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ON DIRAC'S GENERALIZATION OF BROOKS' THEOREM

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1. Introduction. It is easy to verify that any connected graph G with maximum degree s has chromatic number $\chi(G) \leqq 1 + s$. In [**1**], R. L. Brooks proved that $\chi(G) \leqq s$, unless $s = 2$ and G is an odd cycle or $s > 2$ and G is the complete graph K_{s+1}. This was the first significant theorem connecting the structure of a graph with its chromatic number. For $s \geqq 4$, Brooks' theorem says that every connected s-chromatic graph other than K_s contains a vertex of degree $> s - 1$. An equivalent formulation can be given in terms of s-critical graphs. A graph G is said to be *s-critical* if $\chi(G) = s$, but every proper subgraph has chromatic number less than s. Each s-critical graph has minimum degree $\geqq s - 1$. We can now restate Brooks' theorem: if an s-critical graph, $s \geqq 4$, is not K_s and has p vertices and q edges, then $2q \geqq (s-1)p + 1$. Dirac [**2**] significantly generalized the theorem of Brooks by showing that $2q \geqq (s-1)p + s - 3$ and that this result is best possible. Dirac's theorem has several important applications. For example, Dirac [**3**] used his result to show that if a graph G with genus $n \geqq 1$ has

$$\chi(G) = \left[\frac{7 + (1 + 48n)^{\frac{1}{2}}}{2}\right] = H(n),$$

then G contains $K_{H(n)}$ as a subgraph. The object of this note is to present a new proof of Dirac's theorem.

2. Dirac's theorem. In this section we state and prove Dirac's theorem. Although our proof is not particularly short, it is considerably shorter than the original one [**2**]. The first part of the proof below was suggested by Melnikov and Vizing's [**4**] elegant new proof of Brooks' theorem.

THEOREM. *If G is an s-critical graph, $s \geqq 4$, which is not complete, then*

$$2q \geqq (s-1)p + s - 3.$$

Proof. Suppose the theorem is false. Then there exists an s-critical, $s \geqq 4$, graph $H \neq K_s$, with $2q \leqq (s-1)p + s - 4$. Since H is s-critical, $2q \geqq (s-1)p$, so that at most $s - 4$ vertices of H have degree $\geqq s$. Let v be a vertex of degree $s - 1$ and let $H' = H - v$. The graph H' has $\chi(H') = s - 1$. In each $(s-1)$-coloring of H' the vertices $v_1, v_2, \ldots, v_{s-1}$ adjacent to v must necessarily be colored in different colors, say $1, 2, \ldots, s-1$, respectively.

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Also each v_i must be adjacent to a vertex colored $j \neq i$, $1 \leqq j \leqq s - 1$.

Assume the graph H' has been colored as above, then:

1. *Vertices v_i and v_j $(i, j = 1, \ldots, s - 1, i \neq j)$ are in the same component C_{ij} of the subgraph induced by vertices colored i and j.* Otherwise the interchanging of colors i and j in the component containing v_i would give an $(s - 1)$-coloring of H' in which v_i and v_j have the same color. A v_iv_j-path in C_{ij} will be denoted by P_{ij}.

We let N denote the number of colors which are assigned to vertices having degree $\geqq s$ in H. Also let $n = s - 1 - N$ and assume that $1, 2, \ldots, n$ are the colors assigned only to vertices having degree $s - 1$ in H. We note that $n \geqq 3$, since $0 \leqq N \leqq s - 4$. Also, for $1 \leqq i \leqq$ n, v_i is adjacent to exactly one vertex of each color $j \neq i$, $1 \leqq j \leqq s - 1$.

2. *The component C_{ij} $(i, j = 1, 2, \ldots, n;\ i \neq j)$ is a path.* All the vertices of C_{ij} have degree $s - 1$ in H. It follows from above that the vertices v_i and v_j have degree 1 in C_{ij}. All other vertices must have degree 2; otherwise in moving from v_i to v_j along P_{ij}, the first vertex u having degree > 2 in C_{ij} would have degree $\geqq s$ in H. If u had degree $s - 1$, then it could be recolored in a color different from i and j so that v_i and v_j would lie in different components, in contradiction to 1.

3. *The paths P_{ij} and P_{ik} $(i, j, k = 1, 2, \ldots, n;\ i \neq j \neq k \neq i)$ have no common vertex except v_i.* If they had a common vertex different from v_i of degree $s - 1$ in H, then it could be recolored in a color different from i, j, k so that v_i and v_j would not be joined by a P_{ij} path.

