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ESSENTIALLY COMMUTATIVE C^*-ALGEBRAS WITH ESSENTIAL SPECTRUM HOMEOMORPHIC TO S^{2n-1}

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Abstract

This paper gives a complete classification of essentially commutative C^*-algebras whose essential spectrum is homeomorphic to S^{2n-1} by their characteristic numbers. Let $\mathscr{A}_1$, $\mathscr{A}_2$ be such two C^*-algebras; then they are C^*-isomorphic if and only if they have the same n-th characteristic number. Furthermore, let $\gamma_n(\mathscr{A}) = m$; then $\mathscr{A}$ is C^*-isomorphic to $C^*(M_{z_1}, \dots, M_{z_n})$ if $m = 0$, $\mathscr{A}$ is C^*-isomorphic to $C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m})$ if $m \neq 0$. Some examples are given to show applications of the classification theorem. We finally remark that the proof of the theorem depends on a construction of a complete system of representatives of $\operatorname{Ext}(S^{2n-1})$.

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1. Introduction

Let $\mathscr{A}$ be a C^*-algebra of operators on a separable Hilbert space H. In what follows we assume always that $\mathscr{A}$ contains the identity operator I and the ideal $\mathscr{K}$ of compact operators. We say that $\mathscr{A}$ is *essentially commutative* if $AB - BA$ is compact for all $A, B \in \mathscr{A}$. A natural problem is how to classify essentially commutative C^*-algebras in C^*-isomorphism sense. Then the problem is to find invariants and models. First if two such C^*-algebras are C^*-isomorphic, then the isomorphism is necessarily implemented by a unitary operator [Dou]. Let $\mathscr{A}$ be essentially commutative, and $M_{\mathscr{A}}$ be the maximal ideal space of $\mathscr{A}/\mathscr{K}$ which is called the *essential spectrum* of $\mathscr{A}$. For a compact metrizable space X, let Σ_X denote the class of all essentially commutative

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C^*-algebras $\mathscr{A}$ whose essential spectrum is homeomorphic to X. Now taking $\mathscr{A}$ in Σ_X, one hence has a natural extension of $\mathscr{K}$ by $C(X)$

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{A} \xrightarrow{\phi} C(X) \longrightarrow 0.$$

The classification problem thus is equivalent to the classification of extensions of $\mathscr{K}$ by $C(X)$ in the following sense. Let $(\mathscr{A}_1, \phi_1)$ and $(\mathscr{A}_2, \phi_2)$ be two extensions of $\mathscr{K}$ by $C(X)$. We call them *weakly equivalent* if there exists the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{A}_1 & \xrightarrow{\phi_1} & C(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_2} & & \downarrow{\scriptstyle\theta_3} & & \\ 0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{A}_2 & \xrightarrow{\phi_2} & C(X) & \longrightarrow & 0, \end{array}$$

where θ_1, θ_2 and θ_3 are C^*-isomorphisms. Now let $\mathrm{Ext}_w(X)$ denote the set of the classes of weak equivalence. From Blackadar [Bla], one knows that $\mathrm{Ext}_w(X)$ is a semigroup, but in general, not a group. Hence intuitively, the classification problem for Σ_X is closely related to the BDF-theory [BDF1, BDF2] and homotopy theory. For general compact metrizable space X, it is extremely difficult to classify Σ_X in C^*-isomorphism sense. In [Guo1], we introduce an invariant called the characteristic number to study essentially normal operators. In the present paper, we will develop this invariant, and use it to give $\Sigma_{S^{2n-1}}$ a complete classification, where S^{2n-1} is the boundary of the unit ball B_n in $\mathbb{C}^n$. For convenience, we write Σ_n for $\Sigma_{S^{2n-1}}$. Firstly, we use the mapping degrees on the unit sphere to give a complete system of representatives of $\mathrm{Ext}(S^{2n-1})$, and hence shows that the n-th characteristic number γ_n is a complete invariant for the class Σ_n in C^*-isomorphism sense. Some examples are given to show the applications of the classification theorem. Since the generalized Poincaré conjecture is true in the case $n \neq 3$ (see [Sma1, Sma2]), our example shows that Toeplitz algebra $C^*(\Omega)$ on Poincaré domain $\Omega (\subset \mathbb{C}^n,\ n \neq 2)$ is necessarily C^*-isomorphic to Toeplitz algebra $C^*(B_n)$ on the unit ball in $\mathbb{C}^n$. In the case $n = 2$, $C^*(\Omega)$ is isomorphic to $C^*(B_2)$ if and only if the Poincaré conjecture is true for $\partial\Omega$. This fact is proved by the different method in [Guo2].

