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On Complemented Subspaces of Non-Archimedean Power Series Spaces

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Abstract. The non-archimedean power series spaces, $A_1(a)$ and $A_\infty(b)$, are the best known and most important examples of non-archimedean nuclear Fréchet spaces. We prove that the range of every continuous linear map from $A_p(a)$ to $A_q(b)$ has a Schauder basis if either $p = 1$ or $p = \infty$ and the set $M_{b,a}$ of all bounded limit points of the double sequence $(b_i/a_j)_{i,j\in\mathbb{N}}$ is bounded. It follows that every complemented subspace of a power series space $A_p(a)$ has a Schauder basis if either $p = 1$ or $p = \infty$ and the set $M_{a,a}$ is bounded.

1 Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|\colon \mathbb{K} \to [0,\infty)$. For fundamentals of locally convex Hausdorff spaces and normed spaces we refer to [9–11].

Any infinite-dimensional Banach space of countable type is isomorphic, *i.e.,* linearly homeomorphic, to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm), so it has a Schauder basis [10, Theorem 3.16]. It is also known that any infinite-dimensional Fréchet space of finite type is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the product topology [13, Theorem 7], so it has a Schauder basis, too.

Hence every closed subspace of c_0 and $\mathbb{K}^{\mathbb{N}}$ has a Schauder basis. By [15, Proposition 9], we have a similar fact for $c_0 \times \mathbb{K}^{\mathbb{N}}$. For $c_0^{\mathbb{N}}$ it is not true, since there exist Fréchet spaces of countable type without a Schauder basis [14, Theorem 3] and every Fréchet space of countable type is isomorphic to a closed subspace of $c_0^{\mathbb{N}}$ [4, Remark 3.6]. In fact, every infinite-dimensional Fréchet space which is not isomorphic to any of the following spaces (c_0, $\mathbb{K}^{\mathbb{N}}$, $c_0 \times \mathbb{K}^{\mathbb{N}}$) contains a closed subspace without a Schauder basis [15, Theorem 7].

One of the most important problems for Fréchet spaces is the following one:

> Let E be a Fréchet space with a Schauder basis. Does every complemented subspace F of E have a Schauder basis?

For nuclear Fréchet spaces over the field of real or complex numbers, this problem was posed by Pełczyński in 1970, and it is still open.

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In [17, Proposition 9], it was shown that every quotient of $c_0^{\mathbb{N}}$ has a Schauder basis. Thus every complemented subspace of $c_0^{\mathbb{N}}$ has a Schauder basis [17, Corollary 10].

The power series spaces of finite type and infinite type, $A_1(a)$ and $A_\infty(b)$, are the best known and most important examples of nuclear Fréchet spaces with a Schauder basis. In this paper we show that the range of every continuous linear operator from $A_p(a)$ to $A_q(b)$ has a Schauder basis if either $p = 1$ or $p = \infty$ and the set $M_{b,a}$ of all finite limit points of the double sequence $(b_i/a_j)_{i,j\in\mathbb{N}}$ is bounded (Corollary 3.11). It follows that every complemented subspace of a power series space $A_p(a)$ has a Schauder basis, if either $p = 1$ or $p = \infty$ and the set $M_{a,a}$ is bounded (Corollary 3.13).

In this paper we use and develop some ideas of [8] (see also [6]).

2 Preliminaries

The linear span of a subset A of a linear space E is denoted by $[A]$.

Let E, F be locally convex spaces. A map $T\colon E \to F$ is called an isomorphism if it is linear, bijective and the maps T, T^{-1} are continuous. If there exists an isomorphism $T\colon E \to F$, then we say that E is isomorphic to F and write $E \simeq F$. The family of all continuous linear maps from E to F we denote by $L(E, F)$. The *range* of $T \in L(E, F)$ is the subspace $T(E)$ of F.

Sequences (x_n) and (y_n) in a locally convex space E are:

- *equivalent* if there exists an isomorphism P between the closed linear spans of (x_n) and (y_n) in E, such that $Px_n = y_n$ for every $n \in \mathbb{N}$;
- *quasi-equivalent* if there exist $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ and a permutation π of $\mathbb{N}$ such that the sequences $(\alpha_n x_{\pi(n)})$ and (y_n) are equivalent.

A finite sequence $(x_1, \dots, x_n)$ in a finite-dimensional locally convex space E is a *Schauder basis* in E if there exist $f_1, \dots, f_n \in E'$ such that $x = \sum_{i=1}^n f_i(x)x_i$ for every $x \in E$, and $f_i(x_j) = \delta_{i,j}$ for all $1 \le i, j \le n$; clearly, every Hamel basis in E is a Schauder basis in E.

A sequence (x_n) in an infinite-dimensional locally convex space E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^\infty \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$, and the coefficient functionals $f_n\colon E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous.

By a *seminorm* on a linear space E we mean a function $p\colon E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \le \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

The set of all continuous seminorms on a locally convex space E is denoted by $\mathcal{P}(E)$. A nondecreasing sequence (p_n) of continuous seminorms on a metrizable locally convex space E is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there are $C > 0$ and $k \in \mathbb{N}$ such that $p(x) \le C p_k(x)$ for all $x \in E$.

A complete metrizable locally convex space is called a *Fréchet space*. Let (x_n) be a sequence in a Fréchet space E. The series $\sum_{n=1}^\infty x_n$ is convergent in E if and only if $\lim x_n = 0$.

A normable Fréchet space is a *Banach space*.

A metrizable locally convex space E is of *countable type* if it contains a linearly dense sequence (x_n). A metrizable locally convex space E is of *finite type* if

$$\dim(E/\ker p) < \infty$$

for every $p \in \mathcal{P}(E)$. Put $B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$. Let A be a subset of a locally convex space E. The set

$$\operatorname{co} A = \Big\{ \sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in B_{\mathbb{K}}, a_1, \dots, a_n \in A \Big\}$$

is the absolutely convex hull of A; its closure in E is denoted by $\overline{\operatorname{co}}^E A$. A subset A of a locally convex space E is *absolutely convex* if $\operatorname{co} A = A$.

