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Modules with Unique Closure Relative to a Torsion Theory

S. Doğruöz, A. Harmanci, and P. F. Smith

Abstract. We consider when a single submodule and also when every submodule of a module M over a general ring R has a unique closure with respect to a hereditary torsion theory on Mod-R.

1 Introduction

In this note all rings are associative with identity and all modules are unitary right modules. Let R be a ring. A submodule K of an R-module M is called *closed in* M provided K has no proper essential extension in M. By a *closure of a submodule* N *of* M we mean a closed submodule K of M such that N is an essential submodule of K. Note that K is a closure of N in M if and only if K is a maximal essential extension of N in M, and such a K always exists by Zorn's Lemma. This is (i) of the following well-known result.

Lemma 1.1 *Let K, L, N be submodules of a module M with $K \subseteq L$.*

(i) *There exists a closed submodule H of M such that N is an essential submodule of H.*
(ii) *The submodule L is closed in M if and only if N/L is an essential submodule of the module M/L for every essential submodule N of M containing L.*
(iii) *If L is a closed submodule of M, then L/K is a closed submodule of M/K.*
(iv) *If K is a closed submodule of L and L is a closed submodule of M, then K is a closed submodule of M.*

Proof See [2, p. 6]. ■

In [3], the module M is called a UC-*module* provided every submodule of M has a unique closure in M and necessary and sufficient conditions are given for M to be a UC-module. Further conditions for M to be a UC-module are given in [5].

Let R be a ring and let τ be a hereditary torsion theory on Mod-R. (For basic information concerning hereditary torsion theories see [4].) Let M be an R-module. For any submodule N of M, $T_\tau(N)$ will denote the submodule H of M containing N such that H/N is the τ-torsion submodule of M/N.

Given a module M, a submodule L of M is called τ-*essential* provided L is an essential submodule of M and M/L is a τ-torsion module. Moreover, a submodule K of M is called τ-*closed in* M provided K has no proper τ-essential extension in M, *i.e.,*

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if N is a submodule of M such that K is a τ-essential submodule of N, then $K = N$. (Some authors call a submodule K of a module M τ-closed provided the module M/K is τ-torsion-free.) Note that if K is a submodule of M such that either M/K is τ-torsion-free or K is a closed submodule of M, then K is a τ-closed submodule of M. The first result taken from [1, Lemma 3.6] describes τ-closed submodules; we give its proof for completeness.

Lemma 1.2 *The following statements are equivalent for a submodule K of a module M.*

(i) *K is a τ-closed submodule of M.*
(ii) *K is a closed submodule of $T_\tau(K)$.*
(iii) *There exists a submodule L of M containing K such that K is a closed submodule of L and M/L is τ-torsion-free.*

Proof (i) $\Rightarrow$ (ii). Suppose that K is an essential submodule of a submodule N of $T_\tau(K)$. Then K is a τ-essential submodule of N, so that $K = N$.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Let H be a submodule of M such that K is a τ-essential submodule of H. Then H/K is τ-torsion, so that $(H + L)/L$ is also τ-torsion. Hence $H \subseteq L$. Because K is an essential submodule of H, we have $K = H$. ■

Corollary 1.3 *Let $K \subseteq L$ be submodules of a module M such that L is a τ-closed submodule of M. Then L/K is a τ-closed submodule of M/K.*

Proof By Lemma 1.2 there exists a submodule H of M containing L such that L is a closed submodule of H and M/H is τ-torsion-free. By Lemma 1.1(iii), L/K is a closed submodule of H/K, and by Lemma 1.2, L/K is a τ-closed submodule of M/K. ■

Corollary 1.4 *The following statements are equivalent for a module M.*

(i) *Whenever $K \subseteq L \subseteq N$ are submodules of M such that K is a τ-closed submodule of L and L is a τ-closed submodule of N, then K is a τ-closed submodule of N.*
(ii) *Whenever $K \subseteq L \subseteq N$ are submodules of M such that L/K is τ-torsion-free and L is a closed submodule of N, then K is a τ-closed submodule of N.*

