1983-073-9 Published online by Cambridge University Press

congruence ψ on L by defining

$$\psi = \{(x, y) \in L \times L \mid (x \wedge b, y \wedge b) \in \Theta \text{ and } x \vee b = y \vee b\}.$$

If $\Theta \neq \Delta$ then so is ψ. In this case we may pick elements $c, d \in D$ such that $(c, d) \in \psi$ but $c \neq d$. It follows that $(c \wedge b, d \wedge b) \in \Theta \cap D' \times D'$, so we need only verify that $c \wedge b \neq d \wedge b$. Assume $c \wedge b = d \wedge b$. Since we also have $c \vee b = d \vee b$ by definition of ψ, we obtain the contradiction $c = (c \wedge b) \vee c = (d \wedge b) \vee c = (d \vee c) \wedge (b \vee x) = (d \vee c) \wedge (d \vee b) = d \vee (c \wedge b) = d \vee (d \wedge b) = d$. Hence, $\downarrow b$ is an essential extension of D' and $\downarrow b$ is essentially metrizable.

Using a dual argument, we find that $\uparrow a$ is essentially metrizable, too. The essential metrizability of $[a, b]$ is now an easy consequence of the first two cases. □

PROPOSITION 4. *Every compact, essentially metrizable, Hausdorff topological lattice is metrizable.*

Proof. Suppose that the compact topological lattice L is an essential extension of the countable lattice D. Then since every compact topological lattice can be imbedded in a product of compact metric lattices [Using the techniques of Chapter A, Section 8 of [5], it is easily seen that this sort of approximation by metric objects will hold much more generally than for our particular situation.*], it follows that there is a point-separating family $\{p_\gamma : L \to L_\gamma \mid \gamma \in \Gamma\}$ of continuous lattice homomorphisms from L onto metric topological lattices. A countable subfamily $\{p_\gamma : L \to L_\gamma \mid \gamma \in \Gamma'\}$ from this collection will suffice to separate points in D. Then the canonical map $p : L \to \pi\{L_\gamma : \gamma \in \Gamma'\}$ associated with this subfamily must be an isomorphism when it is restricted to D. But L is an essential extension of D which implies that p must be an algebraic and topological imbedding. The product space $\pi\{L_\gamma : \gamma \in \Gamma'\}$ is metrizable. Therefore L is metrizable. □

DEFINITION 5. A sublattice D of a lattice L is said to be doubly order-generating in L if for each $x \in L$, $\inf(\uparrow x \cap D) = x = \sup(\downarrow x \cap D)$.

In the terminology of Banaschewski and Bruns, a doubly order-generating set is said to be meet and join dense. This next result is essentially due to Banaschewski and Bruns. Even though we state it in terms of distributive lattices, the proof really is not dependent upon any lattice identity.

PROPOSITION 6. *If D is a doubly order-generating sublattice of the distributive lattice L, then L is an essential extension of D.*

Proof. Suppose that $\varphi : L \to M$ is a lattice homomorphism, but not an isomorphism from L onto the lattice M. Then there are elements $a, b \in L$ such

* See the Appendix for details.

that $\varphi(a)=\varphi(b)$—we may assume that $a<b$. Because D is doubly order-generating in L, there are elements $c, d \in D$ such that $c \in \uparrow a \setminus \uparrow b$ and $d \in \downarrow b \setminus \downarrow c$. Then $d \not\geq c$, $b \wedge d = d$, $a \vee c = c$, $\varphi(c)=\varphi(b \vee c)$ and $\varphi(d)=\varphi(a \wedge d)$ which permit us to make the following calculations:

$$\begin{aligned}\varphi(c \wedge d) &= \varphi((b \vee c) \wedge (a \wedge d)) = \varphi(a \wedge d)\\ &= \varphi(d).\end{aligned}$$

Since $d \not\geq c$, the elements d and $c \wedge d$ are distinct and φ identifies these two elements of D. Therefore L is an essential extension of D. □

It is easy to find lattices which are essential extensions of nondoubly order-generating sublattices. For example, let $A=\{(x, y) \in I \times I \mid x=1$ or $y=0\}$. Then since A is a maximal chain in $I \times I$, it is a complete subset of $I \times I$. Hence it cannot order-generate $I \times I$. Nevertheless $I \times I$ is an essential extension of A.

We now have assembled all of the ingredients for our purely order-theoretic characterizations of metrizability for completely distributive lattices.

