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THE BLOCH SPACE AND BESOV SPACES OF ANALYTIC FUNCTIONS

KAREL STROETHOFF

We shall give an elementary proof of a characterisation for the Bloch space due to Holland and Walsh, and obtain analogous characterisations for the little Bloch space and Besov spaces of analytic functions on the unit disk in the complex plane.

1. INTRODUCTION

We let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and use dA for the normalised Lebesgue area measure on $\mathbb{D}$. For $1 < p < \infty$, an analytic function f is in the *Besov space* $\mathcal{B}_p$ if

$$\|f\|_{\mathcal{B}_p} := \left(\int_{\mathbb{D}} |f'(z)|^p \left(1 - |z|^2\right)^{p-2} dA(z)\right)^{1/p} < \infty,$$

that is, the function $\left(1 - |z|^2\right) f' \in L^p(\mathbb{D}, d\lambda)$, where $d\lambda(z) = \left(1 - |z|^2\right)^{-2} dA(z)$. We note that the measure λ is not a finite measure on $\mathbb{D}$; its importance stems from the fact that it is Möbius–invariant. To make this precise we need more notation. For $w \in \mathbb{D}$ the Möbius transformation φ_w is defined by

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}, \text{ for } z \in \mathbb{D}.$$

The following identity is easily verified:

$$1 - |\varphi_w(z)|^2 = \frac{\left(1 - |w|^2\right)\left(1 - |z|^2\right)}{|1 - \overline{w}z|^2}. \tag{1}$$

So, the function φ_w maps $\mathbb{D}$ into itself. It is furthermore easy to verify that φ_w is its own inverse. Noting that $\varphi_w'(z) = \left(|w|^2 - 1\right)/(1 - \overline{w}z)^2$, the above identity states:

$$\left(1 - |z|^2\right) |\varphi_w'(z)| = 1 - |\varphi_w(z)|^2, \tag{2}$$

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and thus $d\lambda(\varphi_w(z)) = \left(1-|\varphi_w(z)|^2\right)^{-2}|\varphi_w'(z)|^2\,dA(z) = \left(1-|z|^2\right)^{-2}dA(z) = d\lambda(z)$. Hence we have the following change-of-variable formula:

$$\int_{\mathbb{D}} h(\varphi_w(z))\,d\lambda(z) = \int_{\mathbb{D}} h(u)\,d\lambda(u), \tag{3}$$

where h is a positive measurable function on $\mathbb{D}$. Using (3) it is easily seen that $\|f\circ\varphi_w\|_{\mathcal{B}_p} = \|f\|_{\mathcal{B}_p}$, and consequently, if $f\in\mathcal{B}_p$, then $f\circ\varphi_w\in\mathcal{B}_p$, for all $w\in\mathbb{D}$. The Besov space $\mathcal{B}_1$ is defined differently: it is the set of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}_1} := \int_{\mathbb{D}} |f''(z)|\,dA(z) < \infty.$$

Even though the above semi-norm is not Möbius-invariant, the Besov space $\mathcal{B}_1$ is; that is, if $f\in\mathcal{B}_1$, then $f\circ\varphi_w\in\mathcal{B}_1$, for all $w\in\mathbb{D}$. Another Möbius–invariant space of analytic functions on $\mathbb{D}$ is the *Bloch space* $\mathcal{B}$; it is the set of all analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f'(z)| < \infty.$$

That $\mathcal{B}$ is Möbius–invariant is most easily seen from the observation that $\left(1-|z|^2\right)|f'(z)| = \left|(f\circ\varphi_z)'(0)\right|$, so that $\|f\|_{\mathcal{B}} = \sup\{\left|(f\circ\varphi_z)'(0)\right| : z\in\mathbb{D}\}$, and thus $\|f\circ\varphi_w\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$, for all $w\in\mathbb{D}$. Note that $\mathcal{B}_1\subset\mathcal{B}_p\subset\mathcal{B}$ for each $1<p<\infty$. It is easy to prove that all these spaces are Banach spaces. Rubel and Timoney [**9**] have actually shown that the Bloch space $\mathcal{B}$ is maximal among all Möbius–invariant Banach spaces of analytic functions on $\mathbb{D}$ (provided there are so-called "decent" linear functionals). In [**2**] Arazy and Fisher have shown that the Besov space $\mathcal{B}_1$ is minimal among all Möbius–invariant Banach spaces of analytic functions on $\mathbb{D}$ (see also [**3**]). The Besov space $\mathcal{B}_2$, more often referred to as the *Dirichlet space*, is easily seen to be a Hilbert space. In [4] Arazy and Fisher proved that $\mathcal{B}_2$ is the only Möbius–invariant Hilbert space of analytic functions on $\mathbb{D}$.