2. Some basic lemmas

Let $\mathscr{A}$ be essentially commutative. If a family $\{T_\lambda | \lambda \in \Lambda\}$, $\mathscr{K}$ and the identity operator I generate $\mathscr{A}$, the family $\{T_\lambda | \lambda \in \Lambda\}$ is called *a set of generators* of $\mathscr{A}$. The *rank* of $\mathscr{A}$, by definition, is the minimum cardinality of such a family, and is denoted by $\mathrm{rank}(\mathscr{A})$. A C^*-algebra is said to be *finitely generated* if $\mathrm{rank}(\mathscr{A})$ is finite. Let rank$(A) = n$, and $\{T_1, T_2, \dots, T_n\}$ be a set of generators of $\mathscr{A}$. This induces a natural homeomorphism

$$\tau : M_{\mathscr{A}} \to \Delta$$

by $\tau(m) = (\hat{T}_1(m), \dots, \hat{T}_n(m))$, where $\hat{T}$ denotes the Gelfand transform of T onto $C(M_{\mathscr{A}})$ and $\Delta = \{(\hat{T}_1(m), \dots, \hat{T}_n(m)) \mid m \in M_{\mathscr{A}}\}(\subset \mathbb{C}^n)$. It is obvious that the topological dimension of Δ $(\leq 2n)$ is uniquely determined by $\mathscr{A}$. For the unit sphere S^{2n-1} of $\mathbb{C}^n$, we have the following basic fact.

LEMMA 2.1. *Let the essential spectrum $M_{\mathscr{A}}$ of $\mathscr{A}$ be homeomorphic to S^{2n-1}. Then* rank$(\mathscr{A}) = n$*, and there exists a set $\{T_1, T_2, \dots, T_n\}$ of generators of $\mathscr{A}$ such that*

$$\tau : M_{\mathscr{A}} \to S^{2n-1}; \qquad \tau(m) = \left(\hat{T}_1(m), \dots, \hat{T}_n(m)\right)$$

is a homeomorphism.

PROOF. If the essential spectrum $M_{\mathscr{A}}$ of $\mathscr{A}$ is homeomorphic to S^{2n-1}, then one has a natural extension of $\mathscr{K}$ by $C(S^{2n-1})$

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{A} \xrightarrow{\phi} C(S^{2n-1}) \longrightarrow 0. \tag{2.1}$$

Now take T_i in $\phi^{-1}(z_i)$ for $i = 1, 2, \dots, n$. It is easily checked that the family $\{T_1, T_2, \dots, T_n\}$ is a set of generators of $\mathscr{A}$, and

$$\tau : M_{\mathscr{A}} \to S^{2n-1}; \qquad \tau(m) = \left(\hat{T}_1(m), \dots, \hat{T}_n(m)\right)$$

is a homeomorphism. Since

$$2\,\text{rank}(\mathscr{A}) \geq 2n - 1,$$

this implies rank$(\mathscr{A}) = n$. □

From Lemma 2.1, each $\mathscr{A}$ in Σ_n yields an extension (2.1) of $\mathscr{K}$ by $C(S^{2n-1})$ and hence yields the following exact sequence

$$0 \longrightarrow \mathscr{K} \otimes M_k \longrightarrow \mathscr{A} \otimes M_k \xrightarrow{\phi\otimes 1} C(S^{2n-1}) \otimes M_k \longrightarrow 0 \tag{2.2}$$

for the algebra M_k of $k \times k$ complex matrices. For $A \in \mathscr{A} \otimes M_k$, $\tilde{A}$, the image of A in $C(S^{2n-1}) \otimes M_k$, is called the symbol of A. It is easily seen that A is Fredholm if and only if $\tilde{A}$ has non-vanishing determinant.

LEMMA 2.2. *Let $n > 1$ and $k < n$. Then for any Fredholm operator A in $\mathscr{A} \otimes M_k$, we have* index$(A) = 0$.