A subset B of a locally convex space E is *compactoid* (or a *compactoid*) if for each neighbourhood U of 0 in E there exists a finite subset A of E such that $B \subset U + \operatorname{co} A$.

By a *Fréchet-Montel space* we mean a Fréchet space E such that every bounded subset of E is compactoid.

Let E and F be locally convex spaces. An operator $T \in L(E, F)$ is *compact* if for some neighbourhood U of zero in E the set $T(U)$ is compactoid in F.

For any seminorm p on a locally convex space E the map $\overline{p}\colon E/\ker p \to [0, \infty)\, x+\ker p \to p(x)$ is a norm on $E_p = E/\ker p$.

A locally convex space E is nuclear if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geq p$ such that the map $\varphi_{q,p}\colon (E_q, \overline{q}) \to (E_p, \overline{p})\,, x + \ker q \to x + \ker p$ is compact. Any nuclear Fréchet space is a Fréchet-Montel space.

Let U be an absolutely convex neighbourhood of zero in a locally convex space E. The Minkowski functional of U, $p_U : E \to [0, \infty)$, $p_U(x) = \inf\{|\alpha| : \alpha \in \mathbb{K}$ and $x \in \alpha U\}$, is a continuous seminorm on E.

Let E be a locally convex space. If $A \subset E$ and $B \subset E'$, then we put $A^\circ = \{f \in E' : |f(x)| \leq 1$ for every $x \in A\}$ and ${}^\circ B = \{x \in E : |f(x)| \leq 1$ for every $f \in B\}$. For $A \subset E$ we put $A^e = \bigcap\{\lambda A : \lambda \in \mathbb{K}$ and $|\lambda| > 1\}$ if the set $|\mathbb{K}| = \{|\alpha| : \alpha \in \mathbb{K}\}$ is dense in $[0, \infty)$, and $A^e = A$ otherwise.

An infinite matrix $A = (a_{n,k})$ of real numbers is a *Köthe matrix* if $0 \leq a_{n,k} \leq a_{n,k+1}$ for all $n, k \in \mathbb{N}$, and $\sup_k a_{n,k} > 0$ for every $n \in \mathbb{N}$.

Let A be a Köthe matrix. The space

$$K(A) = \{(\alpha_n) \subset \mathbb{K} : \lim_{n\to\infty} |\alpha_n| a_{n,k} = 0 \text{ for every } k \in \mathbb{N}\}$$

with the base (p_k) of seminorms, where $p_k((\alpha_n)) = \max_n |\alpha_n| a_{n,k}$, $k \in \mathbb{N}$, is a Fréchet space. The sequence (e_j), where $e_j = (\delta_{j,n})$, is an unconditional Schauder basis in $K(A)$.

A Fréchet space E with a Schauder basis has *the quasi-equivalence property* if every two Schauder bases in E are quasi-equivalent.

Any infinite-dimensional Fréchet space E with a Schauder basis is isomorphic to $K(A)$ for some Köthe matrix (see [1], Proposition 2.4 and its proof).

Let Γ be the family of all non-decreasing sequences $a = (a_n)$ of positive real numbers with $\lim a_n = \infty$. Let $a = (a_n) \in \Gamma$. Then the following Fréchet spaces are nuclear (see [1, 18]:

- $A_1(a) = K(B)$ with $B = (b_{n,k})$, $b_{n,k} = e^{-a_n/k}$;
- $A_\infty(a) = K(B)$ with $B = (b_{n,k})$, $b_{n,k} = e^{ka_n}$.

$A_1(a)$ and $A_\infty(a)$ are the *power series spaces* (of *finite type* and *infinite type*, respectively).

The power series spaces have the quasi-equivalence property [16, Corollary 6].

Let E be a locally convex space. A linearly dense sequence (x_n) in E is an *orthogonal basis* in E if $(x_n) \subset (E \setminus \{0\})$ and if there is a base (p_k) in $\mathcal{P}(E)$ such that for all $k, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ we have $p_k(\sum_{i=1}^n \alpha_i x_i) = \max_{1 \le i \le n} p_k(\alpha_i x_i)$.

Every orthogonal basis in a locally convex space E is a Schauder basis and every Schauder basis in a Fréchet space is an orthogonal basis [4, Propositions 1.4 and 1.7].

Let $(E, \|\cdot\|)$ be a normed space and let $t \in (0,1)$. A sequence $(x_n) \subset E$ is *t-orthogonal* if for all $m \in \mathbb{N}, \alpha_1, \ldots, \alpha_m \in \mathbb{K}$ we have

$$\Big\| \sum_{i=1}^m \alpha_i x_i \| \ge t \max_{1 \le i \le m} \|\alpha_i x_i \Big\|.$$

If $(x_n) \subset (E \setminus \{0\})$ is t-orthogonal and linearly dense in E, then it is *t-orthogonal basis* in E. Every t-orthogonal basis in E is a Schauder basis.

3 Results

We start with the following.

Theorem 3.1 *Let E and F be Fréchet spaces and let $T \in L(E,F)$. Assume that there exists a linearly dense absolutely convex compactoid K in E and an absolutely convex neighbourhood U of zero in F such that p_U is a norm on F and the set*

$$\mathcal{W}_T = \{S \in L(E,F) : S(K) \subset T(K) \text{ and } T^{-1}(U) \subset S^{-1}(U)\}$$

is equicontinuous. Then the range of T has a Schauder basis.