Proof (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). By Lemma 1.2 there exists a submodule F of L containing K such that K is closed in F and L/F is τ-torsion-free, and there exists a submodule G of N containing L such that L is closed in G and N/G is τ-torsion-free. By hypothesis F is τ-closed in G. Again using Lemma 1.2 there exists a submodule H of G containing F such that F is closed in H and G/H is τ-torsion-free. It follows that K is closed in H and N/H is τ-torsion-free. Finally Lemma 1.2 gives that K is τ-closed in N. ■

A submodule K of a module M is called a τ-*closure* of a submodule N of M provided N is a τ-essential submodule of K and K is a τ-closed submodule of M. By Zorn's Lemma, every submodule N of M has a τ-closure in M. The module M is called a τ-UC-*module* provided every submodule N of M has a unique τ-closure in M.

Let us consider some simple examples. First, let τ_0 denote the hereditary torsion theory on Mod-R, such that 0 is the only τ-torsion module. Then a submodule N of a module M is τ_0-essential if and only if $N = M$ and every submodule of M is τ_0-closed. Thus every module is τ_0-UC. Secondly, let τ_1 denote the hereditary torsion theory on Mod-R such that every module is τ-torsion. Then the τ_1-essential submodules of M coincide with the essential submodules of M and the module M is a τ_1-UC-module if and only if M is a UC-module. Next, let τ_G denote the Goldie torsion theory on Mod-R (see [4] for details). It is clear that a submodule L of an R-module M is τ_G-essential in M if and only if L is essential in M. Thus a submodule K of M is a τ_G-closure of a given submodule N of M if and only if K is a closure of N in M. Thus a module is τ_G-UC if and only if it is a UC-module.

Let τ be any hereditary torsion theory on Mod-R. Let M be any R-module. Let N be any essential submodule of M. Clearly $T_\tau(N)$ is the unique τ-closure of N in M. Thus every uniform R-module is τ-UC. (Recall that a module U is *uniform* if $U \neq 0$ and every non-zero submodule of U is essential.)

Let $\mathbb{Z}$ denote the ring of integers and let p be any prime in $\mathbb{Z}$. Then τ_p will denote the hereditary torsion theory on Mod-$\mathbb{Z}$ given by p-torsion, *i.e.,* a $\mathbb{Z}$-module M is τ_p-torsion if and only if M is an abelian p-group. We shall see after Corollary 3.5 that if n is any positive integer coprime to p, then the $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}n)$ is τ_p-UC but not UC.

Example 1.5 Let p be any prime in $\mathbb{Z}$ and let N be any submodule of the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}p)$. Then N has a unique τ_p-closure in M or $N = \mathbb{Z}(pm, 0)$ for some non-zero $m \in \mathbb{Z}$.

Proof Let $K = \mathbb{Z}(c, u)$ for some $c \in \mathbb{Z}$ and $0 \neq u \in \mathbb{Z}/\mathbb{Z}p$. If $c = 0$, then K is a direct summand of M. Suppose that $c \neq 0$. Then $K \cong \mathbb{Z}$. If K is τ_p-essential in a submodule L of M, then K is essential in L, so that L is also a cyclic submodule of M. It is easy to prove that $K = L$. Thus, in any case, K is a τ_p-closed submodule of M.

