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SPLIT GRAPHS HAVING DILWORTH NUMBER TWO

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1. Introduction. All graphs considered in this paper are finite, undirected, loopless and without multiple edges.

The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. Thus we have

$$E(G) = \{\{x, y\} | x, y \in V(G), x \text{ and } y \text{ are adjacent in } G\}.$$

A set $S \subseteq V(G)$ will be called *independent* if no edge of G has both end vertices in S. The set $S \subseteq V(G)$ will be called *complete* if any two distinct vertices of S are adjacent. If the complement of G is denoted by $\bar{G}$, then S is independent in G if and only if it is complete in $\bar{G}$.

For $x \in V(G)$ we denote by $N(x)$ the set of vertices adjacent to x. The *vicinal preorder* $\precsim$ of G is defined on $V(G)$ by

$$x \precsim y \quad \text{if and only if} \quad N(x) \subseteq N(y) \cup \{y\}.$$

It is easy to see that $\precsim$ is in fact a preorder, i.e. a reflexive and transitive relation.

For any preorder $\precsim$, a subset S of the underlying set is called a *chain*, if for any two elements x and y of S, $x \precsim y$ or $y \precsim x$ holds. S is an *antichain* if for any $x, y \in S$, $x \precsim y$ implies $x = y$. The *dual preorder* $\precsim^*$ is defined by

$$x \precsim^* y \quad \text{if and only if} \quad y \precsim x.$$

The *Dilworth number* $\nabla(G)$ of a graph is the minimum number of chains of the vicinal preorder covering $V(G)$. According to the well-known theorem of Dilworth [**3**], $\nabla(G)$ also equals the cardinality of the maximum size antichains in the vicinal preorder. For $S \subseteq V(G)$, $\nabla_G(S)$ will denote the minimum number of chains of the vicinal preorder of G covering S, which is also equal to the maximum number of elements of S that are pairwise uncomparable in the vicinal preorder of G. Thus $\nabla_G(V(G)) = \nabla(G)$, but generally the Dilworth number $\nabla(H)$ of an induced subgraph H of G is strictly less than $\nabla_G(V(H))$.

It is easy to see that the vicinal preorder of $\bar{G}$ is the dual of that of G. Therefore $\nabla(G) = \nabla(\bar{G})$ and also for every $S \subseteq V(G)$, $\nabla_G(S) = \nabla_{\bar{G}}(S)$.

Some connections between properties of the graph G and the structure of its vicinal preorder have been studied in [**2**], [**5**], and explicitly in [**6**]. The number

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$\nabla(G)$ appears to have a particular significance. For example, *threshold graphs* have been characterized in [**2**] as graphs with Dilworth number 1, i.e. graphs that do not have an induced subgraph isomorphic to $2K_2$, C_4 or P_4 (Figure 1).

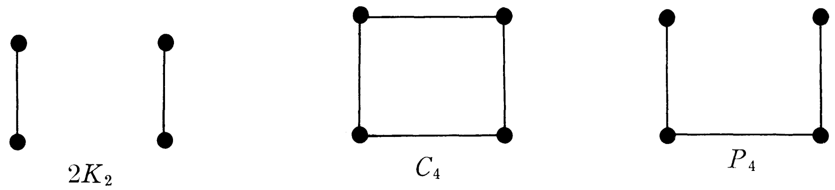

FIGURE 1

G is a *comparability graph* if a strict partial order $<$ (irreflexive, antisymmetric and transitive relation) can be defined on $V(G)$ in such a way that

$$\{x, y\} \in E(G) \quad \text{if and only if} \quad (x < y \text{ or } y < x).$$

G is an *interval graph* if there is a mapping i, called a *representation of* G, that associates to every $x \in V(G)$ a non-empty bounded interval of the naturally ordered set $\mathbf{Z}$ of all integers, such that $\{x, y\} \in E(G)$ if and only if ($x \neq y$ and $i(x) \cap i(y) \neq \emptyset$). The mapping i does not have to be injective (but if we required it to be injective, we would not obtain a different definition). Also, $\mathbf{Z}$ could be replaced by any other infinite linearly ordered set.

A graph is *chordal* if it does not have an induced subgraph of finite girth $\geqq 4$. Interval graphs are the classical examples of chordal graphs.

2. Split graphs.

The following result was proved in [**4**].