In the next section we shall state and prove a criterion for containment in the Bloch space obtained by Holland and Walsh [**8**]. In section 3 we shall prove a characterisation for the analytic Besov spaces analogous to the criterion of Holland and Walsh for containment in the Bloch space.

2. The Bloch Space

In [**8**] Holland and Walsh obtained the following characterisation for the Bloch space.

Theorem 1. *For an analytic function f on $\mathbb{D}$:*

$$f\in\mathcal{B} \iff \sup\left\{\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}\left|\frac{f(z)-f(w)}{z-w}\right| : z,w\in\mathbb{D},\ z\neq w\right\} < \infty.$$

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Holland and Walsh's proof of the above result is quite complicated. We shall show how the Möbius–invariance can be exploited to obtain a very easy proof of the above theorem.

PROOF OF THEOREM 1: The implication "$\Leftarrow$" is trivial. To prove the other implication, suppose that $f \in \mathcal{B}$. Then

$$|f(u) - f(0)| \leqslant \int_0^1 |u|\,|f'(tu)|\,dt \leqslant \|f\|_{\mathcal{B}} \int_0^1 \frac{|u|}{1 - t^2\,|u|^2}\,dt = \|f\|_{\mathcal{B}}\,\tfrac{1}{2}\log\frac{1+|u|}{1-|u|},$$

for each $u \in \mathbb{D}$. Now

$$\begin{aligned}\frac{1}{2}\log\frac{1+|u|}{1-|u|} &= \tfrac{1}{2}\log\frac{(1+|u|)^2}{1-|u|^2} = \log\frac{1+|u|}{\left(1-|u|^2\right)^{1/2}}\\ &\leqslant \frac{1+|u|}{\left(1-|u|^2\right)^{1/2}} - 1 = \frac{1+|u| - \left(1-|u|^2\right)^{1/2}}{\left(1-|u|^2\right)^{1/2}}\\ &\leqslant \frac{1+|u|-(1-|u|)}{\left(1-|u|^2\right)^{1/2}} = \frac{2\,|u|}{\left(1-|u|^2\right)^{1/2}},\end{aligned}$$

where we used the inequality $\left(1-|u|^2\right)^{1/2} \geqslant 1-|u|$. It follows that

$$|f(u) - f(0)| \leqslant \|f\|_{\mathcal{B}}\,\frac{2\,|u|}{\left(1-|u|^2\right)^{1/2}},$$

for all $u \in \mathbb{D}$. For $z, w \in \mathbb{D}$, replace f in the above inequality by $f \circ \varphi_w$ and let $u = \varphi_w(z)$. Using $\varphi_w(\varphi_w(z)) = z$, $\|f \circ \varphi_w\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ and identity (1), we get

$$|f(z) - f(w)| \leqslant \|f\|_{\mathcal{B}}\,\frac{2\,|\varphi_w(z)|}{\left(1-|\varphi_w(z)|^2\right)^{1/2}} = 2\,\|f\|_{\mathcal{B}}\,\frac{|z-w|}{\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}},$$

and thus

$$\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}\left|\frac{f(z)-f(w)}{z-w}\right| \leqslant 2\,\|f\|_{\mathcal{B}}, \tag{4}$$

for distinct $z, w \in \mathbb{D}$, completing the proof of Theorem 1. ∎

Before we consider the analytic Besov spaces in the next section, we briefly discuss the *little Bloch space* $\mathcal{B}_0$, the set of all analytic functions f on $\mathbb{D}$ for which

$$\lim_{|z| \to 1^-} \left(1-|z|^2\right) f'(z) = 0.$$

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For an analytic function f on $\mathbb{D}$ and $0 < t < 1$ the *dilate* f_t is the function defined by $f_t(z) = f(tz)$. In [1] it is shown that for an analytic function f on $\mathbb{D}$: $f \in \mathcal{B}_0$ if and only if $\|f - f_t\|_{\mathcal{B}} \to 0$ as $t \to 1^-$.

In analogy to Theorem 1 we have the following result.