Let Y denote the direct summand $0 \oplus (\mathbb{Z}/\mathbb{Z}p)$ of M. If $N \cap Y \neq 0$, then $N = A \oplus (\mathbb{Z}/\mathbb{Z}p)$, for some ideal A of $\mathbb{Z}$, and either N is a direct summand of M (if $A = 0$) or N is an essential submodule of M, so that in either case N has a unique τ_p-closure in M (see the above remarks about essential submodules). Now suppose that $N \cap Y = 0$. Then N embeds in $\mathbb{Z}$ so that $N = \mathbb{Z}(a, v)$ for some $0 \neq a \in \mathbb{Z}$ and $v \in \mathbb{Z}/\mathbb{Z}p$. If $v \neq 0$, then N is τ_p-closed in M by the above remarks. Suppose that $v = 0$. Suppose that p does not divide a. If N is τ_p-essential in a submodule L, then again L is cyclic and it is easy to check that $L = N$. Thus, in this case, N is τ_p-closed in M. Now suppose that $a = p^k b$ for some positive integer k and some integer b which is coprime to p. Then it can be proved that the τ_p-closures of N in M are the submodules $\mathbb{Z}(b, 0)$ and $\mathbb{Z}(p^i b, u)$ for all $0 \leq i \leq k - 1$ and $0 \neq u \in \mathbb{Z}/\mathbb{Z}p$. Thus in this case N does not have a unique τ_p-closure in M. ■

2 Submodules with Unique Closures

Let R be an arbitrary ring and let τ be an arbitrary hereditary torsion theory on Mod-R. In this section we shall examine when a given submodule of an R-module M

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has a unique τ-closure in M.

Lemma 2.1 *Let $N \subseteq K \subseteq H$ be submodules of a module M such that K is a τ-closure of N in H. Then there exists a τ-closure K' of N in M such that $K = K' \cap H$.*

Proof Note that N is τ-essential in K. By Zorn's Lemma there exists a submodule K' of M such that K' is maximal with respect to the properties $K \subseteq K'$ and N is τ-essential in K'. Then K' is a τ-closure of N in M. Moreover, N τ-essential in K' implies that K is τ-essential in $K' \cap H$. It follows that $K = K' \cap H$. ■

Lemma 2.2 *Let N and K be submodules of a module M. Then K is a τ-closure of N in M if and only if K is a closure of N in $T_\tau(N)$.*

Proof Let $T = T_\tau(N)$. Suppose first that K is a τ-closure of N in M. Then K/N is a τ-torsion module and hence $K \subseteq T$. Note that N is an essential submodule of K. By Lemma 1.2 K is a closed submodule of T and hence a closure of N in T.

Conversely, suppose that K is a closure of N in T. Then N is τ-essential in K. By Lemma 1.2 again K is τ-closed in M. It follows that K is a τ-closure of N in M. ■

Corollary 2.3 *Let N be a submodule of a module M such that the module M/N is τ-torsion. Then the τ-closures of N in M coincide with the closures of N in M.*

Proof By Lemma 2.2. ■

Lemma 2.4 *Let $K \subseteq L$ be submodules of a module M such that K is a τ-closed submodule of M and L is a τ-essential submodule of M. Then L/K is τ-essential in M/K.*

Proof Suppose that L/K is not τ-essential in M/K. Then L/K is not essential in M/K. There exists a submodule N of M, properly containing K such that $K = L \cap N$. Note that $N/K \cong (N + L)/L$, so that N/K is τ-torsion. Thus K is not essential in N. Let H be a non-zero submodule of N such that $K \cap H = 0$. Note that $L \cap H = L \cap N \cap H = K \cap H = 0$. But L is essential in M. Thus $H = 0$, a contradiction. ■

Corollary 2.5 *Let $K \subseteq L$ be submodules of a module M such that K is a τ-closed submodule of M and L/K is a τ-closed submodule of M/K. Then L is a τ-closed submodule of M.*

Proof Suppose that L is a τ-essential submodule of a submodule N of M. By Lemma 2.4, L/K is a τ-essential submodule of N/K and hence $L/K = N/K$. Thus $L = N$. ■

Lemma 2.6 *Let $K \subseteq N$ be submodules of a module M such that K is a τ-closed submodule of M. Then each τ-closure of N/K in M/K is a submodule of the form L/K, where L is a τ-closure of N in M. Moreover the converse holds in case K is a closed submodule of M.*

Proof Let L be any submodule of M containing K such that L/K is a τ-closure of N/K in M/K. Clearly N is τ-essential in L. Moreover, by Corollary 2.5 L is a τ-closed submodule of M. Thus L is a τ-closure of N in M. Now suppose that K is closed in M and that L is a τ-closure of N in M. By Lemma 1.1(ii), N/K is essential in L/K and hence N/K is τ-essential in L/K. By Corollary 1.3, L/K is τ-closed in M/K, so that L/K is a τ-closure of N/K in M/K. ■

