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A NOTE ON EPI-CONVERGENCE

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ABSTRACT. Let LSC(X) denote the set of extended real valued lower semicontinuous functions on a metrizable space X. If $f, f_1, f_2, f_3, \ldots$ is a sequence in LSC(X), we say $\langle f_n \rangle$ is *epi-convergent* to f provided the sequence of epigraphs $\langle \mathrm{epi} f_n \rangle$ is Kuratowski-Painlevé convergent to epi$f$. In this note we address the following question: what conditions on f and/or on X are necessary and sufficient for this mode of convergence to force epigraphical convergence with respect to the stronger Hausdorff metric and Vietoris topologies?

1. Introduction.

Let 2^X be the closed subsets of a metric space $\langle X, d \rangle$, and let CL(X) be the nonempty closed subsets. Classical convergence for sequences in 2^X attributed to Painlevé by Hausdorff [Ha], is now often called *Kuratowski-Painlevé convergence*. Given a sequence $A_1, A_2, A_3, A_4, \ldots$ of (possibly empty) closed subsets of $\langle X, d \rangle$, we write

$$\begin{aligned} \mathrm{Li}\, A_n &= \{x \in X : \text{there exists a sequence } \langle a_n \rangle \text{ convergent to } x \text{ with} \\ &\qquad a_n \in A_n \text{ for all but finitely many integers } n\}, \\ \mathrm{Ls}\, A_n &= \{x \in X : \text{there exist positive integers } n_1 < n_2 < n_3 < \cdots \\ &\qquad \text{and } a_k \in A_{n_k} \text{ such that } \langle a_k \rangle \longrightarrow x\}. \end{aligned}$$

Clearly, the sets $\mathrm{Li}\, A_n$ and $\mathrm{Ls}\, A_n$ are closed, and $\mathrm{Li}\, A_n \subset \mathrm{Ls}\, A_n$. The sequence $\langle A_n \rangle$ is declared *Kuratowski-Painlevé convergent* [Ku, AF] to a (closed) subset A of X if $A = \mathrm{Li}\, A_n = \mathrm{Ls}\, A_n$, or equivalently, if both inclusions $\mathrm{Ls}\, A_n \subset A$ and $A \subset \mathrm{Li}\, A_n$ hold. When this is satisfied we write $A = K - \lim A_n$.

Kuratowski-Painlevé convergence plays a fundamental role in modern one-sided analysis, where the basic functional objects are extended real valued lower semicontinuous functions rather than continuous ones, and functions are associated with their epigraphs rather than their graphs [At, AF, DG, RW, DM]. Recall the *epigraph* of an extended real valued function $f: X \longrightarrow [-\infty, +\infty]$ on a metrizable space X is the set

$$\mathrm{epi} f \equiv \{(x, \alpha) : x \in X, \alpha \in R, \text{ and } \alpha \geq f(x)\}.$$

In this context, a sequence $\langle f_n \rangle$ of lower semicontinuous functions is called *epi-convergent* to a lower semicontinuous function f provided $\mathrm{epi}\, f = K - \lim \mathrm{epi}\, f_n$.

It is well-known that for sequences of nonempty closed sets, $A = K - \lim A_n$ provided $\langle A_n \rangle$ converges to A in *Hausdorff distance* [CV, KT], defined on CL(X) by the formula

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$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$. Furthermore, the converse holds if and only if X is compact [Be2]. If we equip $X \times R$ with a metric compatible with the product uniformity, one might guess that when X is compact, then Kuratowski-Painlevé convergence of sequences of epigraphs forces their convergence in Hausdorff distance. In fact, it was observed in [Be1] that for a sequence $f, f_1, f_2, \ldots$ of bounded real valued lower semicontinuous functions defined on a compact metric space X, Kuratowski-Painlevé convergence of epigraphs implies Hausdorff metric convergence. However, this fails in LSC(X). We characterize here those limit functions f for which this implication is true.