Proof Clearly, we can assume that the range of T is infinite-dimensional. The completion D of the normed space $F_U = (F, p_U)$ is a Banach space and the set $V = T(K)$ is an absolutely convex compactoid in D. The closed linear span G of V in D is a Banach space of countable type. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$ and let $t \in \mathbb{R}$ with $|\alpha|^{-1} < t < 1$. Using [10, Lemma 4.36, Theorem 4.37], we infer that there exists a t-orthogonal sequence (g_n) in G with $(g_n) \subset (\alpha V) \setminus \{0\}$ such that the closure A of $\operatorname{co}\{g_n : n \in \mathbb{N}\}$ in G includes V and $\lim g_n = 0$ in G. Clearly, (g_n) is linearly dense in G, so it is a t-orthogonal basis in G. Let $(g_n^*) \subset G^*$ be the sequence of coefficient functionals associated with the Schauder basis (g_n) in G. Since $T(E) \subset \overline{[V]}^F \subset G$ we have $T(x) = \sum_{n=1}^\infty g_n^*(T(x))g_n$ in G for every $x \in E$. It is easy to check that $A = \{\sum_{n=1}^\infty \alpha_n g_n : (\alpha_n) \subset B_{\mathbb{K}}\}$. Thus $|g_n^* T(x)| \le 1$ for all $x \in K, n \in \mathbb{N}$.

The set $W = \alpha \overline{V}^F$ is an absolutely convex complete metrizable compactoid in F. By [12, Theorem 3.2], we get $\tau|_W = \tau_U|_W$, where τ and τ_U are topologies of F and F_U, respectively. Hence $\lim g_n = 0$ in F. Thus the series $\sum_{n=1}^\infty g_n^* T(x) g_n$ is convergent

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in F for every $x \in [K]$. Since $T(x) = \sum_{n=1}^{\infty} g_n^* T(x) g_n$ in F_U for every $x \in [K]$ and $\tau_U \subset \tau$ we have $T(x) = \sum_{n=1}^{\infty} g_n^* T(x) g_n$ in F for every $x \in [K]$.

Let $m \in \mathbb{N}$. Put $T_m \colon E \to F$, $T_m(x) = \sum_{n=1}^{m} g_n^* T(x) g_n$. Clearly $T_m \in L(E, F)$. For every $n \in \mathbb{N}$ there exists $z_n \in K$ such that $g_n = \alpha T(z_n)$. If $x \in K$, then $T_m(x) = \alpha T(\sum_{n=1}^{m} g_n^* T(x) z_n) \in \alpha T(K)$; so $T_m(K) \subset \alpha T(K)$. Let $x \in T^{-1}(U)$. Then $p_U(Tx) \le 1$, so $\max_n |g_n^* T(x)| p_U(g_n) \le t^{-1} p_U(Tx) < |\alpha|$. Hence $p_U(T_m(x)) < |\alpha|$, so $T_m(x) \in \alpha U$. Thus $T^{-1}(U) \subset \alpha T_m^{-1}(U)$.

We have shown that $(\alpha^{-1} T_m) \subset \mathcal{W}_T$, so $T_m, m \in \mathbb{N}$, are equicontinuous. Since $\lim_m T_m(x) = T(x)$ in F for every $x \in [K]$, we infer that $\lim_m T_m(x) = T(x)$ in F for every $x \in E$. Hence $T(x) = \sum_{n=1}^{\infty} g_n^* T(x) g_n$ in F for all $x \in E$.

If $\sum_{n=1}^{\infty} \alpha_n g_n = 0$ in F, then $\sum_{n=1}^{\infty} \alpha_n g_n = 0$ in G, so $\alpha_n = 0, n \in \mathbb{N}$. Thus (g_n) is a Schauder basis in $T(E)$. ■

By the first part of the proof of Theorem 3.1 we get the following.

Proposition 3.2 *Let F be a Fréchet space with a continuous norm. Then the linear span of every compactoid in F has a Schauder basis.*

Remark 3.3 Let F be a Fréchet space of countable type with a continuous norm and without a Schauder basis (see [14]). Let (x_n) be a linearly dense sequence in F. For some $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ we have $\lim_n \alpha_n x_n = 0$ in F. Then the closure X of $\mathrm{co}\{\alpha_n x_n : n \in \mathbb{N}\}$ in F is a closed absolutely convex compactoid in F and $[X]$ has no orthogonal basis. However, by Proposition 3.2, $[X]$ has a Schauder basis.

Using Proposition 3.2 we get the following.

Corollary 3.4 *Let E and F be Fréchet spaces. Assume that F has a continuous norm. Then the range of every compact linear operator T from E to F has a Schauder basis.*

Remark 3.5 We put $x/y = 0$, if $x = y = 0$; $x/y = \infty$, if $x > 0 = y$; and $x \cdot \infty = \infty$, if $x > 0$. If $0 \le a \le c$ and $0 \le d \le b$, then $a/b \le c/d$. If $a > 0, b > 0, c \ge 0$ and $d \ge 0$, then $(ac)/(bd) = (a/b)/(c/d)$.

Let E and F be Fréchet spaces with non-decreasing bases $(\|\cdot\|_s)$ and $(\|\cdot\|_t)$ in $\mathcal{P}(E)$ and $\mathcal{P}(F)$, respectively. Let $U_s = \{x \in E : \|x\|_s \le 1\}$ and $V_t = \{x \in F : \|x\|_t \le 1\}$ for $s, t \in \mathbb{N}$. For $T \in L(E, F)$, $D \subset E$ and $s, t \in \mathbb{N}$ we put $\|T\|_{D,t} = \sup_{y \in D} \|Ty\|_t$ and $\|T\|_{s,t} = \sup_{y \in U_s} \|Ty\|_t$.

Let E and F be Fréchet spaces. We shall write

— $(E, F) \in \mathfrak{R}$ if the range of every continuous linear operator T from E to F has a Schauder basis;

— $(E, F) \in \mathfrak{R}_1$ if there exist non-decreasing bases $(\|\cdot\|_s)$ and $(\|\cdot\|_t)$ in $\mathcal{P}(E)$ and $\mathcal{P}(F)$, respectively, and an absolutely convex compactoid D in E such that

$$\exists \mu \, \forall s \, \exists t \, \exists C \, \forall T \in L(E, F) : \|T\|_{t,s} \le C \max\{\|T\|_{D,t}, \|T\|_{s,\mu}\};$$

— $(E, F) \in \mathfrak{R}_2$ if there exist Köthe matrices A and B with $E \simeq K(A)$ and $F \simeq K(B)$

such that

$$\exists\mu\,\forall k\,\exists m\,\forall n\,\exists C\,\forall i,j:\ b_{j,k}/a_{i,m}\le C\max\{b_{j,m}/a_{i,n},b_{j,\mu}/a_{i,k}\}.$$

Theorem 3.6 *Let E and F be Fréchet spaces of countable type such that* $(E,F)\in\mathfrak{R}_1$. *Then* $(E,F)\in\mathfrak{R}$.