Theorem 2. *For an analytic function f on $\mathbb{D}$:*

$$f \in \mathcal{B}_0 \iff \lim_{|z|\to 1^-} \sup\left\{ \left(1 - |z|^2\right)^{1/2} \left(1 - |w|^2\right)^{1/2} \left| \frac{f(z) - f(w)}{z - w} \right| : w \in \mathbb{D},\ w \neq z \right\} = 0.$$

Proof: The implication "$\Leftarrow$" is again trivial. To prove the other implication, suppose that $f \in \mathcal{B}_0$ and let $0 < t < 1$. Using (4) we see that the dilate f_t satisfies:

$$\begin{aligned} \left(1 - |z|^2\right)^{1/2} \left(1 - |w|^2\right)^{1/2} \left| \frac{f_t(z) - f_t(w)}{z - w} \right| &\leqslant 2t \frac{\left(1 - |z|^2\right)^{1/2} \left(1 - |w|^2\right)^{1/2}}{\left(1 - |tz|^2\right)^{1/2} \left(1 - |tw|^2\right)^{1/2}} \|f\|_{\mathcal{B}} \\ &\leqslant \frac{2t}{1 - t^2} \left(1 - |z|^2\right)^{1/2} \|f\|_{\mathcal{B}}. \end{aligned}$$

Applying inequality (4) to the function $f - f_t$, it follows with the help of the triangle inequality that

$$\begin{aligned} \sup\left\{ \left(1 - |z|^2\right)^{1/2} \left(1 - |w|^2\right)^{1/2} \left| \frac{f(z) - f(w)}{z - w} \right| : w \in \mathbb{D},\ w \neq z \right\} & \\ \leqslant \frac{2t}{1 - t^2} \left(1 - |z|^2\right)^{1/2} \|f\|_{\mathcal{B}} + 2 \|f - f_t\|_{\mathcal{B}}. & \end{aligned}$$

First letting $|z| \to 1^-$ and then $t \to 1^-$, the implication "$\Rightarrow$" follows. ∎

For various other characterisations of the Bloch and little Bloch space we refer the reader to [**1**, **5**, **10**, **11**, **12**].

3. Besov Spaces of Analytic Functions

Another way of interpreting Theorem 1 is that for an analytic function f on $\mathbb{D}$ we have $f \in \mathcal{B}$ if and only if the function $\left(1 - |z|^2\right)^{1/2} \left(1 - |w|^2\right)^{1/2} (f(z) - f(w))/(z - w)$ is in $L^\infty(\mathbb{D} \times \mathbb{D})$. We shall prove the corresponding result for the Besov spaces:

Theorem 3. *If $2 < p < \infty$, then for an analytic function f on $\mathbb{D}$ we have:*

$$f \in \mathcal{B}_p \iff \int_{\mathbb{D}} \int_{\mathbb{D}} \left| \frac{f(z) - f(w)}{z - w} \right|^p \left(1 - |z|^2\right)^{p/2} \left(1 - |w|^2\right)^{p/2} d\lambda(w)\, d\lambda(z) < \infty.$$

Of special interest is the case in which $\left(1 - |z|^2\right)^{p/2} d\lambda(z) = dA(z)$. This is so when $p = 4$, and we have the following special case of Theorem 3:

COROLLARY 4. *For an analytic function f on $\mathbb{D}$ we have:*

$$f \in \mathcal{B}_4 \iff \int_{\mathbb{D}}\int_{\mathbb{D}} \left|\frac{f(z)-f(w)}{z-w}\right|^4 dA(z)\,dA(w) < \infty.$$

In the proof of Theorem 3 we shall need a couple of lemmas. In order to state the first of these lemmas we need to introduce more notation.

For $w \in \mathbb{D}$ and $0 < r < 1$ we write $D(w,r)$ for the set $\{\varphi_w(z) : |z| < r\} = \varphi_w(r\mathbb{D})$. Because φ_w is a Möbius transformation, the set $D(w,r)$ is a Euclidean disk contained in $\mathbb{D}$. Its Euclidean centre and radius are easily determined to be $(1-r^2)w\Big/\left(1-r^2|w|^2\right)$ and $\left(1-|w|^2\right)r\Big/\left(1-r^2|w|^2\right)$ respectively (see, for example [7, Section 1.1]). We refer to the set $D(w,r)$ as the *pseudohyperbolic disk* centred at w with (pseudohyperbolic) radius r.

LEMMA 5. *Let $1 \leqslant p < \infty$, $\alpha > 0$, and $0 < r < 1$. There is a constant C, only depending upon p, α, and r, such that*

$$\int_{D(w,r)} |g(z)|^p \left(1-|z|^2\right)^\alpha d\lambda(z) \geqslant C\,|g(w)|^p \left(1-|w|^2\right)^\alpha,$$

for every analytic function g on $\mathbb{D}$ and $w \in \mathbb{D}$.