THEOREM 1. *For any graph G the following three conditions are equivalent:*
(i) *both G and $\bar{G}$ are chordal;*
(ii) *$V(G)$ can be partitioned into a complete and an independent set;*
(iii) *G does not contain an induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

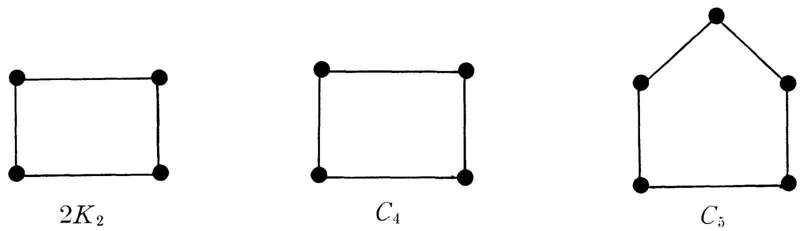

FIGURE 2

A graph satisfying the conditions of Theorem 1 is called a *split graph*. The aim of this paper is to characterize interval and comparability graphs within the class of split graphs.

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3. The good representation and its consequences. We shall need the following:

LEMMA. *Every complete graph K is an interval graph. For any representation i of K, $\bigcap_{x \in V(K)} i(x) \neq \emptyset$.*

Proof. The first part of the lemma is obvious. The second part is proved by induction on $|V(K)|$. If K has only 1 vertex, the statement is trivially true. Assume it is not true for a representation i of a complete graph K, and let K have the smallest possible number of vertices. Let $x \in V(K)$. Then the subgraph induced by $V(K) \backslash \{x\}$ is complete and

$$\bigcap_{\substack{y \in V(K) \\ y \neq x}} i(y)$$

is a non-empty interval $[a, b]$ of $\mathbf{Z}$. Clearly there is a vertex $y \neq x$ with $i(y) = [a, c]$ and also a $z \neq x$ with $i(z) = [d, b]$. Let $i(x) = [e, f]$. If $f < a$, then $i(x) \cap i(y) = \emptyset$ and if $e > b$, then $i(x) \cap i(z) = \emptyset$. Since none of the above intersections can be empty, we must have $f \geqq a$ and $e \leqq b$, i.e. $i(x) \cap [a, b] \neq \emptyset$. But $i(x) \cap [a, b] = \bigcap_{y \in V(K)} i(y)$.

Let G be a split graph with a partition $V(G) = K \cup I$, K complete, I independent. A *good representation* i of G is one for which all the $i(x)$, $x \in I$, are singletons.

PROPOSITION 1. *Every split interval graph G has a good representation.*

Proof. For every $x \in I$, $x \cup N(x)$ induces a complete graph. According to the lemma, $i(x) \cap \bigcap_{y \in N(x)} i(y) \neq \emptyset$. Let $\bar{x}$ be any element of $i(x) \cap \bigcap_{y \in N(x)} i(y)$. A good representation is obtained by replacing each $i(x)$, $x \in I$, by $\{\bar{x}\}$.

It is clear that the complement $\bar{G}$ of a split graph G is a split graph. The following proposition can also be obtained from Theorem 2 of [**7**]. We shall give a direct proof.

PROPOSITION 2. *Let G be a split graph. G is an interval graph if and only if $\bar{G}$ is a comparability graph.*

Proof. Let $V(G) = K \cup I$, $K \cap I = \emptyset$, K complete, I independent. Assume G is an interval graph. Let i be a good representation of G. Define a strict partial order $<$ on $V(G) = V(\bar{G})$ as follows:

$$x < y \quad \text{if and only if} \quad (\forall a \in i(x), \forall b \in i(y), a < b \text{ in the natural order of } \mathbf{Z}).$$

It is easy to check that $\{x, y\} \in E(\bar{G})$ if and only if ($x < y$ or $y < x$). Thus $\bar{G}$ is a comparability graph.

Let $\bar{G}$ be a comparability graph with an appropriate strict order $<$ on $V(\bar{G}) = V(G)$. Since I is complete in $\bar{G}$, its elements can be labelled $x_1, \ldots, x_n$

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in a unique way such that $x_1 < \ldots < x_n$. For each x_k, $1 \leqq k \leqq n$, let $i(x_k) = \{2k\}$. For each $y \in K$, let $i(y) = \{2l-1, 2l, 2l+1 | 1 \leqq l \leqq n$ and $\{y, x\} \notin E(\bar{G})\}$. Then i is an interval representation of G and it is in fact a good representation.

THEOREM 2. *Let G be a split graph,* $V(G) = K \cup I$, $K \cap I = \emptyset$, *K complete, I independent. The following two conditions are equivalent:*
(i) *G is an interval graph;*
(ii) $\nabla_G(I) \leqq 2$.

