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STABILITY OF ALMOST PERIODIC SOLUTIONS OF AN AUTONOMOUS EQUATION

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The purpose of this paper is to extend to almost periodic (*a.p.*) solutions a stability result on the periodic solutions of the autonomous equation

$$x' = F(x),$$

(cf. Coppel [1], p. 82 or Coddington and Levinson [2], p. 323.)

THEOREM. *Let $u(t)$ be a non-constant a.p. solution of*

$$x' = F(x) \tag{1}$$

and let $F(x)$ be continuously differentiable at all points of the closure of the path

$$x = u(t).$$

Suppose that there exist two supplementary projections P_1, P_2(P_2 is 1-dimensional) such that the variational equation

$$y' = F_x[u(t)]y \tag{2}$$

has a fundamental matrix $Y(t)$, $Y(0)=I$, satisfying

$$\begin{aligned} |Y(t)P_1Y^{-1}(s)| &\leq L\exp(-\alpha(t-s)) \quad \text{for} \quad t \geq s, \\ |Y(t)P_2Y^{-1}(s)| &\leq L \qquad\qquad\qquad\quad \text{for} \quad t \leq s, \end{aligned} \tag{3}$$

where L, α are positive constants.

Then there exist positive constants ε,δ such that if a solution $\varphi(t)$ of (1) *satisfies $|\varphi(t_1)-(t_2)|<\varepsilon$ for some t_1 and t_2, then*

$$|\varphi(t-h)-u(t)| \leq \delta\exp(-\alpha t/2) \quad \text{for} \quad t \geq 0,$$

where h is some real constant, depending on φ.

Proof. Setting $x=z+u(t)$ in (1) we obtain

$$z' = F[z+u(t)]-F[u(t)] = F_x[u(t)]z+f(t, z) \tag{4}$$

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where $f(t, 0)\equiv 0$ and for each $\gamma>0$, there exists a $\delta>0$ such that

(5) $$|f(t, z_1)-f(t, z_2)| \leq \gamma\, |z_1-z_2|$$

uniformly in t, if $|z_1|$, $|z_2|\leq\delta$.

From $u'(t)=F[u(t)]$ it follows by differentiation that $u'(t)$ is a solution of (2). We can write

$$u'(t) = Y(t)\xi \quad \text{for some} \quad \xi \neq 0.$$

Choose γ in (5) so that $\theta=4L\gamma\alpha^{-1}<1$. Let T be the transformation of the space Z of continuous functions $z(t)$ with $\|z(t)\|=\sup_{t\geq 0}\exp(\alpha t/2)|z(t)|\leq\delta$ defined by

$$Tz(t) = Y(t)\xi_1+\int_0^t Y(t)P_1Y^{-1}(s)f(s, z(s))\, ds-\int_t^\infty Y(t)P_2Y^{-1}(s)f(s, z(s))\, ds,$$

where $\xi_1 \in P_1X$ and $|\xi_1|<L^{-1}(1-\theta)\delta$, Then $Tz(t)$ is a solution of

$$z' = F_x[u(t)]z+f[t, z(t)].$$

$Tz(t)$ is continuous and

$$|Tz(t)| \leq L\exp(-\alpha t)\, |\xi_1|+L\gamma\int_0^t \exp(-\alpha(t-s))\, \|z\|\, \exp(-\alpha s/2)\, ds$$
$$+L\gamma\int_t^\infty \|z\|\ \exp(-\alpha s/2)\, ds.$$

Therefore

$$|Tz(t)| \leq L\exp(-\alpha t/2)\, |\xi_1|+4L\gamma\alpha^{-1}\exp(-\alpha t/2)\ \|z\|,$$

or

(6) $$\|Tz(t)\| \leq L\, |\xi_1|+\theta\, \|z\| < (1-\theta)\, \delta+\theta\delta = \delta.$$

Similarly for any two functions $z_1(t)$, $z_2(t)$ in Z we find

$$\|Tz_1(t)-Tz_2(t)\| \leq \theta\|z_1-z_2\|$$

It follows from the contraction mapping principle that the equation $z=Tz$ has a unique solution $z=z(t, \xi_1)$ and hence the equation (1) has a unique solution

$$x(t, \xi_1) = z(t, \xi_1)+u(t).$$

For $t=0$ we have

$$x(0, \xi_1)-u(0) = z(0, \xi_1) = \xi_1-\int_0^\infty P_2Y^{-1}(s)f(s, z(s))\, ds = \xi_1+o(\xi_1),$$

since $|f(t, z)|=o(|z|)$ uniformly in t for $|z|\to 0$ and, by (6),

$$\|z\| \leq (1-\theta)^{-1}L\, |\xi_1|.$$

Let $x(t, \eta)$ denote the solution of (1) with $x(0, \eta)=\eta$. Then

$$x'[0, u(0)] = u'(0) = \xi.$$

For $\eta=u(0)$ the equation

(7) $$x(t, \eta)-z(0, \xi_1)-u(0) = 0$$

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has the solution $t=0$, $\xi_1=0$. It follows by one form of the implicit function theorem that if $|\eta-u(0)|<\sigma$ for some $\sigma>0$, then (7) admits a solution $t=t'$, $\xi_1=\xi_1'$ where

$$|t'| < l \quad \text{and} \quad |\xi_1'| < L^{-1}(1-\theta)\,\delta \tag{8}$$

By the theorem on continuous dependence of solutions on initial values, there exists a constant $\varepsilon>0$ such that if a solution $\psi(t)$ of (1) satisfies

$$|\psi(t_0)-u(t_0)| < 3\varepsilon$$

for some $t_0(0\le t_0\le l)$, then $\psi(t)$ is defined for all $|t|\le l$ and

$$|\psi(0)-u(0)| < \sigma.$$

Hence for some t', ξ' satisfying (8) we can write $\psi(t')$ in the form

$$\psi(t') = z(0, \xi_1')+u(0).$$

Now let $\varphi(t)$ be any solution of (1) such that

$$|\varphi(t_1)-u(t_2)| < \varepsilon,$$

for some t_1, t_2.

Since $u(t)$ is *a.p.*, it is vniformly continuous, and so

$$|u(s)-u(s_1)| \le \varepsilon \quad \text{for} \quad |s-s_1| \le \beta = \beta(\varepsilon).$$

Let $t_0 \in [0, l]$. For any t_2, we can define a translation number τ such that $|t_2+\tau-t_0|\le\beta$. Then

$$|u(t_2+\tau)-u(t_0)| \le \varepsilon.$$

It follows that

$$|\varphi(t_1)-u(t_0)| \le |\varphi(t_1)-u(t_2)|+|u(t_2)-u(t_2+\tau)|+|u(t_2+\tau)-u(t_0)| < \varepsilon+\varepsilon+\varepsilon = 3\varepsilon.$$

Then $\psi(t)=\varphi(t-t_0+t_1)$ is also a solution of (1) and

$$|\psi(t_0)-u(t_0)| < 3\varepsilon$$

Since the solution $\psi(t+t')$ of (1) takes the same value at $t=0$ as the solution $z(t, \xi_1')+u(t)$, we have

$$\psi(t+t') = z(t, \xi_1')+u(t)$$

for all $t\ge0$. Set $h=t_0-t_1-t'$ and we obtain for $t\ge0$,

$$|\varphi(t-h)-u(t)| = |z(t, \xi_1')| \le \delta \exp(-\alpha t/2).$$

This completes the proof.

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