THEOREM 7. *Let L be a completely distributive lattice. Then the following conditions are equivalent:*

(i) *L is metrizable.*

(ii) *L is essentially metrizable*

(iii) *L has a countable doubly order-generating sublattice.*

(iv) *L is an essential extension of a countable chain.*

Proof. After appealing to Propositions 4 and 6 and calling attention to Proposition 5 of [1] (which states that "Every at most countable distributive lattice L is an essential extension of a chain"), we need only show that (i) implies (iii). The interval topology on L will have a countable neighborhood base of compact sublattices, say $\{U_n : n=1, 2, \ldots\}$. For each n let $z_n = \inf U_n$ and $u_n = \sup U_n$. Define M to be the sublattice generated by all of the z_n's and u_n's. Without difficulty one sees that M is a countable doubly order-generating sublattice of L. (See also [4, III-4.2].)

Although it might seem to be so, our story is not yet complete. Essential metrizability has an impact upon the theory of complete distributive lattices beyond the metrizability situation. It will be seen that it leads to conditions which decide whether a distributive lattice is completely distributive or, when intrinsic topologies are introduced, in deciding when these topologies have bases of sublattices.

To get an idea of the situation we are dealing with, we will consider an already well-discussed example. Let us assume that we have a complete, meet-and-join continuous distributive lattice L. (A distributive lattice L is meet continuous iff $a \wedge \sup A = \sup(A \wedge a)$ for every subset $A \subseteq L$. Join continuity is defined dually.) If we assume further that L is essentially metrizable, must it then be completely distributive? The answer is negative as we see from:

EXAMPLE. Let $L = \mathcal{O}_{\text{reg}}([0, 1])$ be the complete Boolean algebra of regular open subsets of the unit interval. Then L is doubly order-generated by the open intervals with rational end-points and their complements in L. Thus, L is essentially metrizable by Proposition 5. But L is not completely distributive. One reason for this may be found in the paper by E. E. Floyd [3] in the course of which he shows that: Let τ be any topology on $\mathcal{O}_{\text{reg}}([0, 1])$ in which lattice ideals and lattice filters converge to their respective suprema and infima. Then every neighborhood of 1 contains a filter with infimum 0; i.e., from a topological point of view, the lattice of regular open sets of $[0, 1]$ is pathological. A similar idea is contained in the dissertation of D. J. Clinkenbeard (see [2]).

At this point we will try to describe the difficulties that arise in the examples of Floyd and Clinkenbeard in a definition. First, we must recall the definition of a Scott open set.

DEFINITION 8. Let L be a complete lattice. A subset $U \subseteq L$ is called Scott open provided that

(i) $U = \uparrow U$.

(ii) U intersects every ideal $I \subseteq L$ such that $\sup I \in U$.

DEFINITION 9. A complete lattice L is called a topologically incompatible lattice, i-lattice for short, if every Scott open neighborhood of 1 contains a filter having 0 as its infimum. An interval of a lattice is called an i-interval if it is an i-lattice.

PROPOSITION 10. *Let L be a compact topological meet-semilattice. Then L contains no i-intervals. Moreover if L has a largest element, then L is not an i-lattice.*

Proof. It is enough to verify the second statement. Since L is compact, the largest element 1 has an order convex open neighborhood U such that $0 \notin \bar{U}$. Now notice that an order convex neighborhood of 1 must be an upper end. Hence the set U will be Scott open, since in a compact semilattice, ideals and filters converge to their suprema and infima respectively. Finally, for every filter $F \subseteq U$ we have $\inf F \in \bar{U}$, i.e., $\inf F \neq 0$. □

At this point it is necessary to bring in a few topological concepts. In a lattice L we say that an element y is way below an element x, written $y \ll x$, if whenever J is an ideal of L such that $\sup J$ belongs to $\uparrow x$, then $J \cap \uparrow y \neq \varnothing$. The concept way above is defined and denoted in a dual fashion. If the compact topological lattice L has a neighborhood base of lattices, then for each $x \in L$

we must have $\inf\{y \in L \mid y \gg x\} = x = \sup\{y \in L \mid y \ll x\}$. For further discussion on this topic, we suggest that the reader consult [4, VI-3.4, p. 282 ff.].