When X is compact, the Hausdorff metric topology τ_{H_d} on CL(X) coincides with the *Vietoris topology* τ_V, also called the *finite topology*, having as a subbase all sets of form

$$V^{\text{hit}} \equiv \{A \in 2^X : A \cap V \neq \emptyset\}, \quad F^{\text{miss}} \equiv \{A \in 2^X : A \cap F = \emptyset\}$$

where V runs over the open subsets of X and F runs over the closed subsets of X [Mi, KT]. Like the Hausdorff metric topology, we have $A = \tau_V - \lim A_n \Rightarrow A = K - \lim A_n$ [FLL] and the converse holds if and only if X is compact. The class of lower semicontinuous functions f for which $\operatorname{epi} f = K - \lim f_n \Rightarrow \operatorname{epi} f = \tau_V - \lim \operatorname{epi} f_n$ differs from the class for which $\operatorname{epi} f = K - \lim f_n \Rightarrow \operatorname{epi} f = H_d - \lim \operatorname{epi} f_n$. We also characterize this class.

2. **Preliminaries.** Let $\langle X, d\rangle$ be a metric space. If $x \in X$ and $\alpha > 0$, let $U_\alpha[x]$ denote the open ball with center x and radius α, and if $A \subset X$, write $U_\alpha[A]$ for the open enlargement $\bigcup_{a\in A} U_\alpha[a]$. It is clear that the Hausdorff distance between A and B in CL(X) can be rewritten as

$$H_d(A,B) = \inf\{\alpha > 0 : U_\alpha[A] \supset B \text{ and } U_\alpha[B] \supset A\}.$$

Hausdorff distance so defined is an infinite valued metric on CL(X), that inherits completeness and compactness of the underlying metric space [CV, KT]. The induced Hausdorff metric topology is not changed provided we replace d by a metric that defines the same uniformity. Thus if replace d by $d' = \min\{d, 1\}$ we get a finite valued metric compatible with τ_{H_d}. For a metric on $X \times R$, we find it simplest to use *box metric* ρ defined by $\rho[(x_1,\alpha_1),(x_2,\alpha_2)] = \max\{d(x_1,x_2), |\alpha_1 - \alpha_2|\}$. As we have said, $\tau_{H_d} = \tau_V$ on CL(X) if and only if X is compact; more precisely, $\tau_{H_d} \supset \tau_V$ if and only if the gap $\inf\{d(a,b) : a \in A, b \in B\}$ between disjoint elements of A and B of CL(X) is positive, whereas $\tau_{H_d} \subset \tau_V$ if and only if $\langle X, d\rangle$ is totally bounded [Mi].

It is known (see, *e.g.*, [FLL, Be2, DM]) that in any metric space—in fact, in any first countable space—Kuratowski-Painlevé convergence is compatible with a topology of the Vietoris type called the *Fell topology* τ_F [Fe], having as a subbase all sets of the form

$$V^{\text{hit}} \equiv \{A \in 2^X : A \cap V \neq \emptyset\}, \quad K^{\text{miss}} \equiv \{A \in 2^X : A \cap K = \emptyset\}$$

where V runs over the open subsets of X and K runs over the compact subsets of X. This means that in 2^X, $A = K - \lim A_n$ if and only if $A = \tau_F - \lim A_n$. The Fell topology has a remarkable property: it is always compact, independent of the character of the underlying space (for three different proofs, see [At, Fe, No]). On the other hand, assuming the continuum hypothesis, the topology is sequentially compact if and only if $\langle X, d\rangle$ is separable [Si]. The following are equivalent [Po]: (1) X is locally compact; (2) $\langle 2^X, \tau_F\rangle$ is Hausdorff. In this case, $\langle 2^X, \tau_F\rangle$ is compact Hausdorff and $\langle \mathrm{CL}(X), \tau_F\rangle$ is locally compact Hausdorff.

By a *lower semicontinuous function* $f\colon \langle X, d\rangle \longrightarrow [-\infty, +\infty]$, we mean a function with closed epigraph. Equivalently, f is a lower semicontinuous function if and only if for each $\alpha \in R$, its *sublevel set at height* α $\mathrm{slv}(f;\alpha) \equiv \{x \in X : f(x) \leq \alpha\}$ is a closed subset of X. We denote the set of lower semicontinuous functions on X by $\mathrm{LSC}(X)$. If $f \in \mathrm{LSC}(X)$, we write $\mathrm{dom} f$ for $\{x \in X : f(x)$ is finite$\}$. We call f *proper* provided $f(x) > -\infty$ for each x, and $\mathrm{dom} f \neq \emptyset$. $\mathrm{LSC}_0(X)$ will denote the set of proper lower semicontinuous functions on X.

