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NECESSARY AND SUFFICIENT FIXED POINT CRITERA INVOLVING ATTRACTORS

JACEK JACHYMSKI

Let f be a continuous self-map on a complete metric space X and $p \in X$. Let c be a positive real. Equivalent conditions are given for the singleton $\{p\}$ to be an attractor of a set of c-fixed points of f. We also establish equivalent conditions for the existence of a contractive fixed point of f. These results subsume a body of fixed point theorems.

1. INTRODUCTION

In recent years many papers have appeared offering conditions of a contractive type which a self-map f on a metric space X is to fulfil to ensure the convergence of successive approximations and the existence of a fixed point. These conditions usually have the form of inequalities in which some auxiliary functions occur, and thus they are too special to be necessary also for the existence of a fixed point. Since few papers deal with equivalent conditions, it may be of interest to find some necessary and sufficient fixed point criterions in such a way they could easily yield some of these special contractive theorems. This is our purpose here - we offer in Section 3 a principle which will enable us to give some equivalent conditions for a point p to attract a set of c-fixed points of f (Theorem 3) and to attract each point of X under f (Theorem 2). As will be indicated in Section 5, both criteria subsume a body of fixed point theorems.

2. PRELIMINARIES

Let f be a self-map on a topological space X and let A, B be subsets of X. Let $\mathbb{N}$ be the set of all positive integers. A is an *attractor* for B under f if A is nonempty compact and f-invariant, and for any open set G containing A there exists $k \in \mathbb{N}$ such that

$$f^n(B) \subseteq A, \quad \text{for all } n \geqslant k.$$

The concept of attractor was first introduced by Nussbaum [**10**] who considered attractors for compact sets. Later several authors examined the case when the singleton

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$\{p\}$ was an attractor for some family $\mathbb{F}$ of sets under f. The following families $\mathbb{F}$ were considered:

(a) $\mathbb{F} = \{B \subseteq X : B \text{ is compact}\}$, (Janos, Ko, Tan [5]);
(b) $\mathbb{F} = \{X\}$, (Leader [8]);
(c) $\mathbb{F} = \{\{x\} : x \in X\} \cup \{U_p\}$, where U_p is a neighbourhood of p (Leader [7]).

It is clear that if p attracts some nonempty subset of X then $p = fp$. We say that p is a *contractive fixed point* of f, if p attracts each point of X ([8]). We shall be employing the concept of attractor in Section 4.

Throughout this note $\mathbb{Z}_+$ denotes the set of all non-negative integers, $\mathbb{R}_+$ is the set of all non-negative reals. The set $\{x, fx, f^2x, \ldots\}$ is called an *orbit* of a point x and it is denoted by $O(x)$. Occasionally, we use the notation $x^k = f^k x$, for the sake of brevity. For $c \in \mathbb{R}_+$, $\mathrm{Fix}_c f$ denotes the set of all c-fixed points of f, that is points x with $d(x, fx) \leqslant c$ ([2]). The Hausdorff metric for sets is denoted by d_H.

3. A necessary and sufficient uniform Cauchy criterion and corollaries

THEOREM 1. *Let f be a self-map on a metric space X, $c > 0$ and let F be a non-empty subset of $\mathrm{Fix}_c f$. Define the set P by $(x, y) \in P$ if and only if there exist $i, j \in \mathbb{Z}_+$ and $z \in F$ such that $x = f^i z$, $y = f^j z$ and $d(x, y) \leqslant c$. Then the following statements are equivalent:*

(i) *For $z \in F$, the sequences $\{f^n z\}$ are uniformly Cauchy;*
(ii) *$d(f^n x, f^n y) \to 0$ uniformly for all $(x, y) \in P$;*
(iii) *For some increasing sequence $\{k_n\}$ of positive integers $d(f^{k_n} x, f^{k_n} y) \to 0$ uniformly for all $(x, y) \in P$, and $d(f^n z, f^{n+1} z) \to 0$ uniformly for all $z \in F$.*

PROOF: We shall verify implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Given (i), observe that $d(f^n x, f^n y) \to 0$ uniformly for all $x, y \in O(z)$ and $z \in F$, so (ii) holds. Given (ii), since $(z, fz) \in P$ for all $z \in F$, we have that $d(f^n z, f^{n+1} z) \to 0$ uniformly for all $z \in F$, so (iii) holds.