Proof Let D be an absolutely convex compactoid in E such that

$$\exists\mu\,\forall k\,\exists m\,\exists C\,\forall T\in L(E,F):\ \|T\|_{m,k}\le C\max\{\|T\|_{D,m},\|T\|_{k,\mu}\}. \tag{3.1}$$

Consider three cases.

Case 1: D is not linearly dense in E. Then F is normable. Indeed, the closure G of the linear span of D in E is weakly closed in E [2, p. 257]. Thus there exists $f\in(E'\setminus\{0\})$ with $f(G)=\{0\}$. Let $k\ge\mu$ with $f(U_k)\subset\gamma B_{\mathbb{K}}$ for some $\gamma\in\mathbb{K}$. Then for some m and C we have

$$\forall T\in L(E,F):\|T\|_{m,k}\le C\max\{\|T\|_{D,m},\|T\|_{k,\mu}\}. \tag{3.2}$$

Let $y\in V_\mu$. Put $T\colon E\to F,T(x)=f(x)y$. Clearly, $T\in L(E,F),\|T\|_{D,m}=0$ and $\|T\|_{k,\mu}\le|\gamma|$. Thus $\|T\|_{m,k}\le C|\gamma|$. Let $\beta\in(f(U_m)\setminus\{0\})$. Then $\beta y=Tz$ for some $z\in U_m$, so $\|\beta y\|_k\le\|T\|_{m,k}\le C|\gamma|$. Hence $\|y\|_k\le C|\gamma\beta^{-1}|$, thus $V_\mu\subset\lambda V_k$ for some $\lambda\in\mathbb{K}$. It follows that for every $k\ge\mu$ the seminorm $\|\cdot\|_k$ is equivalent to $\|\cdot\|_\mu$, so F is normable. Thus $(E,F)\in\mathfrak{R}$, since every normed space of countable type has a t-orthogonal basis for $t\in(0,1)$ [10, Theorem 3.16 and its proof].

Case 2: $\|\cdot\|_\mu$ is not a norm. Then E is finite-dimensional. Indeed, let $y\in(F\setminus\{0\})$ with $\|y\|_\mu=0$. Let $k\in\mathbb{N}$ with $y\notin\lambda V_k$ for some $\lambda\in(\mathbb{K}\setminus\{0\})$. For some m and $C>1$ we have (3.2). Let $\beta\in\mathbb{K}$ with $|\beta|>C$ such that $y\in\beta V_m$. Let $f\in D^\circ$ and $T\colon E\to F,T(x)=f(x)y$. Clearly, $T\in L(E,F),\|T\|_{k,\mu}=0$ and $\|T\|_{D,m}\le|\beta|$. Thus $\|T\|_{m,k}\le|\beta|^2$, so $f(U_m)y\subset\beta^2V_k$. Hence $f\in(\lambda\beta^{-2}U_m)^\circ$, since $y\notin\lambda V_k$. Thus $D^\circ\subset(\lambda\beta^{-2}U_m)^\circ$, so $\lambda\beta^{-2}U_m\subset{}^\circ(D^\circ)=(\overline{D})^e\subset\beta\overline{D}$ [11, Corollary 4.9, Proposition 4.10]. It follows that E has a compactoid neigbourhood of zero. By [3, Proposition 0.3], E is finite-dimensional; so $(E,F)\in\mathfrak{R}$.

Case 3: D is linearly dense in E and $\|\cdot\|_\mu$ is a norm on F. Let $\beta\in\mathbb{K}$ with $|\beta|>1$. Let $T\in L(E,F)$ and $\mathcal{W}_T=\{S\in L(E,F):S(D)\subset T(D)\text{ and }T^{-1}(V_\mu)\subset S^{-1}(V_\mu)\}$. For all $k,m\in\mathbb{N}$ and $S\in\mathcal{W}_T$ we have $\|S\|_{D,m}\le\|T\|_{D,m}$ and $\|Sx\|_\mu\le|\beta|\|Tx\|_\mu,x\in E$, so $\|S\|_{k,\mu}\le|\beta|\|T\|_{k,\mu}$. Clearly, $\|T\|_{D,m}<\infty$ for every $m\in\mathbb{N}$ and there exists $k_0\in\mathbb{N}$ such that $\|T\|_{k,\mu}<\infty$ for every $k\ge k_0$. Hence, using (3.1), we infer that

$$\exists k_0\forall k\ge k_0\,\exists m\,\exists C\,\forall S\in\mathcal{W}_T:\ \|S\|_{m,k}\le C\max\{\|T\|_{D,m},|\beta|\|T\|_{k,\mu}\}.$$

Thus the set $\mathcal{W}_T$ is equicontinuous. By Theorem 3.1, the range of T has a Schauder basis. It follows that $(E,F)\in\mathfrak{R}$. ■

Theorem 3.7 *Let E be a Fréchet-Montel space and let F be a Fréchet space. Assume that* $(E,F)\in\mathfrak{R}_2$. *Then* $(E,F)\in\mathfrak{R}$.

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Proof Let A and B be Köthe matrices with $E \simeq K(A)$ and $F \simeq K(B)$ such that

$$\exists \mu\, \forall k\, \exists m\, \forall n\, \exists C\, \forall i, j: \; b_{j,k}/a_{i,m} \le C \max\{b_{j,m}/a_{i,n}, b_{j,\mu}/a_{i,k}\} \tag{3.3}$$

Without loss of generality we can assume that $K(B)$ is non-normable (see the proof of Theorem 3.6). and $a_{k,n}, b_{k,n} \in |\mathbb{K}|$ for all $k, n \in \mathbb{N}$. Clearly, it is enough to show that $(K(A), K(B)) \in \mathfrak{R}$.