LEMMA 11. *Let L be a complete distributive lattice without i-intervals. If L is meet continuous, join continuous and essentially metrizable, then* $x = \sup\{y : y \ll x\}$ *for every* $x \in L$; *i.e., using the terminology of* [4] *L is a continuous lattice.*

Proof. Let $b \in L$ and let $a = \sup\{y \in L : y \ll b\}$. Then $a \le b$. We have to show that $a = b$. Assume that $a < b$. First of all, using Proposition 3 and the meet continuity of L, we may take b to be 1. Now let x be any element of L such that $x \ll_a 1$ where $\ll_a$ denotes the way below relation on the lattice $\uparrow a$. We define $\mathbf{x} \in L$ by $\mathbf{x} = \inf\{c : a \vee c \ge x\}$. The distributivity and the join continuity of L imply $\mathbf{x} \vee a = x$. Now let J be an ideal of L such that $\sup J = 1$. Then $J \vee a = \{c \vee a : c \in J\}$ is an ideal of $\uparrow a$ having supremum 1, whence $x \in J \vee a$. Pick an element $c \in J$ such that $c \ge x \vee a$. Then $c \ge \mathbf{x}$ and we conclude that $\mathbf{x} \in J$ showing that $\mathbf{x} \ll 1$. Now the definition of a implies $\mathbf{x} \le a$; i.e., $x = \mathbf{x} \vee a = a$.

Appealing to Proposition 3 again, we therefore may assume that $a = 0$ and that $x \ll 1$ if and only if $x = 0$. We now show that the essential metrizability of L implies that L is an i-lattice. The arguments we shall use here may be viewed as a Baire category theorem for the Scott topology on L:

Let $D \subseteq L$ be a countable sublattice such that L is an essential extension of D and define

$$Q = \{\inf\{x : x \vee d \ge c\} : c, d \in D, d \in \downarrow c \setminus \{c\}\}.$$

Then Q is countable. Moreover, for every $x \in L \setminus \{0\}$ there is a $q \in Q$ such that $q \le x$. Indeed, if $x \ne 0$ then $\Theta = \{(u, v) : x \vee u = x \vee v\}$ is nontrivial. Hence, since L is an essential extension of D, we can find elements $c, d \in D$ such that $d \in \downarrow c \setminus \{c\}$ and $x \vee d = x \vee c$ and this implies that $c = c \wedge (x \vee c) = c \wedge (x \vee d) \le x \vee d$. For these elements $c, d \in D$ let $q = \inf\{x : x \vee d \ge c\}$. Then $q \le x$. Since the join continuity of L implies $0 \notin Q$, we may conclude that $L \setminus \{0\} = \bigcup \{\uparrow q : q \in Q\}$.

We now arrange the elements of Q in a sequence $q_1, q_2, q_3, \ldots$ and note that none of the q_n is way below 1, since no element of $L \setminus \{0\}$ is. Let U be a fixed Scott open neighborhood of 1. By induction, we will pick elements $u_n \in U$ such that

$$(*) \qquad q_n \not\le u_1 \wedge \cdots \wedge u_n \in U.$$

Indeed, let $n = 1$. Then $q_1 \not\ll 1$ and hence we can find an ideal I such that $\sup I = 1$, but $q_1 \notin I$. Since U is Scott open, we can pick $u_1 \in U \cap I$.

Now assume that we already picked $u_1, \ldots, u_n \in U$ such that (*) holds. Since q_{n+1} is not way below 1, it cannot be way below $u_1 \wedge \cdots \wedge u_n \in U$ either. Hence there is an ideal I of L such that $\sup I \ge u_1 \wedge \cdots \wedge u_n$, but $q_{n+1} \notin I$. The meet

https://doi.org/10.4153/CMB-1983-073-9 Published online by Cambridge University Press

continuity of L implies that we may assume $u_1 \wedge \cdots \wedge u_n = \sup I$. Now pick $u_{n+1} \in I \cap U$. Then $q_{n+1} \not\leq u_{n+1} = u_1 \wedge \cdots \wedge u_n \wedge u_{n+1} \in U$. The elements $u_1, u_2, \ldots$ now generate a filter F which is contained in U and we have $\inf F = \inf_{n \in \mathbb{N}} u_n$. Assume that $0 \neq \inf_{n \in \mathbb{N}} u_n$. Then $q_m \leq \inf_{n \in \mathbb{N}} u_n$ for a certain $m \in \mathbb{N}$ since $L \setminus \{0\} = \uparrow q_1 \supseteq \uparrow q_2 \cup \cdots$, contradicting the fact that $q_m \not\leq u_1 \wedge \cdots \wedge u_m$. Hence $\inf F = 0$ and therefore L is an i-lattice. We have arrived at a contradiction, which concludes the proof of Lemma 11. □

On the surface, Lemma 11 does not seem to be self-dual since the assumption of having no i-intervals as well as the definition of i-lattices are not self-dual. However, a joint continuous distributive continuous lattice is a compact topological lattice by Corollary VII.2.4 of [4]. Therefore, using Proposition 10 we may apply Lemma 11 to the dual lattice L^{op} and find that L^{op} is also a continuous lattice. Now Theorem I.3.15 of [4] yields.