PROOF: Let g be an analytic function on $\mathbb{D}$. It is easy to verify that

$$\int_{r\mathbb{D}} g(z)\left(1-|z|^2\right)^\alpha d\lambda(z) = g(0)\int_{r\mathbb{D}} \left(1-|z|^2\right)^\alpha d\lambda(z),$$

and by Jensen's inequality we have:

$$\int_{D(0,r)} |g(z)|^p \left(1-|z|^2\right)^\alpha d\lambda(z) \geqslant |g(0)|^p \int_{D(0,r)} \left(1-|z|^2\right)^\alpha d\lambda(z) = C_0\,|g(0)|^p,$$

so the inequality holds for $w = 0$ with $C_0 = \int_0^{r^2} (1-x)^{\alpha-2}\,dx$. Now if $w \in \mathbb{D}$, apply the above inequality to $g \circ \varphi_w$. Using change-of-variable formula (3) and identity (1) we have:

$$\begin{aligned} C_0\,|g(w)|^p &\leqslant \int_{D(0,r)} |g(\varphi_w(z))|^p \left(1-|z|^2\right)^\alpha d\lambda(z) \\ &= \int_{D(w,r)} |g(u)|^p \left(1-|\varphi_w(u)|^2\right)^\alpha d\lambda(u) \\ &\leqslant \int_{D(w,r)} |g(u)|^p \frac{\left(1-|u|^2\right)^\alpha \left(1-|w|^2\right)^\alpha}{|1-\overline{w}u|^{2\alpha}}\,d\lambda(u). \end{aligned}$$

If $u \in D(w,r)$, then $u = \varphi_w(z)$, for some $z \in r\mathbb{D}$, and thus $|1-\overline{w}u| = \left(1-|w|^2\right)/|1-\overline{w}z| \geqslant \left(1-|w|^2\right)/(1+r)$. It follows that

$$\int_{D(w,r)} |g(u)|^p \frac{\left(1-|u|^2\right)^\alpha\left(1-|w|^2\right)^\alpha}{|1-\overline{w}u|^{2\alpha}}\,d\lambda(u) \leqslant \frac{(1+r)^{2\alpha}}{\left(1-|w|^2\right)^\alpha}\int_{D(w,r)} |g(u)|^p\left(1-|u|^2\right)^\alpha d\lambda(u),$$

proving the lemma (with $C = C_0/(1+r)^{2\alpha}$). ∎

Towards the Proof of Theorem 3. Let f be analytic on $\mathbb{D}$. For $w \in \mathbb{D}$ write $D(w)$ for $D(w,1/2)$, and apply Lemma 5 to the function $g(z) = (f(z)-f(w))/(z-w)$ to get:

$$\int_{D(w)}\left|\frac{f(z)-f(w)}{z-w}\right|^p\left(1-|z|^2\right)^{p/2} d\lambda(z) \geqslant C\left(1-|w|^2\right)^{p/2}|f'(w)|^p.$$

It follows that

$$\begin{aligned}\int_{\mathbb{D}}\int_{\mathbb{D}}\left|\frac{f(z)-f(w)}{z-w}\right|^p&\left(1-|z|^2\right)^{p/2}\left(1-|w|^2\right)^{p/2} d\lambda(z)\,d\lambda(w)\\ &\geqslant \int_{\mathbb{D}}\int_{D(w)}\left|\frac{f(z)-f(w)}{z-w}\right|^p\left(1-|z|^2\right)^{p/2}\left(1-|w|^2\right)^{p/2} d\lambda(z)\,d\lambda(w)\\ &\geqslant C\int_{\mathbb{D}}|f'(w)|^p\left(1-|w|^2\right)^p d\lambda(w).\end{aligned}$$

This proves the implication "$\Leftarrow$" in Theorem 3. To prove the converse we shall use the following lemma.

Lemma 6. *Let $1 \leqslant p < \infty$, and $\alpha > 1$. There is a finite constant C, only depending on p and α such that*

$$\int_{\mathbb{D}}\left|\frac{g(u)-g(0)}{u}\right|^p\left(1-|u|^2\right)^\alpha d\lambda(u) \leqslant C\int_{\mathbb{D}}|g'(u)|^p\left(1-|u|^2\right)^{p+\alpha} d\lambda(u),$$

for every analytic function g on $\mathbb{D}$.