Let $k \in \mathbb{N}$. For some $t > k$ we have $\sup_j b_{j,t}/b_{j,\mu} = \infty$. By (3.3) there exists $s > t$ such that

$$\forall n\, \exists \overline{C_n} > 0\, \forall i, j: \; \frac{b_{j,t}}{a_{i,s}} \le \overline{C_n} \max\Big\{\frac{b_{j,s}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,t}}\Big\}$$

and there is $m \ge \mu$ such that

$$\forall n\, \exists \hat{C}_n > 0\, \forall i, j: \; \frac{b_{j,s}}{a_{i,m}} \le \hat{C}_n \max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,s}}\Big\}.$$

Let $n \in \mathbb{N}$. For some $j_0 \in \mathbb{N}$ we have $b_{j_0,t}/b_{j_0,\mu} > \overline{C_n}\hat{C}_n$; clearly $b_{j_0,t} > 0$. Let $i, j \in \mathbb{N}$. Then $b_{j_0,t}/a_{i,s} \le \overline{C_n} \max\{b_{j_0,s}/a_{i,n}, b_{j_0,\mu}/a_{i,t}\}$. Hence

$$\frac{b_{j,\mu}}{a_{i,s}} \le \overline{C_n} \max\Big\{\frac{b_{j_0,s} b_{j,\mu}}{b_{j_0,t} a_{i,n}}, \frac{b_{j_0,\mu} b_{j,\mu}}{b_{j_0,t} a_{i,t}}\Big\},$$

so

$$\hat{C}_n \frac{b_{j,\mu}}{a_{i,s}} \le \max\Big\{D_n \frac{b_{j,\mu}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,t}}\Big\} \le \max\Big\{D_n \frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}$$

for $D_n = \hat{C}_n \overline{C_n} b_{j_0,s}/b_{j_0,t}$. It follows that

$$\frac{b_{j,k}}{a_{i,m}} \le \frac{b_{j,s}}{a_{i,m}} \le \max\Big\{C'_n \frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\},$$

where $C'_n = \max\{\hat{C}_n, D_n\}$.

We have shown that

$$\exists \mu\, \forall k\, \exists m\, \forall n\, \exists C_{k,n} > 1\, \forall i, j: \; \frac{b_{j,k}}{a_{i,m}} \le \max\Big\{C_{k,n} \frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}. \tag{3.4}$$

Let $\beta \in \mathbb{K}$ with $|\beta| > 1$. Let $C_t = \max(\{C_{k,n} : k \le t, n \le t\} \cup \{a_{i,k} : i \le t, k \le t\})$ for all $t \in \mathbb{N}$. Then $C_{k,n} \le C_k C_n$ for all $k, n \in \mathbb{N}$, and $d_i = \inf_t C_t/a_{i,t} > 0$ for every $i \in \mathbb{N}$. For $i \in \mathbb{N}$ let $x_i \in \mathbb{K}$ with $d_i < |x_i| \le |\beta| d_i$.

We shall prove that $x = (x_i) \in K(A)$. Let $k \in \mathbb{N}$. Then $|x_i| a_{i,t} \le C_t |\beta|$ for all $i, t \in \mathbb{N}$. Let W be an infinite subset of $\mathbb{N}$. The space $K(A)$ has no infinite-dimensional normable closed subspace [5, Corollary 6.7]. Thus $\sup_{i \in W} a_{i,\bar{k}}/a_{i,k} = \infty$ for some $\bar{k} > k$. Hence $\inf_{i \in W} |x_i| a_{i,k} = \inf_{i \in W} |x_i| a_{i,\bar{k}} (a_{i,k}/a_{i,\bar{k}}) \le C_{\bar{k}} |\beta| \inf_{i \in W} (a_{i,k}/a_{i,\bar{k}}) = 0$. It follows that $\lim_i |x_i| a_{i,k} = 0$ for every $k \in \mathbb{N}$, so $x \in K(A)$.

The set $D = \{(y_i) \in K(A) : |y_i| \le |x_i| \text{ for every } i \in \mathbb{N}\}$ is compactoid i n $K(A)$ [7, Theorem 2.5]. Let $k \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that

$$\forall n \forall i,j: \frac{b_{j,k}}{a_{i,m}} \le \max\Big\{C_{k,n}\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}.$$

Let $i, j \in \mathbb{N}$. For some $n \in \mathbb{N}$ we have $C_n/a_{i,n} < |x_i|$. Hence

$$C_{k,n}/a_{i,n} = (C_n/a_{i,n})(C_{k,n}/C_n) < |x_i|C_k;$$

thus

$$\forall i,j: \frac{b_{j,k}}{a_{i,m}} \le \max\Big\{C_k|x_i|b_{j,m}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}.$$

We have shown that

$$\forall k \exists m \exists C \forall i,j: \frac{b_{j,k}}{a_{i,m}} \le C\max\Big\{|x_i|b_{j,m}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}.$$

Let $T \in L(K(A), K(B))$. Let (e_i) and (f_j) be the coordinate Schauder bases in $K(A)$ and $K(B)$, respectively. For every $i \in \mathbb{N}$ there exists $(T_{i,j})_{j=1}^{\infty} \subset \mathbb{K}$ such that $Te_i = \sum_{j=1}^{\infty} T_{i,j}f_j$; clearly, $\|Te_i\|_t = \max_j |T_{i,j}|b_{j,t}$ for all $i, t \in \mathbb{N}$. Let $s, t \in \mathbb{N}$. Put $d_{s,t} = \sup_{i,j} |T_{i,j}|b_{j,t}/a_{i,s}$.