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PROOF. Let $GL(n, C)$ denote the complex linear group. Consider a continuous map

$$F : S^{2n-1} \to GL(n, C).$$

The first column F_1 of the matrix F defines a map

$$F_1 : S^{2n-1} \to C^n - \{0\}$$

so that $f = F_1/|F_1|$ is a map from S^{2n-1} to S^{2n-1}. This map has a degree, $\deg(f)$, up to a sign, the number of points in $h^{-1}(p)$, where h is a differentiable approximation to f and p is a general point (see [Ati] or [Hir]). For F, we then define the degree of F by

$$\deg(F) = \frac{(-1)^{n-1}\deg(f)}{(n-1)!}.$$

Defining $\operatorname{index}(\tilde{A})$ by $\operatorname{index}(A)$, then $\operatorname{index}(\tilde{A}) = \operatorname{index}(\tilde{A}, I_{n-k})$, where I_{n-k} is the $(n-k) \times (n-k)$ identity matrix, and $(\tilde{A}, I_{n-k})$ denotes the matrix

$$(\tilde{A}, I_{n-k}) = \begin{pmatrix} \tilde{A} & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

Let F_1 be the first column of the matrix $(\tilde{A}, I_{n-k})$. It is obvious that the image of $f = F_1/|F_1| : S^{2n-1} \to S^{2n-1}$ is a proper closed subset of S^{2n-1}. One thus concludes $\deg(f) = 0$ by [BT] or [Hir]. Let continuous maps $\hat{A}$ and $\hat{I}_n$ from S^{2n-1} to $GL(n, C)$ be given respectively by $(\tilde{A}, I_{n-k})$ and the $n \times n$ identity matrix I_n. Since

$$\deg(\hat{A}) = \deg(\hat{I}_n) = 0,$$

the theorem of Bott implies that $\hat{A}$ can be continuously deformed to $\hat{I}_n$ (see [Ati]). Combining the above discussion with Douglas [Dou], we see that

$$\operatorname{index}(A) = \operatorname{index}(\tilde{A}) = \operatorname{index}(\tilde{A}, I_{n-k}) = \operatorname{index}(I_n) = \operatorname{index}(I) = 0. \qquad \square$$

In [Guo1], we introduced an invariant called the characteristic number to study essentially normal operators. Lemma 2.2 motivates us to introduce characteristic numbers for C^*-algebras. For any essentially commutative C^*-algebra $\mathscr{A}$, since the image of all Fredholm operators in $\mathscr{A}$ is a multiplicative group in the Calkin algebra, it follows that the indices of all Fredholm operators in $\mathscr{A}$ form a subgroup Γ of the integer group $\mathbb{Z}$, that is, there exists a unique non-negative integer m such that $\Gamma = m\mathbb{Z}$. The *characteristic number* $\gamma(\mathscr{A})$ of $\mathscr{A}$, by definition, is the above m. We also define the *n-th characteristic number* $\gamma_n(\mathscr{A})$ of $\mathscr{A}$ by $\gamma(\mathscr{A} \otimes M_n)$. By the inclusion of $\mathscr{A} \otimes M_n$ in $\mathscr{A} \otimes M_{n+1}$ which sends A to (A, I), this forces that $\gamma_{n+1}(\mathscr{A})$ is a factor of $\gamma_n(\mathscr{A})$ for any natural number n.

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LEMMA 2.3. *Let $\mathscr{A}$ be in the class Σ_n. Then*

$$\gamma_1(\mathscr{A}) = \gamma_2(\mathscr{A}) = \cdots = \gamma_{n-1}(\mathscr{A}) = 0$$

and

$$\gamma_n(\mathscr{A}) = \gamma_{n+1}(\mathscr{A}) = \cdots .$$

PROOF. From Lemma 2.2, we only need to show that

$$\gamma_n(\mathscr{A}) = \gamma_{n+1}(\mathscr{A}) = \cdots .$$

Let $F : S^{2n-1} \to GL(N, C)$ be a continuous map, here $N > n$. Then by Atiyah [Ati], there is a continuous map

$$G : [0, 1] \times S^{2n-1} \to GL(N, C)$$

such that $G(0, z) = F(z)$ and

$$G(1, z) = \begin{pmatrix} H(z) & 0 \\ 0 & I_{N-n} \end{pmatrix},$$

where I_{N-n} is the $(N-n) \times (N-n)$ identity matrix. The argument used in the proof of Lemma 2.2 can then be exploited to show that $\gamma_N(\mathscr{A}) = \gamma_n(\mathscr{A})$. □

To understand the importance of characteristic numbers a little better we shall see in Section 3 that γ_n is a complete invariant of C^*-algebras in Σ_n in C^*-isomorphism sense.