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PROOF: This follows easily from Theorem 5.6 of [6] and the fact that for the measure $d\mu(u) = \left(1-|u|^2\right)^{\alpha} d\lambda(u)$ the linear operator V, defined on $L^p(\mathbb{D}, d\mu)$ by $(Vg)(u) = (g(u)-g(0))/u$, is bounded (as can be shown using the closed graph theorem). ∎

COMPLETION OF THE PROOF THEOREM 3: Fix $w \in \mathbb{D}$. Using identity (1) and making the change-of-variables $z = \varphi_w(u)$ we have:

$$\begin{aligned}\int_{\mathbb{D}} \left|\frac{f(z)-f(w)}{z-w}\right|^p \left(1-|z|^2\right)^{p/2}\left(1-|w|^2\right)^{p/2} d\lambda(z) \\ &= \int_{\mathbb{D}} \left|\frac{f(z)-f(w)}{\varphi_w(z)}\right|^p \left(1-|\varphi_w(z)|^2\right)^{p/2} d\lambda(z) \\ &= \int_{\mathbb{D}} \left|\frac{(f\circ\varphi_w)(u)-(f\circ\varphi_w)(0)}{u}\right|^p \left(1-|u|^2\right)^{p/2} d\lambda(u).\end{aligned}$$

Applying Lemma 6 to the function $f \circ \varphi_w$ we see that

$$\begin{aligned}\int_{\mathbb{D}} \left|\frac{(f\circ\varphi_w)(u)-(f\circ\varphi_w)(0)}{u}\right|^p \left(1-|u|^2\right)^{p/2} d\lambda(u) \\ &\leqslant C\int_{\mathbb{D}} |f'(\varphi_w(u))|^p |\varphi_w'(u)|^p \left(1-|u|^2\right)^p \left(1-|u|^2\right)^{p/2} d\lambda(u) \\ &= C\int_{\mathbb{D}} |f'(\varphi_w(u))|^p \left(1-|\varphi_w(u)|^2\right)^p \left(1-|u|^2\right)^{p/2} d\lambda(u) && \text{[using (2)]} \\ &= C\int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^p \left(1-|\varphi_w(z)|^2\right)^{p/2} d\lambda(z) && \text{[using (3)]} \\ &= C\int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^{(3/2)p-2} \frac{\left(1-|w|^2\right)^{p/2}}{|1-\overline{w}z|^p} dA(z) && \text{[using (1)].}\end{aligned}$$

Integrating with respect to w now yields:

$$\begin{aligned}\int_{\mathbb{D}}\int_{\mathbb{D}} \left|\frac{f(z)-f(w)}{z-w}\right|^p \left(1-|z|^2\right)^{p/2}\left(1-|w|^2\right)^{p/2} d\lambda(z)\,d\lambda(w) \\ &\leqslant C\int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^{(3/2)p-2} \frac{\left(1-|w|^2\right)^{(p/2)-2}}{|1-\overline{w}z|^p} dA(z)\,dA(w) \\ &= C\int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^{(3/2)p-2} \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{(p/2)-2}}{|1-\overline{w}z|^p} dA(w)\,dA(z).\end{aligned}$$

As a special case of one of the Forelli-Rudin estimates (see, for example, [**12**, Lemma 4.2.2]),the inner integral is bounded by a constant times $\left(1-|z|^2\right)^{-p/2}$. (Note that this estimate requires $(p/2)-2>-1$, that is, $p>2$.) Thus, there is a constant C' such that

$$\begin{aligned}\int_{\mathbb{D}}\int_{\mathbb{D}}\left|\frac{f(z)-f(w)}{z-w}\right|^p\left(1-|z|^2\right)^{p/2}\left(1-|w|^2\right)^{p/2}d\lambda(z)\,d\lambda(w)\\ \leqslant C'\int_{\mathbb{D}}|f'(z)|^p\left(1-|z|^2\right)^{(3/2)p-2}\left(1-|z|^2\right)^{-p/2}dA(z)\\ = C'\int_{\mathbb{D}}|f'(z)|^p\left(1-|z|^2\right)^p d\lambda(z),\end{aligned}$$

and the proof of Theorem 3 is completed. □

REMARK. For $1\leqslant p<\infty$ and $\alpha\leqslant -1$, an analytic function h on $\mathbb{D}$ satisfies $\int_{\mathbb{D}}|h(z)|^p\left(1-|z|^2\right)^\alpha dA(z)<\infty$ only when $h\equiv 0$. It is easily seen that for $1\leqslant p\leqslant 2$ the condition of Theorem 3 involving the iterated integral implies that the function f must be constant. Thus, the conclusion of Theorem 3 does not hold for $1\leqslant p\leqslant 2$.

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Department of Mathematical Sciences
The University of Montana
Missoula MT 59812-1032
United States of America