Consider two cases:

Case 1: There exists $i \in \mathbb{N}$ with $a_{i,s} = 0$ such that $\|Te_i\|_t > 0$. Then for every $\alpha \in \mathbb{K}$ we have $\alpha e_i \in U_s$, so $\|T\|_{s,t} = \sup_{y \in U_s} \|Ty\|_t = \infty = \|Te_i\|_t/a_{i,s} \le d_{s,t}$. Hence $\|T\|_{s,t} = d_{s,t}$.

Case 2: For every $i \in \mathbb{N}$ with $a_{i,s} = 0$, we have $\|Te_i\|_t = 0$. Put $W = \{i \in \mathbb{N} : a_{i,s} > 0\}$. Let $y \in U_s$. Then $\|y\|_s = \max_{i \in \mathbb{N}} |y_i|a_{i,s} \le 1$ and

$$\|Ty\|_t = \|\sum_{i=1}^{\infty} y_i Te_i\|_t \le \max_{i \in \mathbb{N}} |y_i|\|Te_i\|_t = \max_{i \in W} |y_i|\|Te_i\|_t \le \sup_{i \in W} \frac{\|Te_i\|_t}{a_{i,s}}.$$

For every $i \in \mathbb{N}$ there exists $\alpha_{i,s} \in \mathbb{K}$ with $|\alpha_{i,s}| = a_{i,s}$. Hence for every $i \in W$ we have $\alpha_{i,s}^{-1}e_i \in U_s$ and $\|T(\alpha_{i,s}^{-1}e_i)\|_t = \|Te_i\|_t/a_{i,s}$. If follows that

$$\|T\|_{s,t} = \sup_{y \in U_s} \|Ty\|_t = \sup_{i \in W} \frac{\|Te_i\|_t}{a_{i,s}} = \sup_{i \in \mathbb{N}} \frac{\|Te_i\|_t}{a_{i,s}} = d_{s,t}.$$

We have shown that $\|T\|_{s,t} = \sup_{i,j} |T_{i,j}|b_{j,t}/a_{i,s}$ for all $s, t \in \mathbb{N}$.

Let $t \in \mathbb{N}$. For $y \in D$ we have

$$\|Ty\|_t = \|\sum_{i=1}^{\infty} y_i Te_i\|_t \le \max_{i \in \mathbb{N}} |y_i|\|Te_i\|_t \le \max_{i \in \mathbb{N}} |x_i|\|Te_i\|_t \le \sup_{i,j} |T_{i,j}||x_i|b_{j,t}.$$

Clearly, $x_i e_i \in D$ and $\|T(x_i e_i)\|_t = |x_i|\|Te_i\|_t$ for every $i \in \mathbb{N}$. It follows that $\|T\|_{D,t} = \sup_{y\in D}\|Ty\|_t = \sup_{i,j}|T_{i,j}||x_i|b_{j,t}$.

Let $k \in \mathbb{N}$. Using (3.4) we get $m \in \mathbb{N}$ and C such that

$$\|T\|_{m,k} = \sup_{i,j}\frac{|T_{i,j}|b_{j,k}}{a_{i,m}} \le C\sup_{i,j}\max\Big\{|T_{i,j}||x_i|b_{j,m}, \frac{|T_{i,j}|b_{j,\mu}}{a_{i,k}}\Big\}$$
$$\le C\max\Big\{\sup_{i,j}|T_{i,j}||x_i|b_{j,m}, \sup_{i,j}\frac{|T_{i,j}|b_{j,\mu}}{a_{i,k}}\Big\} = C\max\{\|T\|_{D,m}, \|T\|_{k,\mu}\}$$

for every $T \in L(K(A), K(B))$. Thus we have proved that $(K(A), K(B)) \in \mathfrak{R}_1$. By Theorem 3.6 we infer that $(K(A), K(B)) \in \mathfrak{R}$. ■

By the proof of Theorem 3.7 we get the following.

Corollary 3.8 *Let E be a Fréchet-Montel space and F a non-normable Fréchet space. If $(E,F) \in \mathfrak{R}_2$, then $(E,F) \in \mathfrak{R}_1$.*

Now we shall prove the following result.

Proposition 3.9 *Let A and B be Köthe matrices such that the Fréchet spaces $E = K(A)$ and $F = K(B)$ have the quasi-equivalence property. Then $(E,F) \in \mathfrak{R}_2$ if and only if*

$$\exists\mu\,\forall k\,\exists m\,\forall n\,\exists C\,\forall i,j:\ \frac{b_{j,k}}{a_{i,m}} \le C\max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}.$$

Proof Assume that $(E,F) \in \mathfrak{R}_2$. Then there exist Köthe matrices A' and B' with $K(A') \simeq E$ and $K(B') \simeq F$ such that

$$\exists\mu'\,\forall k\,\exists m\,\forall n\,\exists C\,\forall i,j:\ \frac{b'_{j,k}}{a'_{i,m}} \le C\max\Big\{\frac{b'_{j,m}}{a'_{i,n}}, \frac{b'_{j,\mu'}}{a'_{i,k}}\Big\}. \tag{3.5}$$

Let $T\colon K(A') \to K(A)$ be an isomorphism. Let (e_i) and (e'_i) be the coordinate bases in $K(A)$ and $K(A')$, respectively. Clearly, $(f_i) = (T(e'_i))$ is a Schauder basis in $K(A)$, so it is quasi-equivalent to (e_i). Thus there exist $(\alpha_i) \subset \mathbb{K}\setminus\{0\}$ and a permutation π of $\mathbb{N}$ such that $(\alpha_i f_{\pi(i)})$ and (e_i) are equivalent. Therefore there is an isomorphism $P\colon K(A') \to K(A)$ with $P(\alpha_i e'_{\pi(i)}) = e_i$ for $i \in \mathbb{N}$. Hence

$$\forall k\,\exists t\,\exists C\,\forall i:\ a_{i,k} \le C|\alpha_i|a'_{\pi(i),t} \text{ and } |\alpha_i|a'_{\pi(i),k} \le Ca_{i,t}.$$