Although we will not use the following formulation, epi-convergence in $\mathrm{LSC}(X)$ can be given a local characterization [At, Theorem 1.39]: at each $x \in X$, (1) whenever $\langle x_n\rangle$ is convergent to x, we have $f(x) \leq \liminf_{n\to\infty} f_n(x_n)$, and (2) there exists a sequence $\langle x_n\rangle$ convergent to x such that $f(x) = \lim_{n\to\infty} f_n(x_n)$. Epi-convergence neither implies nor is implied by pointwise convergence; the two modes of convergence are linked by the notion of equi-lower semicontinuity [SW, DSW, Ma].

Identifying elements of $\mathrm{LSC}(X)$ with their epigraphs in $X \times R$, the Fell topology on the lower semicontinuous functions is usually called *the topology of epi-convergence*, but it is also the *inf-vague topology* by the probabilists (see, *e.g.*, [Ve, No]). As $\mathrm{LSC}(X)$ is closed in $\langle 2^{X\times R}, \tau_F\rangle$, the function space $\langle \mathrm{LSC}(X), \tau_F\rangle$ is always compact, too. Compatibility of Kuratowski-Painlevé convergence in $\mathrm{LSC}(X)$ with the Fell topology means that whenever $f, f_1, f_2, f_3, \ldots$ is a sequence in $\mathrm{LSC}(X)$, then $\mathrm{epi} f = K - \lim f_n$ if and only if (i) whenever V is open in $X \times R$ and $\mathrm{epi} f \cap V \neq \emptyset$, then eventually, $\mathrm{epi} f_n \cap V \neq \emptyset$, and (ii) whenever K is compact in $X \times R$ and $\mathrm{epi} f \cap K = \emptyset$, then eventually, $\mathrm{epi} f_n \cap K = \emptyset$.

3. **Epi-convergence versus Hausdorff metric convergence of epigraphs.** As we have defined Hausdorff distance only between nonempty closed subsets, we only investigate the relationship between epi-convergence and Hausdorff metric convergence of epigraphs when the limit function $f \in \mathrm{LSC}(X)$ has nonempty epigraph. Again, we are interested in the question: if $\langle X, d\rangle$ is a compact metric space and ρ is the box metric on $X \times R$, under what conditions on f does $\mathrm{epi} f = K{-}\lim \mathrm{epi} f_n$ imply $\lim_{n\to\infty} H_\rho(\mathrm{epi} f_n, \mathrm{epi} f) = 0$?

Actually, there is no need to assume at the outset that X is compact, for no such function f with $\mathrm{epi} f \neq \emptyset$ can exist more generally. To see this, first observe that f must be bounded below, for otherwise $\mathrm{epi} f = K - \lim \mathrm{epi}(f \vee -n)$, but for each n, $H_\rho\big(\mathrm{epi}(f \vee -n), \mathrm{epi} f\big) = +\infty$. For future reference, notice that for each n, $f \vee -n \in \mathrm{LSC}_0(X)$. Now if X is noncompact, choose $\langle x_n\rangle$ in X with no cluster point. Then if $f \in \mathrm{LSC}(X)$, $\mathrm{epi} f \neq \emptyset$, and $\inf_{x\in X} f(x) = \alpha$ is finite, for each n, define $f_n \in \mathrm{LSC}_0(X)$ by

$$f_n(x) = \begin{cases} \alpha - 1 & \text{if } x = x_n \\ f(x) & \text{otherwise} \end{cases}.$$

Clearly, $\mathrm{epi} f = K - \lim \mathrm{epi} f_n$ but for each n, $H_\rho(\mathrm{epi} f_n, \mathrm{epi} f) \geq 1$.

We now come to our characterization theorem.

THEOREM 1. *Let $\langle X, d\rangle$ be a metric space, and let ρ be the box metric on $X \times R$. Suppose f is a lower semicontinuous function on X with* $\operatorname{epi} f \neq \emptyset$. *The following are equivalent:*

(1) X is compact, f is proper, and $\operatorname{dom} f \equiv \{x \in X : f(x) \in R\}$ *is dense in X;*

(2) whenever $\langle f_n\rangle$ is a sequence in $\mathrm{LSC}(X)$ *with* $\operatorname{epi} f = K - \lim \operatorname{epi} f_n$, *then* $\lim_{n\to\infty} H_\rho(\operatorname{epi} f_n, \operatorname{epi} f) = 0$;

(3) whenever $\langle f_n\rangle$ is a sequence in $\mathrm{LSC}_0(X)$ *with* $\operatorname{epi} f = K - \lim \operatorname{epi} f_n$, *then* $\lim_{n\to\infty} H_\rho(\operatorname{epi} f_n, \operatorname{epi} f) = 0$.