3. The equivalence classes of $\mathrm{Ext}(S^{2n-1})$

Let us begin with facts from the BDF-theory [BDF1, BDF2]. Let X be a compact metrizable space. An *extension* of $\mathscr{K}$ by $C(X)$ is a pair $(\mathscr{E}, \phi)$, where $\mathscr{E}$ is a C^*-subalgebra of operators on some separable Hilbert space which contains $\mathscr{K}$ and the identity operator I, and ϕ is a C^*-homomorphism of $\mathscr{E}$ onto $C(X)$ with kernel $\mathscr{K}$. In the language of homology, an extension $(\mathscr{E}, \phi)$ is a short exact sequence

$$0 \longrightarrow \mathscr{K} \stackrel{i}{\longrightarrow} \mathscr{E} \stackrel{\phi}{\longrightarrow} C(X) \longrightarrow 0.$$

Extensions $(\mathscr{E}_1, \phi_1)$ and $(\mathscr{E}_2, \phi_2)$ are called *equivalent* if there exists a C^*-isomorphism $\psi : \mathscr{E}_2 \to \mathscr{E}_1$ such that $\phi_2 = \phi_1\psi$. The set of equivalence classes of extensions of $\mathscr{K}$ by $C(X)$ is denoted $\mathrm{Ext}(X)$. In [BDF2], they proved that $\mathrm{Ext}(X)$ is a group, and the correspondence $X \mapsto \mathrm{Ext}(X)$ yields a homotopy invariant covariant functor. It is well known that one of the applications of the BDF-theory is to classify essentially normal operators modulo the compacts under unitary equivalence (see [BDF1]).

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Below, we shall concentrate on working out explicitly a complete system of representatives for the equivalence classes of extensions of $\mathscr{K}$ by $C(S^{2n-1})$. In the case $n = 1$, a complete system of representatives of $\mathrm{Ext}(S^1)$ is worked out explicitly by Toeplitz extension on the unit circle S^1 in [BDF1]. In the case $n > 1$, the periodicity theorem [BDF2] implies $\mathrm{Ext}(S^{2n-1}) = \mathbb{Z}$. Let $L^2_a(B_n)$ be the Bergman space on the unit ball B_n in $\mathbb{C}^n$, and let $C^*(B_n)$ be the C^*-algebra generated by all Toeplitz operators on $L^2_a(B_n)$ with symbols in $C(\overline{B}_n)$. The Coburn exact sequence [Cob]

$$0 \longrightarrow \mathscr{K} \xrightarrow{\ i\ } C^*(B_n) \xrightarrow{\ \pi\ } C\left(S^{2n-1}\right) \longrightarrow 0$$

is a natural extension of $\mathscr{K}$ by $C(S^{2n-1})$. From Venugopalkrishna [Ven], we see that this extension is a generator of $\mathrm{Ext}(S^{2n-1})$ (see also [BDF2]). By the theorem of Bott in [Ati], there is a natural isomorphism

$$\pi_{2n-1}(GL(n, C)) \cong K^1\left(S^{2n-1}\right) \cong \mathbb{Z},$$

where $\pi_{2n-1}(GL(n, C))$ is the group of homotopy classes of continuous maps from S^{2n-1} to $GL(n, C)$. Applying the BDF-theory [BDF2], the homomorphism

$$\gamma_\infty : \mathrm{Ext}\left(S^{2n-1}\right) \to \mathrm{Hom}\,(\pi_{2n-1}(GL(n, C)), \mathbb{Z}) \ \ (\cong \mathbb{Z})$$

is surjective, and hence is an isomorphism. For an extension $(\mathscr{E}, \phi)$, $\gamma_\infty(\mathscr{E})$ is defined by

$$\gamma_\infty(\mathscr{E})[f_{ij}] = \mathrm{index}\left[\phi^{-1}(f_{ij})\right].$$