4. *If $i, j = 1, \ldots, n;\ i \neq j$, then v_i and v_j are adjacent.* Assume without loss of generality that v_1 and v_2 are not adjacent. Then the path P_{12} contains a vertex y adjacent with v_1 and different from v_2. Interchange the colors along P_{13}. After this change, the new paths P_{12} and P_{23} will each have the common vertex y, in contradiction to 3.

Since $H \neq K_s$, there exist non-adjacent vertices v_α and v_β $(\alpha, \beta = 1, 2, \ldots, s - 1;\ \alpha \neq \beta)$. At least one color class determined by the $s - 3$ colors different from α and β has only one point adjacent to v_α and one point adjacent to v_β. Otherwise, $\deg v_\alpha + \deg v_\beta \geqq 2(s - 1) + s - 3$ and hence $2q \geqq (s - 1)p + s - 3$. Let the colors that meet this condition be $1', 2', \ldots, t'$.

5. $\{1', 2', \ldots, t'\} \cap \{1, 2, \ldots, n\} \neq \phi$. Otherwise each color $1', \ldots, t'$ is associated with a vertex of degree $\geqq s$. Also, the $s - 3 - t$ colors different from $1', \ldots, t', \alpha, \beta$, each have two vertices adjacent to v_α or two vertices adjacent to v_β. Again we obtain $2q \geqq (s - 1)p + s - 3$. We can assume that $1 \in \{1', \ldots, t'\} \cap \{1, \ldots, n\}$.

6. *v_1 is adjacent to at most one of the vertices v_α and v_β.* If v_1 is adjacent to both v_α and v_β, then $C_{1\alpha} = (v_1, v_\alpha)$ and $C_{1\beta} = (v_1, v_\beta)$. If we now interchange the colors along $C_{1\alpha}$, then we obtain a coloring in which v_β is not adjacent to any vertex colored 1. We assume that v_1 and v_β are not adjacent.

Using arguments like that used for statement 2, it is easy to verify that for each i, $2 \leqq i \leqq n$, such that $C_{\beta i}$ is not a path, there is a vertex colored β in

$C_{\beta i}$ having degree $\geqq s$. These vertices are not necessarily distinct, however, if a vertex occurs k times, then its degree is at least $s - 1 + k$.

7. *There exists at least one* i, $2 \leqq i \leqq n$, *such that* $C_{\beta i}$ *is a path*. Otherwise, by the above remarks, if there are p' vertices colored β, then the sum of the degrees of these vertices is at least $(s-1)p' + n - 1$. We also have at least $N - 1$ vertices having degree $\geqq s$ which are not colored β. Again, we have $2q \geqq (s-1)p + s - 3$.

We let $2, 3, \ldots, m$, denote the colors which satisfy 7. By 6, there is a path $P_{1\beta}$ which contains a vertex u adjacent with v_1 and different from v_β.

8. *The vertex* u *does not belong to each* $C_{\beta i}$, $2 \leqq i \leqq m$. If it did, then u would have degree $\geqq 2m + (s - 1 - (m+1)) = (s-1) + (m-1)$. By the remarks preceding 7, the sum of the degrees of the p' vertices colored β is at least $(s-1)p' + (n-m) + (m-1) = (s-1)p' + n - 1$. There are also $(s-1) - (n+1)$ distinct vertices having degree $\geqq s$ which are colored with colors from the set $\{n+1, n+2, \ldots, s-1\} - \{\beta\}$. Therefore, $2q \geqq (s-1)p + s - 3$.

To produce a final contradiction, we let 2 be a color satisfying statements 7 and 8. Then u is not on $P_{\beta 2}$. By 4, $P_{12} = (v_1, v_2)$ so that if we interchange colors along $P_{\beta 2}$, then we obtain a coloring having the property that v_1 is not adjacent to any vertex colored 2. This is impossible and hence the assumption that $2q \leqq (s-1)p + s - 4$ is false.

References

1. R. L. Brooks, *On coloring the nodes of a network*, Proc. Cambridge Philos. Soc. *37* (1941), 194–197.

2. G. A. Dirac, *A theorem of R. L. Brooks and a conjecture of H. Hadwiger*, Proc. London Math. Soc. *7* (1957), 161–195.

3. ——— *Short proof of a map-colour theorem*, Can. J. Math. *9* (1957), 225–226.

4. L. S. Melnikov and V. G. Vizing, *New proof of Brooks' theorem*, J. Combinatorial Theory *7* (1969), 289–290.

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