PROOF. (1) $\Rightarrow$ (2). Let $\varepsilon > 0$ be arbitrary. Since $X = \operatorname{cl}\big(\bigcup_{k=1}^{\infty} \operatorname{slv}(f;k)\big)$, $\langle \operatorname{slv}(f;k)\rangle$ is Kuratowski-Painlevé convergent to X. Since X is compact, convergence in the Hausdorff metric holds, and we can find $k \in Z^+$ with $X \subset U_{\varepsilon/3}[\operatorname{slv}(f;k)]$.

By the compactness of X, f assumes a minimum value on X which we denote by α. Now let F be a finite $\varepsilon/3$-dense subset of the compact set $\operatorname{epi} f \cap (X \times [\alpha, k])$. By epi-convergence, there exists an index N such that for each $n \geq N$, we have $F \subset U_{\varepsilon/3}[\operatorname{epi} f_n]$. Since $\operatorname{epi} f$ recedes in the vertical direction, we obtain $\operatorname{epi} f \subset U_\varepsilon[\operatorname{epi} f_n]$ for each $n \geq N$.

To show that $\operatorname{epi} f_n \subset U_\varepsilon[\operatorname{epi} f]$ eventually, let K be this nonempty compact subset of $X \times R$:

$$K \equiv (X \times [\alpha - \varepsilon, k]) \cap (U_\varepsilon[\operatorname{epi} f])^c.$$

By the convergence of $\langle \operatorname{epi} f_n\rangle$ to $\operatorname{epi} f$ in the Fell topology, there exist $N_1 \in Z^+$ such that for each $n \geq N_1$, we have $\operatorname{epi} f_n \cap K = \emptyset$. Since the horizontal set $X \times \{\alpha - \varepsilon\}$ lies in K and $\operatorname{epi} f_n$ recedes in the vertical direction, we have

$$\operatorname{epi} f_n \subset \big(X \times (k, +\infty)\big) \cup U_\varepsilon[\operatorname{epi} f] \subset U_{\varepsilon/3}[\operatorname{epi} f] \cup U_\varepsilon[\operatorname{epi} f] = U_\varepsilon[\operatorname{epi} f].$$

Thus, for all sufficiently large indices n, both of the inclusions $\operatorname{epi} f \subset U_\varepsilon[\operatorname{epi} f_n]$ and $\operatorname{epi} f_n \subset U_\varepsilon[\operatorname{epi} f]$ are satisfied, as required.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). We have already observed that if (3) holds, then X must be compact and f must be lower bounded. Since $\operatorname{epi} f \neq \emptyset$, f is proper. Now suppose that $\operatorname{cl} \operatorname{dom} f$ is a proper subset of X. Choose $x_0 \in X$ with $d(x_0, \operatorname{dom} f) > 0$. For each $n \in Z^+$ define $f_n \in \mathrm{LSC}_0(X)$ by

$$f_n(x) = \begin{cases} n & \text{if } x = x_0 \\ f(x) & \text{otherwise} \end{cases}.$$

Although $\operatorname{epi} f = K - \lim \operatorname{epi} f_n$, for each n, we have $H_\rho(\operatorname{epi} f_n, \operatorname{epi} f) \geq d(x_0, \operatorname{dom} f) > 0$, which contradicts (3). ■

4. **Epi-convergence versus Vietoris convergence of epigraphs.** Although the Vietoris topology and the Hausdorff metric topologies agree on the nonempty closed subsets of a compact metric space $\langle X, d\rangle$, this is clearly not the case in $\mathrm{CL}(X \times R)$, even for epigraphs of lower semicontinuous functions. For example, for any metric space $\langle X, d\rangle$, we have $X \times R = \tau_V - \lim X \times [-n, +\infty)$. More generally, if $f \equiv -\infty$ and $\operatorname{epi} f = K - \lim \operatorname{epi} f_n$, then $\operatorname{epi} f = \tau_V - \lim \operatorname{epi} f_n$, so that for noncompact X, we can always find

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a function $f \in \mathrm{LSC}(X)$ satisfying $\mathrm{epi} f = K - \lim \mathrm{epi} f_n \Rightarrow \mathrm{epi} f = \tau_V - \lim \mathrm{epi} f_n$. As it turns out, we can find no such $f \in \mathrm{LSC}_0(X)$ unless X is compact, and in this case, f must be real valued. The precise situation is described in the next result.