Theorem 2.7 *Let R be any ring and let τ be any hereditary torsion theory on* Mod-R. *Then the following statements are equivalent for a submodule N of an R-module M.*

(i) *N has a unique τ-closure in M.*
(ii) *N has a unique τ-closure in L for every submodule L of M containing N.*
(iii) *N has a unique closure in $T_\tau(N)$.*
(iv) *N has a unique τ-closure in the submodule $N + m_1R + m_2R$ for all $m_i \in M$ $(i = 1, 2)$.*
(v) *N/K has a unique τ-closure in M/K for every τ-closed submodule K of M contained in N.*
(vi) *N/L has a unique τ-closure in M/L for some closed submodule L of M contained in N.*
(vii) *If N is τ-essential in a submodule L_i of M for all i in an index set I, then N is τ-essential in $\sum_{i\in I} L_i$.*
(viii) *If N is τ-essential in a submodule L_1 of M and also in a submodule L_2 of M, then N is τ-essential in $L_1 + L_2$.*
(ix) *$N^* = \{m \in M : N$ is τ-essential in $N + mR\}$ is a submodule of M.*

In case (ix) *N^* is the unique τ-closure of N in M.*

Proof (i) ⇒ (ii). By Lemma 2.1.

(ii) ⇒ (iii). Clear.

(iii) ⇒ (i). By Lemma 2.2.

(ii) ⇒ (iv). Clear.

(iv) ⇒ (i). Suppose that (i) is false. Let K_1 and K_2 be distinct τ-closures of N in M. Let m be any element which belongs to K_2 but not to K_1. Let $x \in K_1$. By hypothesis, N has a unique τ-closure K in $N + mR + xR$. Now $N + mR$ and $N + xR$ are both contained in K so that $N + mR + xR$ is contained in K. Thus N is τ-essential in $N + mR + xR$. It follows that N is τ-essential in $K_1 + mR$ and hence K_1 is τ-essential in $K_1 + mR$. Thus $K_1 = K_1 + mR$ and $m \in K_1$, a contradiction. This proves (i).

(i) ⇒ (v). By Lemma 2.6.

(v) ⇒ (vi). Clear.

(vi) ⇒ (i). By Lemma 2.6 again.

(i) ⇒ (ix). Let K be the unique τ-closure of N in M. Note that $0 \in N^*$. Moreover, if m_1 and m_2 are elements of N^* and if $r \in R$, then both m_1 and m_2 belong to K, so that $m_1 - m_2 \in K$ and $m_1 r \in K$. It follows that N is τ-essential in $N + (m_1 - m_2)R$ and also in $N + m_1 rR$. Thus $m_1 - m_2 \in N^*$ and $m_1 r \in N^*$. Hence N^* is a submodule of M.

(ix) ⇒ (viii). Let $m \in L_1 + L_2$. Then $m = m_1 + m_2$ for some $m_i \in L_i$ $(i = 1, 2)$. For $i = 1, 2$, N is τ-essential in $N + m_i R$ and hence N is τ-essential in $N + mR$. It follows that N is τ-essential in $L_1 + L_2$.

(viii) ⇒ (vii). Because L_i/N is τ-torsion for each $i \in I$, $(\sum_{i\in I} L_i)/N$ is τ-torsion. Moreover, (viii) gives that N is essential in $\sum_{i\in F} L_i$, for every finite subset F of I and hence N is essential in $\sum_{i\in I} L_i$. Thus N is τ-essential in $\sum_{i\in I} L_i$.

(vii) ⇒ (viii). Clear.

(viii) ⇒ (i). Let K_1 and K_2 be τ-closures of N in M. By (viii), N is τ-essential in $K_1 + K_2$ and hence K_1 is also τ-essential in $K_1 + K_2$. Thus $K_1 = K_1 + K_2$ and hence $K_2 \subseteq K_1$. Similarly $K_1 \subseteq K_2$. Thus $K_1 = K_2$.