Proof. (i) $\Rightarrow$ (ii). Let i be a good interval representation of G. Let $a \in \bigcap_{x \in K} i(x)$. Define $C_1 = \{x \in I | i(x) = \{b\}$ and $b \leqq a$ in $\mathbf{Z}\}$, $C_2 = I \backslash C_1$. It can be verified that C_1 and C_2 are chains in the vicinal preorder of G.

(ii) $\Rightarrow$ (i). Let C_1 and C_2 be two disjoint chains in the vicinal preorder of G such that $C_1 \cup C_2 = I$. Let $C_1 = \{x_1, \ldots, x_n\}$ and $C_2 = \{y_1, \ldots, y_m\}$ be enumerations of their vertices such that $1 \leqq k \leqq j \leqq n$ implies $x_k \gtrsim x_j$ and $1 \leqq k \leqq j \leqq m$ implies $y_k \gtrsim y_j$. Let $i(x_k) = \{-k\}$ for $1 \leqq k \leqq n$ and $i(y_k) = \{k\}$ for $1 \leqq k \leqq m$. For $z \in K$, let $i(z) = \{i(v) | v \in N(z) \cap I\} \cup \{0\}$. Then i is a good representation of G.

Remark. For the good representation i defined in the second part of the above proof, we have

for $x, y \in K$: $x \lesssim y \Leftrightarrow i(x) \subseteq i(y)$;
for $x, y \in C_1$, $i(x) = \{c\}$, $i(y) = \{b\}$: $x \lesssim y \Leftrightarrow b \geqq c$ in $\mathbf{Z}$; and
for $x, y \in C_2$, $i(x) = \{c\}$, $i(y) = \{b\}$: $x \lesssim y \Leftrightarrow c \geqq b$ in $\mathbf{Z}$.

THEOREM $\bar{2}$. *Let G be a split graph,* $V(G) = K \cup I$, $K \cap I = \emptyset$, *K complete, I independent. Then G is a comparability graph if and only if* $\nabla_G(K) \leqq 2$.

COROLLARY 1. *A split graph G is simultaneously an interval and a comparability graph if and only if* $\nabla(G) \leqq 2$.

Proof. Let $V(G) = K \cup I$, $K \cap I = \emptyset$, K complete, I independent. For every $x \in K$, $y \in I$, we have $x \gtrsim y$ in the vicinal preorder and consequently $\nabla(G) = \max\,(\nabla_G(I), \nabla_G(K))$.

Remark. There exist interval (as well as comparability) split graphs with arbitrarily high Dilworth number.

4. Forbidden subgraphs. It is clear that any induced subgraph of an interval split graph is an interval split graph. Consequently, these graphs are characterized by a set F of forbidden induced subgraphs. This set turns out to be finite, unlike in the case of general interval graphs (every polygon of girth at least 4 is a minimal non-interval graph). The problem is analogous for comparability graphs: every odd polygon of girth at least 5 is a minimal non-comparability graph and therefore no characterization of these graphs can be given by a finite set of forbidden subgraphs. However, it will be shown in this

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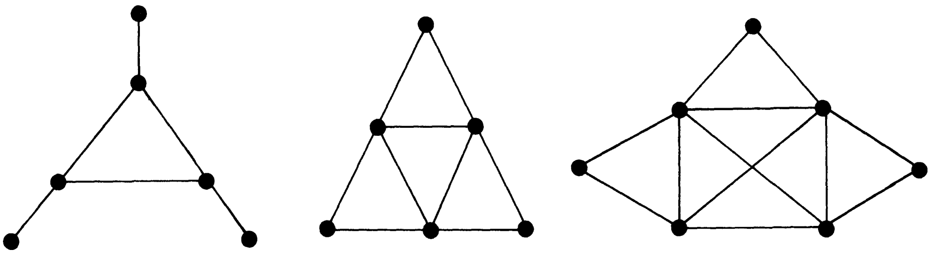
section that such a characterization can be given within the class of split graphs.

THEOREM 3. *A split graph G is an interval graph if and only if it does not contain an induced subgraph isomorphic to any of the graphs G_1, G_2 or G_3 of Figure* 3.



G_1 G_2 G_3

FIGURE 3

Proof. It is easy to see that G_1, G_2 and G_3 are not interval graphs but every proper induced subgraph of any of them is an interval graph. An interval split graph clearly cannot contain any of G_1, G_2 or G_3.