THEOREM 2. *Let $\langle X, d\rangle$ be a metric space, and suppose $f \in \mathrm{LSC}(X)$. The following are equivalent:*

(1) $\mathrm{dom} f$ *is compact and* $\sup_{x\in X} f(x) < +\infty$*;*

(2) whenever $\langle f_n\rangle$ is a sequence in $\mathrm{LSC}(X)$ *with* $\mathrm{epi} f = K - \lim \mathrm{epi} f_n$, *then* $\mathrm{epi} f = \tau_V - \lim \mathrm{epi} f_n$*;*

(3) whenever $\langle f_n\rangle$ is a sequence in $\mathrm{LSC}_0(X)$ *with* $\mathrm{epi} f = K - \lim \mathrm{epi} f_n$, *then* $\mathrm{epi} f = \tau_V - \lim \mathrm{epi} f_n$.

PROOF. (1) $\Rightarrow$ (2). Suppose $f \in \mathrm{LSC}(X)$ satisfies condition (1), $\langle f_n\rangle$ is a sequence in $\mathrm{LSC}(X)$, and $\mathrm{epi} f = K - \lim \mathrm{epi} f_n$, *i.e.*, $\mathrm{epi} f = \tau_F - \lim \mathrm{epi} f_n$. Since the "lower halves" [FLL] of the Fell and Vietoris topologies agree, to show that $\mathrm{epi} f = \tau_V - \lim \mathrm{epi} f_n$, it suffices to show that if $A \in \mathrm{CL}(X)$ and $\mathrm{epi} f \cap A = \emptyset$, then $\mathrm{epi} f_n \cap A = \emptyset$ eventually. Choose $\beta \in R$ with $\sup_{x\in X} f(x) \leq \beta$. Since $\mathrm{dom} f \times [\beta, +\infty) \subset \mathrm{epi} f$ and $(\mathrm{dom} f)^c \times R \subset \mathrm{epi} f$, we have $A \subset \mathrm{dom} f \times (-\infty, \beta)$. Write $\alpha = \min_{x\in \mathrm{dom} f} f(x)$, which exists by compactness, and let K be the following compact subset of $X \times R$:

$$K \equiv (\mathrm{dom} f \times \{\alpha - 1\}) \cup \big(A \cap (\mathrm{dom} f \times [\alpha - 1, \beta])\big).$$

By the choice of α, we have $\mathrm{epi} f \cap K = \emptyset$, and so there exists $N \in Z^+$ such that for each $n \geq N$, we have $\mathrm{epi} f_n \cap K = \emptyset$. We claim that for each such n, we have $\mathrm{epi} f_n \cap A = \emptyset$. We compute

$$\begin{aligned}
&\mathrm{epi} f_n \cap A \\
&= \mathrm{epi} f_n \cap A \cap \big(\mathrm{dom} f \times (-\infty, \beta)\big) \\
&\qquad \subset \big(\mathrm{epi} f_n \cap A \cap (\mathrm{dom} f \times [\alpha - 1, \beta])\big) \cup \Big(\mathrm{epi} f_n \cap A \cap \big(\mathrm{dom} f \times (-\infty, \alpha - 1]\big)\Big) \\
&= \mathrm{epi} f_n \cap A \cap \big(\mathrm{dom} f \times (-\infty, \alpha - 1]\big) \subset \mathrm{epi} f_n \cap \big(\mathrm{dom} f \times (-\infty, \alpha - 1]\big) = \emptyset,
\end{aligned}$$

because $\mathrm{epi} f_n \cap \big(\mathrm{dom} f \times (-\infty, \alpha - 1]\big) \neq \emptyset$ implies $\mathrm{epi} f_n \cap (\mathrm{dom} f \times \{\alpha - 1\}) \neq \emptyset$, which would contradict $\mathrm{epi} f_n \in K^{\mathrm{miss}}$.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Assuming (3), we first show that $\sup_{x\in X} f(x) < +\infty$. If this fails, we can find for each $n \in Z^+$ a point $x_n \in X$ with $f(x_n) > n$ (note that the x_n need not be distinct). Let $A = \{(x_n, n) : n \in Z^+\}$, a closed subset of $X \times R$ disjoint from $\mathrm{epi} f$. For each $n \in Z^+$ define $f_n \in \mathrm{LSC}_0(X)$ by the formula

$$f_n(x) = \begin{cases} n & \text{if } x = x_n \\ \max\{f(x), -n\} & \text{otherwise} \end{cases}.$$