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For the last assertion, let K be the unique τ-closure of N in M. Let $m \in K$. Then N is τ-essential in $N + mR$ and hence $m \in N^*$. It follows that $K \subseteq N^*$. Now let $x \in N^*$. Then N is τ-essential in $N + xR$, thus $N + xR \subseteq K$ so that $x \in K$. It follows that $K = N^*$. ■

Finally, in this section we consider τ-torsion-free modules and for these we have the following result.

Proposition 2.8 *Let M be a τ-torsion-free R-module and let N be any submodule of M. Then $T_\tau(N)$ is the unique τ-closure of N in M.*

Proof If L is a submodule of $T = T_\tau(N)$ such that $L \cap N = 0$, then L embeds in the module T/N and hence L is τ-torsion, so that $L = 0$. Thus N is essential, and hence τ-essential, in T. It follows that T is a τ-closure of N in M and, by Lemma 1.2, T is the unique τ-closure of N. ■

3 τ-UC-Modules

Recall that if R is a ring and τ any hereditary torsion theory on Mod-R, then an R-module M is a τ-UC-module provided every submodule has a unique τ-closure in M. In this section we shall obtain necessary and sufficient conditions for a module to be a τ-UC-module.

Lemma 3.1 *Every submodule of a τ-UC-module is also τ-UC.*

Proof By Lemma 2.1. ■

Lemma 3.2 *An R-module M is a τ-UC-module if and only if there do not exist an R-module X and a proper τ-essential submodule Y of X such that the R-module $X \oplus (X/Y)$ embeds in M.*

Proof First we prove that if Y is a proper τ-essential submodule of an R-module X, then the R-module $Z = X \oplus (X/Y)$ is not τ-UC. Consider the submodule $U = Y \oplus 0$ of Z. Clearly the direct summand $V = X \oplus 0$ of Z is a τ-closure of U. Let $W = \{(x, x + Y) \in Z : x \in X\}$. It is clear that $Z = W \oplus W_0$, where W_0 is the submodule $0 \oplus (X/Y)$ of Z. Moreover, $W/U \cong X/Y$, so that W/U is τ-torsion. It is easy to check that U is an essential submodule of W, so that W is also a τ-closure of U. Thus Z is not a τ-UC-module. By Lemma 3.1, this proves the necessity.

Conversely, suppose that the module M is not τ-UC. Then there exists a submodule N in M such that N has distinct τ-closures K and L in M. Let $T = T_\tau(N)$ and note that $K+L$ is a submodule of T. Since K is a proper submodule of $K+L$, it follows that N is not an essential submodule of $K + L$. Let H be a non-zero submodule of $K + L$ such that $N \cap H = 0$. Note that $K \cap H = 0$, so that H is isomorphic to the submodule $(H + K)/K$ of $(L + K)/K \cong L/(L \cap K)$. Thus there exists a submodule G of L containing $L \cap K$ such that $H \cong G/(L \cap K)$. Note that, because N is τ-essential in L, $L \cap K$ is a proper τ-essential submodule of G. Next $L \cap H = 0$ gives $G \cap H = 0$ and hence $G \oplus H$ embeds in M. ■

Corollary 3.3 *A module M is τ-UC if and only if every 2-generated submodule of M is τ-UC.*

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Proof The necessity follows by Lemma 3.1. Conversely, suppose that M is not a τ-UC-module. By Lemma 3.2 there exist a module X, a proper τ-essential submodule Y of X, and a monomorphism $\alpha\colon X \oplus (X/Y) \to M$. Let x be any element in X but not Y. Then Y is a proper τ-essential submodule of $xR + Y$ and, by Lemma 3.2, the module $Z = xR \oplus (xR + Y)/Y$ is not τ-UC. Then $\alpha(Z)$ is a 2-generated submodule of M which is not τ-UC. ■

Let R be a ring and M an R-module. For any element m in M we set $r(m) = \{r \in R : mr = 0\}$. The next result characterizes which R-modules are τ-UC for a given hereditary torsion theory τ on Mod-R.