Let $L^2(S^{2n-1})$ denote the Hilbert space of square-integrable functions on S^{2n-1}. For $f \in C(S^{2n-1})$, we denote by M_f multiplication operator on $L^2(S^{2n-1})$. It is well known that $(\mathscr{E}_0, \pi_0)$ is the zero element in $\mathrm{Ext}(S^{2n-1})$, where

$$\mathscr{E}_0 = C^*(M_{z_1}, \ldots, M_{z_n}) = \left\{M_f + K \mid f \in C\left(S^{2n-1}\right), K \in \mathscr{K}\right\},$$

and $\pi_0(M_f + K) = f$. Now let $\sigma : S^{2n-1} \to S^{2n-1}$ be a continuous map with the mapping degree $\deg(\sigma) \neq 0$, σ is then surjective. We use $\mathscr{E}_\sigma$ to denote the C^*-algebra $\{T_{\dot{f}\circ\sigma} + K | f \in C(S^{2n-1}), K \in \mathscr{K}\}$, where $T_{\dot{f}\circ\sigma}$ is Toeplitz operator with symbol $\dot{f}\circ\sigma$ on the Bergman space $L^2_a(B_n)$, and $\dot{f}\circ\sigma$ is the standard Poisson extension of $f \circ \sigma$ onto $\overline{B_n}$. This gives an extension $(\mathscr{E}_\sigma, \pi_\sigma)$, where $\pi_\sigma(T_{\dot{f}\circ\sigma} + K) = f$. In fact, it is easily seen that $(\mathscr{E}_\sigma, \pi_\sigma) = \sigma_*(C^*(B_n), \pi)$, and hence if σ_1 and σ_2 are homotopic, the homotopy invariance of Ext then implies that $(\mathscr{E}_{\sigma_1}, \pi_{\sigma_1})$ and $(\mathscr{E}_{\sigma_2}, \pi_{\sigma_2})$ are equivalent.

For $i = \pm 1, \pm 2, \ldots$, take σ_i to be a continuous map from S^{2n-1} to S^{2n-1} with mapping degree $\deg(\sigma_i) = i$. We have thus the following.

LEMMA 3.1. *The extensions $(\mathscr{E}_{\sigma_i}, \pi_{\sigma_i})$ $(i = \pm 1, \pm 2, \ldots)$ together with the trivial extension $(\mathscr{E}_0, \pi_0)$ form a complete system of representatives of* $\mathrm{Ext}(S^{2n-1})$.

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PROOF. Let m be a non-zero integer. By [BDF2], one has

$$\gamma_\infty(\mathscr{E}_{\sigma_m})[f_{ij}] = \operatorname{index}\left[T_{f_{ij}\circ\sigma_m}\right] = \deg(\sigma_m)\operatorname{index}\left[T_{f_{ij}}\right] = m\operatorname{index}\left[T_{f_{ij}}\right]$$

and

$$\gamma_\infty(mC^*(B_n))[f_{ij}] = m\operatorname{index}\left[T_{f_{ij}}\right].$$

It follows that $(\mathscr{E}_{\sigma_m}, \pi_{\sigma_m})$ and $(mC^*(B_n), \pi^{(m)})$ are equivalent. Note that the Toeplitz extension $(C^*(B_n), \pi)$ is a generator of $\operatorname{Ext}(S^{2n-1})$, we thus conclude that the extensions $(\mathscr{E}_{\sigma_i}, \pi_{\sigma_i})$ $(i = \pm 1, \pm 2, \dots)$ together with the extension $(\mathscr{E}_0, \pi_0)$ form a complete system of representatives of $\operatorname{Ext}(S^{2n-1})$. □

LEMMA 3.2. *Let $\sigma : S^{2n-1} \to S^{2n-1}$ be a continuous map. Then we have*

$$\gamma_n(\mathscr{E}_\sigma) = |\deg(\sigma)|.$$

PROOF. Apply Venugopalkrishna [Ven, Theorem 1.5] and the multiplication formula of mapping degree [Hir]. □

LEMMA 3.3. *Let $\sigma', \sigma'' : S^{2n-1} \to S^{2n-1}$ be continuous maps. Then $\mathscr{E}_{\sigma'}$ and $\mathscr{E}_{\sigma''}$ are C^*-isomorphic if and only if*

$$|\deg(\sigma')| = |\deg(\sigma'')|.$$

PROOF. If $\mathscr{E}_{\sigma'}$ and $\mathscr{E}_{\sigma''}$ are C^*-isomorphic, then the isomorphism is implemented by a unitary operator. This implies thus that