Conversely, let G be a minimal non-interval split graph. We claim that G is isomorphic to G_1, G_2 or G_3. Let $V(G) = K \cup I$, $K \cap I = \emptyset$, K complete, I independent. I contains three vertices x_1, x_2 and x_3 forming an antichain in the vicinal preorder of G. Since I is independent, this means precisely that none of the sets $N(x_1)$, $N(x_2)$ and $N(x_3)$ is a subset of any other. Moreover, since G is minimal, $I = \{x_1, x_2, x_3\}$. Let

$$a_1 \in N(x_1)\backslash N(x_2),$$
$$a_2 \in N(x_2)\backslash N(x_3),$$
$$a_3 \in N(x_3)\backslash N(x_1),$$
$$b_1 \in N(x_1)\backslash N(x_3),$$
$$b_2 \in N(x_2)\backslash N(x_1),$$
$$b_3 \in N(x_3)\backslash N(x_2).$$

It is clear that a_1, a_2, a_3 are distinct and so are b_1, b_2, b_3.

We have to distinguish three cases.

Case 1. a_1, a_2, a_3, b_1, b_2, b_3 are all distinct. Then the minimality of G implies that

$$\{x_1, a_2\}, \{x_1, b_3\}, \{x_2, b_1\}, \{x_2, a_3\}, \{x_3, b_2\}, \{x_3, a_1\}$$

all belong to $E(G)$. But then $\{x_1, x_2, x_3, a_1, a_2, a_3\}$ induces a subgraph isomorphic to G_2, in contradiction with the minimality of G.

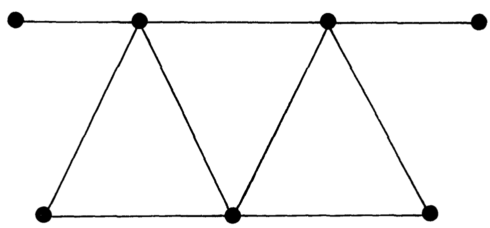
Case 2. $|\{a_1, a_2, a_3, b_1, b_2, b_3\}| < 6$ but $\{a_1, b_1\}$, $\{a_2, b_2\}$ and $\{a_3, b_3\}$ are pairwise disjoint. There exists an i with $a_i = b_i$, say $a_2 = b_2$. If $a_1 \neq b_1$ and

$a_3 \neq b_3$, then $\{x_2, b_1\}, \{x_2, a_3\}, \{x_1, b_3\}, \{x_3, a_1\} \in E(G)$ and $\{x_1, x_2, x_3, b_1, b_3, a_3\}$ induces a proper subgraph isomorphic to G_2, a contradiction. Therefore, $a_1 = b_1$ or $a_3 = b_3$; say, $a_1 = b_1$. If $a_3 \neq b_3$, then G is isomorphic to G_3 and if $a_3 = b_3$, then G is isomorphic to G_1.

Case 3. $\{a_1, b_1\}, \{a_2, b_2\}$ and $\{a_3, b_3\}$ are not pairwise disjoint. We can assume for instance that $\{a_1, b_1\} \cap \{a_3, b_3\} \neq \emptyset$, i.e. that $a_1 = b_3$. If we have $a_2 = b_2$ and $\{b_1, x_2\}, \{a_3, x_2\} \in E(G)$, then $\{x_1, x_2, x_3, b_1, a_1, a_3\}$ induces a proper subgraph isomorphic to G_2, which is impossible. If $a_2 = b_2$ and $\{b_1, x_2\} \notin E(G)$, $\{a_3, x_2\} \notin E(G)$, then $\{x_1, x_2, x_3, b_1, b_2, a_3\}$ induces a subgraph isomorphic to G_1, which is equally impossible. Therefore, if $a_2 = b_2$, then exactly one of the edges $\{b_1, x_2\}$ and $\{a_3, x_2\}$ is present, say $\{b_1, x_2\} \in E(G)$, $\{a_3, x_2\} \notin E(G)$. But in this case G is isomorphic to G_3. Assume now that $a_2 \neq b_2$. The minimality of G implies $\{x_1, a_2\} \in E(G)$ and $\{x_3, b_2\} \in E(G)$. It is then easy to check that $\{x_1, x_2, x_3, a_1, a_2, b_2\}$ induces a proper subgraph isomorphic to G_2, which is impossible. The proof of Theorem 3 is now complete.

Considering Proposition 2 we have the following:

COROLLARY 1. *A split graph G is a comparability graph if and only if it does not have an induced subgraph isomorphic to G_1 or G_2 of Figure 3 or $\bar{G}_3$ of Figure 4.*



$\bar{G}_3$

FIGURE 4

COROLLARY 2. *For a split graph G, $\nabla(G) \leqq 2$ if and only if G does not contain an induced subgraph isomorphic to G_1, G_2, G_3 or $\bar{G}_3$.*

Remark. G_2 is the complement of G_1, and they are neither interval nor comparability graphs. G_3 is a comparability graph, which is not an interval graph and $\bar{G}_3$ is an interval graph without being a comparability graph.

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