Theorem 3.4 *Let R be a ring and let τ be any hereditary torsion theory on* Mod-R. *Then the following statements are equivalent for a module M.*

- (i) *M is a τ-UC-module.*
- (ii) *Every (2-generated) submodule of M is a τ-UC-module.*
- (iii) *Every submodule N has a unique τ-closure in the submodule $N + m_1R + m_2R$ for all $m_i \in M$ $(i = 1, 2)$.*
- (iv) *M/K is a τ-UC-module for every τ-closed submodule K of M.*
- (v) *There do not exist an R-module X and a proper τ-essential submodule Y of X such that the R-module $X \oplus (X/Y)$ embeds in M.*
- (vi) *Given elements m, m' in M with $mR \cap m'R = 0$, $r(m) \subseteq r(m')$, and $r(m')/r(m)$ τ-essential in the R-module $R/r(m)$, then $m' = 0$.*
- (vii) *Given submodules $K \subseteq K'$ and $L \subseteq L'$ of M such that $K' \cap L' = 0$, K'/K is isomorphic to L'/L, and K is τ-essential in K', then L is τ-essential in L'.*
- (viii) *Given any submodule L of M and a homomorphism $\varphi\colon L \to M$ such that $L \cap \varphi(L) = 0$, then* $\ker\varphi$ *is a τ-closed submodule of L.*
- (ix) *Given submodules $L_i (i \in I)$ of M whenever a submodule N of M is a τ-essential submodule of L_i for all $i \in I$, then N is τ-essential in $\sum_{i\in I} L_i$.*
- (x) *Whenever a submodule N of M is τ-essential in a submodule L_1 and a submodule L_2 of M, then N is τ-essential in $L_1 + L_2$.*
- (xi) *$N^* = \{m \in M : N$ is τ-essential in $N + mR\}$ is a submodule of M for every submodule N of M.*

Proof (i) ⇔ (ii). By Lemma 3.1 and Corollary 3.3.

(i) ⇔ (iii) ⇔ (iv) ⇔ (ix) ⇔ (x) ⇔ (xi). Clear from Theorem 2.7.

(i) ⇔ (v). By Lemma 3.2.

(v) ⇒ (vi). Let $X = R/r(m)$ and $Y = r(m')/r(m)$. Note that Y is τ-essential in X and $X \cong mR$ and $X/Y \cong m'R$. Then $X \oplus X/Y$ embeds in M. Hence $m = 0$ or $m' = 0$. By hypothesis $m = 0$ always implies $m' = 0$. Thus $m' = 0$.

(vi) ⇒ (vii). Let $m' \in L'$ and assume that $m'R \cap L = 0$. We shall prove that $m' = 0$. Let α denote the isomorphism $K'/K \to L'/L$. Then $\alpha(m + K) = m' + L$ for some element $m \in K'$. It is clear that $r(m) \subseteq r(m')$ and $R/r(m')$ is τ-torsion because $m'R$ embeds in L'/L. Next we prove $r(m')/r(m)$ is essential in $R/r(m)$. Let $0 \neq t + r(m) \in R/r(m)$. Note that $mt \neq 0$. Because K is essential in K', there exists an element r in R such that $0 \neq mtr \in K$. It follows that $m'tr + L = \alpha(mtr + K) = 0$ and hence $m'tr \in L$. But $m'R \cap L = 0$. Thus $tr \in r(m') \setminus r(m)$. It follows that $r(m')/r(m)$ is an essential submodule of $R/r(m)$. By (vi), $m' = 0$. Thus L is τ-essential in L'.

(vii) $\Rightarrow$ (viii). Let $K = \ker \varphi$. Suppose that K is τ-essential in a submodule N of L. Then $\varphi(N) \cong N/K$ and $N \cap \varphi(N) = 0$. By (vii) 0 is τ-essential in $\varphi(N)$ so that $\varphi(N) = 0$ and hence $K = N$. Thus K is τ-closed in L.