Although, $\mathrm{epi} f = K - \lim \mathrm{epi} f_n$, each $\mathrm{epi} f_n$ hits the closed set A, and so $\langle \mathrm{epi} f_n\rangle$ fails to converge to $\mathrm{epi} f$ in the Vietoris topology, contradicting (3). This shows that f is bounded

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above. To finish the proof, we must show that dom$f$ is a compact subset of X. If this fails, then there exists a sequence $\langle x_n \rangle$ with distinct terms in dom$f$ that has no cluster point in dom$f$, although it might have a cluster point p for which $f(p) = -\infty$. Then $A = \left\{ \left(x_n, -|f(x_n)| - n \right) : n \in Z^+ \right\}$ is a closed subset of $X \times R$ disjoint from epi$f$. For each $n \in Z^+$ define $f_n \in \mathrm{LSC}_0(X)$ by the formula

$$f_n(x) = \begin{cases} -|f(x_n)| - n & \text{if } x = x_n \\ \max\{f(x), -n\} & \text{otherwise} \end{cases}.$$

Again, epi$f = K - \lim \mathrm{epi} f_n$, but each epi$f_n$ hits the closed set A. ∎

COROLLARY. *Let $\langle X, d \rangle$ be a compact metric space. Then the Fell topology, the Hausdorff metric topology, and Vietoris topology all agree on the family of bounded real valued lower semicontinuous functions defined on X, where functions are identified with their epigraphs.*

REFERENCES

[At] H. Attouch, *Variational convergence for functions and operators*, Pitman, New York, 1984.

[AF] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990.

[Be1] G. Beer, *Upper semicontinuous functions and the Stone approximation theorem*, J. Approx. Theory **34**(1982), 1–11.

[Be2] ———, *Metric spaces with nice closed balls and distance functions for closed sets*, Bull. Austral. Math. Soc. **35**(1987), 81–96.

[CV] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Math. **580**, Springer-Verlag, Berlin, 1977.

[DM] G. Dal Maso, *An introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[DG] E. De Giorgi, *Convergence problems for functionals and operators*. In: Recent methods in nonlinear analysis, (eds. E. De Giorgi, E. Magenes and U. Mosco), Pitagora Editrice, Bologna, 1979, 131–188.

[DSW] S. Dolecki, G. Salinetti and R. Wets, *Convergence of functions: equi-semicontinuity*, Trans. Amer. Math. Soc. **276**(1983), 409–429.

[Fe] J. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. **13**(1962), 472–476.

[FLL] S. Francaviglia, A. Lechichki and S. Levi, *Quasi-uniformization of hyperspaces and convergence of nets of semicontinuous multifunctions*, J. Math. Anal. Appl. **112**(1985), 347–370.

[Ha] F. Hausdorff, *Set theory*, Chelsea Publishing Co., New York, New York, 1957.

[KT] E. Klein and A. Thompson, *Theory of correspondences*, Wiley, New York, 1984.

[Ku] K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York, 1966.

[Ma] M.-L. Mazure, *Equi-semicontinuité inférieure, Γ-convergence, et convergence simple*, Séminaire d'Analyse Convexe Montpellier **7**(1981).

[Mi] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71**(1951), 152–182.

[No] T. Norberg, *On Vervaat's sup vague topology*, Ark. Mat. **28**(1990), 139–144.

[Po] H. Poppe, *Eine bemerkung über trennungsaxiome im raum der abgeschlossenen teilmengen eines topologischen raumes*, Arch. Math. **16**(1965), 197–199.

[RW] R. T. Rockafellar and R. Wets, *Variational systems, an introduction*. In: Multifunctions and integrands, (ed. G. Salinetti), Lecture Notes in Math. **1091**, Springer-Verlag, Berlin, 1984, 1–54.

[SW] G. Salinetti and R. Wets, *On the relations between two types of convergence for convex functions*, J. Math. Anal. Appl. **60**(1977), 211–226.

[Si] W. Sierpinski, *Sur l'inversion du théoreme de Bolzano-Weierstrass generalisé*, Fund. Math. **34**(1947), 155–156.

[Ve] W. Vervaat, *Random upper semicontinuous functions and extremal processes*, Report **MS-8801**, Center for Math. and Comp. Sci., Amsterdam, 1988.

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