Similarly there exist $(\beta_j) \subset (\mathbb{K}\setminus\{0\})$ and a permutation σ of $\mathbb{N}$ such that

$$\forall k\,\exists t\,\exists C\,\forall j:\ b_{j,k} \le C|\beta_j|b'_{\sigma(j),t} \text{ and } |\beta_j|b'_{\sigma(j),k} \le Cb_{j,t}.$$

Hence $\exists\mu\,\exists C_1\,\forall j:\ |\beta_j|b'_{\sigma(j),\mu'} \le C_1 b_{j,\mu}$. Let $k \in \mathbb{N}$. Then

$$\exists k'\,\exists C_2\,\forall i,j:\ a_{i,k} \le C_2|\alpha_i|a'_{\pi(i),k'} \text{ and } b_{j,k} \le C_2|\beta_j|b'_{\sigma(j),k'}.$$

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By (3.5) we get $m' \in \mathbb{N}$ such that

$$\forall n\,\exists C_3\,\forall i,j:\ \frac{b'_{j,k'}}{a'_{i,m'}} \le C_3 \max\Big\{\frac{b'_{j,m'}}{a'_{i,n}}, \frac{b'_{j,\mu'}}{a'_{i,k'}}\Big\}.$$

Moreover $\exists \nu\,\exists C_4\,\forall i:\ |\alpha_i|a'_{\pi(i),m'} \le C_4 a_{i,\nu}$ and $\exists m \ge \nu\,\exists C_5\,\forall j:\ |\beta_j|b'_{\sigma(j),m'} \le C_5 b_{j,m}$. Let $n \in \mathbb{N}$. Then $\exists n'\,\exists C_6\,\forall i:\ a_{i,n} \le C_6|\alpha_i|a'_{\pi(i),n'}$. Thus for all $i, j \in \mathbb{N}$ we have

$$\begin{aligned}\frac{b_{j,k}}{a_{i,m}} \le \frac{b_{j,k}}{a_{i,\nu}} &\le \frac{C_2|\beta_j|}{C_4^{-1}|\alpha_i|}\frac{b'_{\sigma(j),k'}}{a'_{\pi(i),m'}}\\ &\le C_2C_3C_4\frac{|\beta_j|}{|\alpha_i|}\max\Big\{\frac{b'_{\sigma(j),m'}}{a'_{\pi(i),n'}}, \frac{b'_{\sigma(j),\mu'}}{a'_{\pi(i),k'}}\Big\}\\ &\le C_2C_3C_4\max\Big\{\frac{C_5 b_{j,m}}{C_6^{-1}a_{i,n}}, \frac{C_1 b_{j,\mu}}{C_2^{-1}a_{i,k}}\Big\}.\end{aligned}$$

Therefore

$$\exists \mu\,\forall k\,\exists m\,\forall n\,\exists C\,\forall i,j:\ \frac{b_{j,k}}{a_{i,m}} \le C\max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\}.$$

The converse implication is obvious. ∎

For the power series spaces we get the following.

Theorem 3.10 *Let $a, b \in \Gamma$ and $p, q \in \{1, \infty\}$. If $p = 1$, then $(A_p(a), A_q(b)) \in \mathfrak{R}_2$. If $p = \infty$, then $(A_p(a), A_q(b)) \in \mathfrak{R}_2$ if and only if the set $M_{b,a}$ of all finite limit points of the double sequence $(b_i/a_j)_{i,j\in\mathbb{N}}$ is bounded.*

Proof Let A and B be the Köthe matrices that define $A_p(a)$ and $A_q(b)$, respectively.

Assume that $p = 1$ and $q = 1$. Let $k \in \mathbb{N}$ and $m = 2k^2$. Let $n, i, j \in \mathbb{N}$. If $a_i < kb_j$, then $-(b_j/k) + (a_i/m) \le -(b_j/m) + (a_i/n)$; if $a_i \ge kb_j$, then $-(b_j/k) + (a_i/m) \le -b_j + (a_i/k)$. Thus for all $n, i, j \in \mathbb{N}$ we have

$$-(b_j/k) + (a_i/m) \le \max\{-(b_j/m) + (a_i/n), -b_j + (a_i/k)\},$$

so $e^{-b_j/k}e^{a_i/m} \le \max\{e^{-b_j/m}e^{a_i/n}, e^{-b_j}e^{a_i/k}\}$. We have shown that

$$\forall k\,\exists m\,\forall n\,\forall i,j:\ \frac{b_{j,k}}{a_{i,m}} \le \max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,1}}{a_{i,k}}\Big\},$$

so $(A_p(a), A_q(b)) \in \mathfrak{R}_2$.

Assume that $p = 1$ and $q = \infty$. Let $k \in \mathbb{N}$ and $m = 2k$. Let $n, i, j \in \mathbb{N}$. If $a_i < 2k^2 b_j$, then $kb_j + (a_i/m) \le mb_j + (a_i/n)$; if $a_i \ge 2k^2 b_j$, then $kb_j + (a_i/m) \le b_j + (a_i/k)$. Thus for all $n, i, j \in \mathbb{N}$ we get $kb_j + (a_i/m) \le \max\{mb_j + (a_i/n), b_j + (a_i/k)\}$, hence $e^{kb_j}e^{a_i/m} \le \max\{e^{mb_j}e^{a_i/n}, e^{b_j}e^{a_i/k}\}$. We have proved that

$$\forall k\,\exists m\,\forall n\,\forall i,j:\ \frac{b_{j,k}}{a_{i,m}} \le \max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,1}}{a_{i,k}}\Big\},$$

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so $(A_p(a), A_q(b)) \in \Re_2$.

Assume that $p = \infty$ and $M_{b,a}$ is bounded. Let $L \in \mathbb{N}$ with $L > \sup M_{b,a}$ and $b_0 = 0$. Then for every $i \in \mathbb{N}$ there exists $t_i \in \mathbb{N}$ such that $b_{t_i-1} \le La_i < b_{t_i}$. By the definition of L and $M_{b,a}$ we get $\lim_i b_{t_i}/a_i = \infty$. If $k, n \in \mathbb{N}$, then there exists $i(k,n) \in \mathbb{N}$ such that $b_{t_i} > 2kna_i$ for all $i \ge i(k,n)$. Put $C_{k,n} = e^{na_{i(k,n)}}$.