THEOREM 12. *Let L be a complete, distributive, essentially metrizable, meet-continuous, join-continuous lattice having no i-intervals, then L is completely distributive.* □

This theorem has a much neater formulation for compact topological lattices.

COROLLARY 13. *Let L be a compact, metric, topological lattice. Then the following statements are equivalent:*

(i) *L is essentially metrizable.*

(ii) *The topology for L has a neighborhood base of lattices.*

(iii) *L is completely distributive.* □

Appendix. We now will fill the hole which we left in the proof of Proposition 4.

THEOREM 14. *Every compact topological lattice L can be imbedded in a product of compact metric lattices.*

Proof. We will use the fact that the toplogy on every compact topological lattice is generated by a uniformity and that the lattice operations $\wedge$ and $\vee$ are uniformly continuous for every such uniformity. Let U be any neighborhood of the diagonal $\Delta \subseteq L \times L$. It is enough to construct a closed lattice congruence Θ on L such that

(i) $\Theta \subseteq U$;

(ii) Θ is a G_δ-subset of $L \times L$, i.e., Θ is the intersection of countably many open sets.

Note that in this case L will admit arbitrary small closed congruences Θ (by (i)) such that L/Θ is metric (by (ii)). By induction, we will pick a sequence $(U_n)_{n \in \mathbb{N}}$

https://doi.org/10.4153/CMB-1983-073-9 Published online by Cambridge University Press

of entourages of the uniform structure of L such that

(i) $\bar{U}_{n+1} \subseteq U_n \subseteq U$ for all $n \in \mathbb{N}$;

(ii) $U_n^{-1} = U_n$;

(iii) $U_{n+1} \circ U_{n+1} \subseteq U_n$;

(iv) $U_{n+1} \wedge U_{n+1} \subseteq U_n$ and $U_{n+1} \vee U_{n+1} \subseteq U_n$, where $U_{n+1} \wedge U_{n+1}$ and $U_{n+1} \vee U_{n+1}$ denote complex products.

Let $U_0 \subseteq U$ be any symmetric neighborhood of the diagonal and assume that we have picked $U_1, \ldots, U_n$ such that (i)–(iv) are satisfied. We have to construct U_{n+1}: Firstly, by the definition of uniform structures there is an entourage V such that $\bar{V} \subseteq U_n$, $V^{-1} = V$ and $V \circ V \subseteq U_n$. Since the lattice operations are uniformly continuous, we can find a symmetric entourage $U_{n+1} \subseteq V$ such that $U_{n+1} \wedge U_{n+1} \subseteq U_n$ and $U_{n+1} \vee U_{n+1} \subseteq U_n$. Obviously, $U_0, U_1, \ldots, U_{n+1}$ satisfy (i)–(iv).

We now use this sequence $(U_n)_{n \in \mathbb{N}}$ in order to define our lattice congruence Θ by

$$\Theta = \bigcap_{n \in \mathbb{N}} U_n.$$

Then Θ is a G_δ-set by definition. Moreover, (i) implies that Θ is closed and (ii) and (iii) imply that Θ is an equivalence relation. Finally, from (iv) above we conclude that Θ is a sublattice of $L \times L$. Hence, Θ behaves as required and the proof is complete. □

COROLLARY 15. *If L is a simple compact lattice, then L is metric.*

With almost no modification of the proof of Theorem 13 we see that this theorem and its corollary holds for compact, topological universal algebras, provided that we only have countably many operation symbols all of which are finitary.

REFERENCES

1. B. Banaschewski and G. Bruns, *Injective hulls in the category of distributive lattices*, J. reine u. angewandte Math. **232** (1968), 102–109.
2. D. J. Clinkenbeard, *The lattice of closed congruences on a topological lattice*, Trans. Amer. Math. Soc. **263** (1981), 457–467.
3. E. E. Floyd, *Boolean algebras with pathological order topologies*, Pacific J. Math **5** (1955), 687–689.
4. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Heidelberg, 1980.
5. K. H. Hofmann and P. S. Mostert, *Elements of Compact Semi-groups*, Merrill Publ. Co., Columbus, Ohio, 1966.
6. D. Papert-Strauss, *Topological lattices*, Proc. London Math. Soc. **18** (1968), 217–230.

UNIVERSITY OF CALIFORNIA AT RIVERSIDE, CALIFORNIA 92521

LOUISANA STATE UNIVERSITY, BATON ROUGE, LA, 70808