(viii) $\Rightarrow$ (v). Suppose that (v) does not hold. Then there exist non-zero submodules L and K of M such that $L \cap K = 0$ and a homomorphism $\theta \colon L \to K$ such that $\ker \theta$ is τ-essential in L. Thus (viii) does not hold. This completes the proof of the theorem. ■

Let M be any R-module. Then $Z_\tau(M)$ will denote the set of elements m in M such that $mE = 0$ for some τ-essential right ideal E of R. Note that $Z_\tau(M)$ is a submodule of the singular submodule $Z(M)$ of M.

Corollary 3.5 *Let M be a module such that $Z_\tau(M) = 0$. Then M is a τ-*UC*-module.*

Proof Let Y be a proper τ-essential submodule of an R-module X. Let $x \in X \setminus Y$ and let $E = \{r \in R : xr \in Y\}$. Then E is a right ideal of R and $R/E \cong (xR+Y)/Y \subseteq X/Y$, so that R/E is a τ-torsion module. Moreover, E is an essential right ideal of R by a standard proof. Thus E is a τ-essential right ideal of R. Because $Z_\tau(M) = 0$, $(xR + Y)/Y$, and hence X/Y, cannot be embedded in M. By Theorem 3.4, M is a τ-UC-module. ■

Note that if M is a τ-torsion-free module, then $Z_\tau(M) = 0$ and hence M is a τ-UC-module, a fact we already knew by Proposition 2.8. In particular, if p is any prime in $\mathbb{Z}$ and n any positive integer coprime to p, then the $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}n)$ is τ_p-torsion-free and hence τ_p-UC but M is not UC by [3]. More generally, let A be any $\mathbb{Z}$-module which is neither torsion nor torsion-free (in the usual sense) but such that A does not contain any element of order p. Then A satisfies $Z_{\tau_p}(A) = 0$ so that A is τ_p-UC by Corollary 3.5. However A is not UC for the following reason. There exist elements a and b in A such that a has infinite order and b has order n for some positive integer n. Note that $\mathbb{Z}a \cap \mathbb{Z}b = 0$ and that $\mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}n) \cong (\mathbb{Z}a) \oplus (\mathbb{Z}b)$ which is a submodule of A. By [3] A is not UC.

Finally, we consider modules whose submodules have unique closures with respect to different torsion theories. Recall that if τ and ρ are hereditary torsion theories on Mod-R, then we write $\tau \leq \rho$ provided every τ-torsion module is also a ρ-torsion module. In this situation we have the following further consequence of Theorem 3.4.

Proposition 3.6 *Let τ and ρ be hereditary torsion theories on* Mod-R *such that $\tau \leq \rho$. Then every ρ-*UC*-module is a τ-*UC*-module. In particular, every* UC*-module is a τ-*UC*-module.*

Proof Suppose that M is an R-module such that M is not τ-UC. By Theorem 3.4, there exist an R-module X and a proper τ-essential submodule Y of X such that the module $X \oplus (X/Y)$ embeds in M. Now $\tau \leq \rho$ gives that Y is a ρ-essential submodule of X. By Theorem 3.4 M is not ρ-UC. ■

We have already seen that a module M is τ_1-UC if and only if M is τ_G-UC. It would be interesting to know for which hereditary torsion theories $\tau \leq \rho$ on Mod-R every τ-UC-module is ρ-UC, and in particular which hereditary torsion theories τ have the property that every τ-UC-module is UC.

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Adnan Menderes University, Department of Mathematics, Science and Art Faculty, Aydin, Türkiye
e-mail: sdogruoz@adu.edu.tr

Hacettepe University, Department of Mathematics, Ankara, Türkiye
e-mail: harmanci@hacettepe.edu.tr

University of Glasgow, Department of Mathematics, Glasgow G12 8QW Scotland, UK
e-mail: pfs@maths.gla.ac.uk