$$\gamma_n(\mathscr{E}_{\sigma'}) = \gamma_n(\mathscr{E}_{\sigma''})$$

and hence by Lemma 3.2,

$$|\deg(\sigma_1)| = |\deg(\sigma_2)|.$$

If $\deg(\sigma') = \deg(\sigma'')$, then Hopf lemma ([Hir]) implies that the maps $\sigma', \sigma'' : S^{2n-1} \to S^{2n-1}$ are homotopic, and hence the homotopy invariance of Ext shows that $\mathscr{E}_{\sigma'}$ and $\mathscr{E}_{\sigma''}$ are isomorphic as C^*-algebras. If $\deg(\sigma') = -\deg(\sigma'')$, write $\sigma'' = (\phi_1, \phi_2, \dots, \phi_n)$ and define σ''' by $(\phi_1, \phi_2, \dots, \overline{\phi}_n)$, then

$$\deg(\sigma') = \deg(\sigma''')$$

and hence $\mathscr{E}_{\sigma'}$ and $\mathscr{E}_{\sigma'''}$ are C^*-isomorphic. Since $\mathscr{E}_{\sigma''} = \mathscr{E}_{\sigma'''}$, it follows that $\mathscr{E}_{\sigma'}$ and $\mathscr{E}_{\sigma''}$ are C^*-isomorphic. □

THEOREM 3.4. *Let $\mathscr{E}$ and $\mathscr{F}$ be in Σ_n. Then $\mathscr{E}$ and $\mathscr{F}$ are C^*-isomorphic if and only if*

$$\gamma_n(\mathscr{E}) = \gamma_n(\mathscr{F}).$$

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PROOF. Assume first that $\mathscr{E}$ and $\mathscr{F}$ are C^*-isomorphic. Then the equality $\gamma_n(\mathscr{E}) = \gamma_n(\mathscr{F})$ is immediate. Conversely, since $\mathscr{E}$ and $\mathscr{F}$ are respectively C^*-isomorphic to one of $\mathscr{E}_0, \mathscr{E}_{\sigma_1}, \mathscr{E}_{\sigma_2}, \dots$, Lemmas 3.1–3.3 imply that if $\gamma_n(\mathscr{E}) = \gamma_n(\mathscr{F})$, then $\mathscr{E}$ and $\mathscr{F}$ are isomorphic as C^*-algebras. □

From Theorem 3.4, we see that n-th characteristic number is a complete invariant for C^*-algebras in Σ_n in C^*-isomorphism sense. In next section we will give examples to show applications of the classification Theorem 3.4.

Now we consider Toeplitz algebras on pseudoregular domains ($\subset \mathbb{C}^n$) with smooth boundary. As pointed out in [Sal, SSU], pseudoregular domains include the strongly pseudoconvex domains, pseudoconvex domains with real-analytic boundary, and more generally, domains of finite type. Let Ω be pseudoregular domain with smooth boundary. Following [Sal, SSU], on the Bergman space $L_a^2(\Omega)$, the C^*-algebra $C^*(\Omega)$ generated by Toeplitz operators with symbols in $C(\overline{\Omega})$ is essentially commutative, and its essential spectrum is $\partial\Omega$. In [SSU], they proved that for each $\lambda \in \Omega$, Toeplitz tuple $T_z - \lambda = \{T_{z_1} - \lambda_1, \dots, T_{z_n} - \lambda_n\}$ is Fredholm, and index$(T_z - \lambda) = (-1)^n$. By [Cur] and Lemma 2.3, one sees that if $\partial\Omega$ is homeomorphic to S^{2n-1}, then $\gamma_n(C^*(\Omega)) = 1$. Since $\gamma_n(C^*(B_n)) = 1$, Theorem 3.4 immediately yields the following.

EXAMPLE 1. Let Ω be a pseudoregular domain with smooth boundary. Then $\partial\Omega$ and S^{2n-1} are homeomorphic if and only if $C^*(\Omega)$ is isomorphic to $C^*(B_n)$ as C^*-algebras.

For a pseudoregular domain Ω in $\mathbb{C}^n$, we say that Ω is a Poincaré domain if its boundary $\partial\Omega$ is homotopy equivalent to the unit sphere S^{2n-1}, that is, $\partial\Omega$ is a homotopy $(2n-1)$-sphere. The generalized Poincaré conjecture says if every closed n-manifold M which is a homotopy n-sphere is homeomorphic to the n-sphere (see [Sma1, Sma2]). Smale [Sma2] showed that the generalized Poincaré conjecture is true in the case $n > 4$. Freedman [Fre] proved the case $n = 4$. For $n = 1, 2$, it is well known that the generalized conjecture is true (see [Hir]). Therefore the famous Poincaré conjecture says that every closed 3- manifold which is a homotopy 3-sphere is homeomorphic to the 3-sphere. This has never been answered. Therefore, for each Poincaré domain Ω in $\mathbb{C}^n$ ($n \neq 2$), its boundary $\partial\Omega$ is actually homeomorphic to the $(2n-1)$-sphere. Example 1 shows thus the following.