Case 1: $q = 1$. Let $k \in \mathbb{N}$ and $m = 2k + L$. Let $n \in \mathbb{N}$. Let $i \in \mathbb{N}$. If $i < i(k,n)$, then $-(b_j/k) - ma_i \le na_{i(k,n)} - (b_j/m) - na_i$ for all $j \in \mathbb{N}$. Let $i \ge i(k,n)$; then $-(b_j/k) - ma_i \le -(b_j/m) - na_i$ for all $j \ge t_i$ and $-(b_j/k) - ma_i \le -b_j - ka_i$ for all $j < t_i$. Hence for all $i, j \in \mathbb{N}$ we have

$$e^{-b_j/k}e^{-ma_i} \le C_{k,n} \max\{e^{-b_j/m}e^{-na_i}, e^{-b_j}e^{-ka_i}\}.$$

Thus

$$\forall k\, \exists m\, \forall n\, \exists C\, \forall i, j : \frac{b_{j,k}}{a_{i,m}} \le C \max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,1}}{a_{i,k}}\Big\},$$

so $(A_p(a), A_q(b)) \in \Re_2$.

Case 2: $q = \infty$. Let $k \in \mathbb{N}$ and $m = k(L + 1)$. Let $n \in \mathbb{N}$. Let $i \in \mathbb{N}$. If $i < i(k,n)$, then $kb_j - ma_i \le na_{i(k,n)} + mb_j - na_i$ for all $j \in \mathbb{N}$. Let $i \ge i(k,n)$; then $kb_j - ma_i \le mb_j - na_i$ for all $j \ge t_i$, and $kb_j - ma_i \le b_j - ka_i$ for all $j < t_i$. Hence for all $i, j \in \mathbb{N}$ we have $e^{kb_j}e^{-ma_i} \le C_{k,n} \max\{e^{mb_j}e^{-na_i}, e^{b_j}e^{-ka_i}\}$. Thus

$$\forall k\, \exists m\, \forall n\, \exists C\, \forall i, j : \frac{b_{j,k}}{a_{i,m}} \le C \max\left\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,1}}{a_{i,k}}\right\},$$

so $(A_p(a), A_q(b)) \in \Re_2$.

Assume that $p = \infty$ and $(A_p(a), A_q(b)) \in \Re_2$. Let $(s_k) = (-1/k)$ if $q = 1$ and $(s_k) = (k)$ if $q = \infty$. By Proposition 9 and [16], Corollary 6, we get

$$\exists \mu\, \forall k\, \exists m\, \forall n\, \exists C\, \forall i, j : \frac{b_{j,k}}{a_{i,m}} \le C \max\Big\{\frac{b_{j,m}}{a_{i,n}}, \frac{b_{j,\mu}}{a_{i,k}}\Big\};$$

hence

$$\exists \mu \forall k \exists m \forall n \exists C_1 \forall i, j : s_k b_j - ma_i \le C_1 + \max\{s_m b_j - na_i, s_\mu b_j - ka_i\}.$$

Thus for $k = \mu + 1$ we have

$$\exists m\, \forall n\, \exists C_1\, \forall i, j : s_{\mu+1}\frac{b_j}{a_i} - m \le \frac{C_1}{a_i} + \max\Big\{s_m\frac{b_j}{a_i} - n, s_\mu\frac{b_j}{a_i}\Big\}.$$

Hence for every $x \in M_{b,a}$ we get $s_{\mu+1}x - m \le \max\{s_m x - n, s_\mu x\}$ for all $n \in \mathbb{N}$. Taking enough large n we obtain $s_{\mu+1}x - m \le s_\mu x$, so $x \le m/(s_{\mu+1} - s_\mu)$. Thus $M_{b,a}$ is bounded. ∎

By Theorems 3.7 and 3.10 we get the following two corollaries.

Corollary 3.11 *Let $a, b \in \Gamma$ and $p, q \in \{1, \infty\}$. Then the range of every continuous linear map from $A_p(a)$ to $A_q(b)$ has a Schauder basis, if either $p = 1$ or $p = \infty$ and the set $M_{b,a}$ is bounded.*

Corollary 3.12 *Let $a, b \in \Gamma$ and $p, q \in \{1, \infty\}$. Let F be a closed subspace of $A_q(b)$. Assume that F is isomorphic to a quotient of $A_p(a)$. Then F has a Schauder basis, if either $p = 1$ or $p = \infty$ and the set $M_{b,a}$ is bounded.*

Using Corollary 3.12 we obtain our next result.

Corollary 3.13 *Let $b \in \Gamma$ and $p \in \{1, \infty\}$. Every complemented subspace of $A_p(b)$ has a Schauder basis, if either $p = 1$ or $p = \infty$ and the set $M_{b,b}$ is bounded.*

By Corollary 3.13 and the quasi-equivalence property of $A_p(b)$ [16, Corollary 6] we get the following.

Proposition 3.14 *Let $b \in \Gamma$ and $p \in \{1, \infty\}$. Then every complemented subspace F of $A_p(b)$ is isomorphic to $A_p(a)$ for some subsequence a of b, if either $p = 1$ or $p = \infty$ and the set $M_{b,b}$ is bounded.*

Proof Let G be a complement of F in $A_p(b)$. By Corollary 3.13, F and G have Schauder bases (x_n) and (y_n), respectively. Put $z_{2n} = x_n$ and $z_{2n-1} = y_n$ for $n \in \mathbb{N}$. Clearly, (z_n) is a Schauder basis in $A_p(b)$. Thus there exist $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ and a permutation π of $\mathbb{N}$ such that (z_n) is equivalent to $(\alpha_n e_{\pi(n)})$. Hence F is isomorphic to the closed linear span H of $(e_{\pi(2n)})$; clearly, H is isomorphic to $A_p(a)$, where a is the non-decreasing rearrangement of $(b_{\pi(2n)})$. ■

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