EXAMPLE 2. Let Ω be a Poincaré domain in $\mathbb{C}^n$ ($n \neq 2$). Then

$$C^*(\Omega) \cong C^*(B_n).$$

Example 2 is proved by different method in [Guo2]. Example 2 suggests that for the Poincaré conjecture in the case of $\partial\Omega$, an operator algebraic proof is perhaps possible. Of course, the validity of the Poincaré conjecture for $\partial\Omega$ remains unknown.

EXAMPLE 3. Let $0 < p, q < \infty$ and $\Omega_{p,q} = \{z \in \mathbb{C}^2 \mid |z_1|^p + |z_2|^q < 1\}$. $\Omega_{p,q}$ is pseudoconvex (because $\log(\Omega_{p,q})_+$ is convex); when $p, q \geq 2$, $\Omega_{p,q}$ is Levi pseudoconvex; and $\Omega_{p,q}$ is strongly pseudoconvex if and only if $p = q = 2$. From [CS], on the Bergman space $L_a^2(\Omega_{p,q})$, one sees that Toeplitz algebra $C^*(\Omega_{p,q})$ (generated by Toeplitz operators with symbols in $C(\overline{\Omega}_{p,q})$) is essentially commutative, and its essential spectrum is $\partial\Omega_{p,q}$, also for each $\lambda \in \Omega_{p,q}$, $\operatorname{index}(T_z - \lambda) = 1$. Then by the radial projection, $\partial\Omega_{p,q}$ and S^3 are homeomorphic, and hence $C^*(\Omega_{p,q})$ and $C^*(B_2)$ are isomorphic as C^*-algebras. However, for p or $q \neq 2$, it is easy to prove that there does not exist any proper holomorphic mapping that maps the unit ball B_2 onto $\Omega_{p,q}$. In this example, we restricted ourselves to $n = 2$, but it is clear that all results hold for $n \geq 2$.

EXAMPLE 4. Considering the domain $\Omega = \{z \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 + |z_1 z_2|^2 < 1\}$, it is easy to check that Ω is a strongly pseudoconvex domain with smooth boundary. Then by the radial projection, $\partial\Omega$ and S^3 are homeomorphic, and hence $C^*(\Omega)$ and $C^*(B_2)$ are isomorphic as C^*-algebras. However, by the Cartan Theorem, the unit ball B_2 and Ω are never holomorphically equivalent.

4. The construction of representatives of the class Σ_n

In this section, we shall construct explicitly a complete system of representatives of the class Σ_n. Let m be a positive integer. Define the map $\sigma_m : S^{2n-1} \to S^{2n-1}$ by

$$\sigma_m(z_1, \dots, z_n) = \left(z_1, \dots, z_{n-1}, |z_n| \frac{z_n^m}{|z_n|^m}\right)$$

and the map $\sigma_{-m} : S^{2n-1} \to S^{2n-1}$ by

$$\sigma_m(z_1, \dots, z_n) = \left(z_1, \dots, z_{n-1}, |z_n| \frac{\bar{z}_n^m}{|z_n|^m}\right).$$

We claim

$$\deg(\sigma_m) = m; \quad \deg(\sigma_{-m}) = -m.$$

Write Ω_m for pseudoconvex domain

$$\left\{(z_1, \dots, z_n) \mid |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^{2/m} < 1\right\}.$$

The domain above is pseudoconvex because $\log(\Omega_m)_+$ is convex. Denote by $\partial\Omega_m$ the boundary of Ω_m, that is,

$$\partial\Omega_m = \left\{(z_1, \dots, z_n) \mid |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^{2/m} = 1\right\}.$$

Since $\overline{\Omega}_m$ has the standard orientation, its boundary $\partial\Omega_m$ inherits an orientation (except some special points), also called 'standard'. This means that $(e_1, \dots, e_{2n-1})$ is an

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orienting basis for $\partial\Omega_m$ if $(e_1, \dots, e_{2n-1}, e_{2n})$ is an orienting basis for $\overline{\Omega}_m$ and e_{2n} points into Ω_m at $z \in \partial\Omega_m$. Define a map δ_m from S^{2n-1} onto $\partial\Omega_m$ by

$$\delta_m(z_1, \dots, z_n) = \left(z_1, \dots, z_{n-1}, z_n^m\right).$$

A straightforward calculation of the Jacobi matrix yields that the mapping degree $\deg(\delta_m) = m$. Furthermore, we establish an orientation-preserving homeomorphism $\eta_m : \partial\Omega_m \to S^{2n-1}$ by

$$\eta_m(z_1, \dots, z_{n-1}, z_n) = \left(z_1, \dots, z_{n-1}, |z_n|^{1/m}\frac{z_n}{|z_n|}\right).$$

It is easily checked that

$$\sigma_m = \eta_m \circ \delta_m$$

and hence

$$\deg(\sigma_m) = \deg(\eta_m)\deg(\delta_m) = 1m = m.$$

Similarly, define an anti-orientation homeomorphism $\eta_{-m} : \partial\Omega_m \to S^{2n-1}$ by

$$\eta_{-m}(z_1, \dots, z_{n-1}, z_n) = \left(z_1, \dots, z_{n-1}, |z_n|^{1/m}\frac{\bar{z}_n}{|z_n|}\right).$$

It is easily seen that $\sigma_{-m} = \eta_{-m} \circ \delta_m$, and $\deg(\sigma_{-m}) = -m$ follows. The claim is proved.

Let $C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m})$ be the C^*-algebra generated by $T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m}$, the identity operator and all compact operators on the Bergman space $L_a^2(B_n)$. We define extensions

$$\left(C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m}), \pi_m\right) \quad \text{and} \quad \left(C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m}), \pi_{-m}\right)$$

of $\mathscr{K}$ by $C(S^{2n-1})$ respectively by

$$\pi_m(T_{f(z_1,\dots,z_{n-1},z_n^m)} + K) = f|_{\partial\Omega_m} \circ \eta_m^{-1}$$

and

$$\pi_{-m}(T_{f(z_1,\dots,z_{n-1},z_n^m)} + K) = f|_{\partial\Omega_m} \circ \eta_{-m}^{-1},$$

here $f \in C(\overline{\Omega}_m)$, $K \in \mathscr{K}$.

We are now in a position to give a main result in this section.

THEOREM 4.1. (1) $\mathscr{E}_{\sigma_m} = \mathscr{E}_{\sigma_{-m}} = C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m})$.
(2) *The extensions*

$$\left(C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m}), \pi_m\right) \quad \textit{and} \quad \left(C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m}), \pi_{-m}\right)$$

(here $m = 1, 2, \dots$*) and the trivial extension* $(\mathscr{E}_0, \pi_0)$ *form a complete system of representatives of* $\mathrm{Ext}(S^{2n-1})$.

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PROOF. (1). It is obvious that the relation $\mathscr{E}_{\sigma_m} = \mathscr{E}_{\sigma_{-m}}$ is true. Then an operator $A \in \mathscr{E}_{\sigma_m}$ if and only if A has form $A = T_{f\circ\sigma_m} +$ compact, $f \in C(S^{2n-1})$, and $B \in C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m})$ if and only if B has form $B = T_{f(z_1,\dots,z_{n-1},z_n^m)} +$ compact, $f \in C(\overline{\Omega}_m)$. The orientation-preserving homeomorphism η_m and the relation $\sigma_m = \eta_m \circ \delta_m$ imply then

$$\mathscr{E}_{\sigma_m} = C^*\left(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m}\right).$$

(2). Apply Lemma 3.1 and the above (1). □

From Lemma 3.2, Theorem 3.4, and Theorem 4.1, we immediately obtain the following:

COROLLARY 4.2. *Let* $\mathscr{E} \in \Sigma_n$. *Then*

(1) *if* $\gamma_n(\mathscr{E}) = 0$, *then* $\mathscr{E}$ *and* $\mathscr{E}_0$ *are* C^**-isomorphic;*
(2) *if* $\gamma_n(\mathscr{E}) = m > 0$, *then* $\mathscr{E}$ *and* $C^*(T_{z_1}, \dots, T_{z_{n-1}}, T_{z_n^m})$ *are* C^**-isomorphic.*

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