To prove (iii) implies (i) take $r \in \mathbb{N}$ such that $d(f^r x, f^r y) \leqslant c/2$ for all (x, y) in P. It means that

$$\text{(1)} \qquad \text{if } d(z^i, z^j) \leqslant c \text{ for some } z \in F \text{ and } i, j \in \mathbb{Z}_+ \text{ then } d(z^{i+r}, z^{j+r}) \leqslant c/2.$$

Consider a subsequence $\{z^{rn}\}_{n=1}^{\infty}$. From (iii) and by the triangle inequality, we get that

$$d\left(z^{rn}, z^{r(n+1)}\right) \to 0, \text{ as } n \to \infty, \text{ uniformly for all } z \text{ in } F,$$

so there exists n_0 in $\mathbb{N}$ such that

$$d\left(z^{rn_0}, z^{r(n_0+1)}\right) \leqslant c/2 \text{ for all } z \text{ in } F. \tag{2}$$

By induction we shall prove that, for any $n \geqslant n_0$ and all z in F,

$$d(z^{rn_0}, z^{rn}) \leqslant c. \tag{3}$$

The case $n = n_0$ is obvious. Assuming (3) to hold for some $n \geqslant n_0$ we shall prove it for $n+1$. Observe that (1) and our induction hypothesis give

$$d\left(z^{r(n_0+1)}, z^{r(n+1)}\right) \leqslant c/2. \tag{4}$$

So apply (2), (4) and the triangle inequality to get that (3) holds for $n+1$.

We shall prove that for $z \in F$ the sequences $\{z^{rn}\}_{n=1}^{\infty}$ are uniformly Cauchy. From (iii), given $0 < \varepsilon \leqslant c$, we can choose ℓ in $\mathbb{N}$ such that

$$d\left(z^{\ell+i}, z^{\ell+j}\right) < \varepsilon, \quad \text{for all } z \text{ in } F \text{ and } i, j \text{ in } \mathbb{Z}_+ \text{ with } d\left(z^i, z^j\right) \leqslant c. \tag{5}$$

In particular, from (3) we get that

$$d\left(z^{rn_0+\ell}, z^{rn+\ell}\right) < \varepsilon, \quad \text{for all } z \text{ in } F \text{ and } n \geqslant n_0.$$

Since $\varepsilon \leqslant c$, we can use (5) again to obtain after r steps that

$$d\left(z^{r(n_0+\ell)}, z^{r(n+\ell)}\right) < \varepsilon, \quad \text{for all } z \text{ in } F \text{ and } n \geqslant n_0.$$

Hence and by the triangle inequality,

$$d(z^{rn}, z^{rm}) < 2\varepsilon, \quad \text{for all } z \text{ in } F \text{ and } n, m \geqslant n_0 + 1.$$

Thus, for $z \in F$, $\{z^{rn}\}_{n=1}^{\infty}$ is uniformly Cauchy.

That $\{z^n\}$ are uniformly Cauchy for all z in F follows easily from $\{z^{rn}\}_{n=1}^{\infty}$ being uniformly Cauchy and $d\left(z^n, z^{n+1}\right) \to 0$ uniformly for $z \in F$. ∎

REMARK. Theorem 1 easily yields and extends Leader's Fixed Point Principle (Theorem 1 in [6]). It is also possible to generalise Theorem 1 for two mappings in such a way it yields results of Som and Mukherjee from [12]. Let us notice here that, in our opinion, the assumptions of Theorem 1 [12] are susceptible to various interpretations and they need some reformulation. In particular, condition (4) of [12] should be of the following form:

"the sequences $\{x_n\}$ generated by the generalised orbit of the points x with $d(x, f_l x) \leqslant c$, $l = 1, 2$, are uniformly Cauchy",

analogous to its counterpart (number (5) in [6]) from Leader's Theorem 1.

Corollary 1. (Cauchy Criterion for a Sequence of Iterates). *Let f be a self-map on a metric space X and $z \in X$. For $n \in \mathbb{Z}_+$, define*

$$c_n := d(f^n z, f^{n+1} z) \text{ and } P_n := \{(x, y) : d(x, y) \leqslant c_n \text{ and } x, y \in O(f^n z)\}.$$

Then the following statements are equivalent:

(i) *The sequence $\{f^n z\}$ is Cauchy;*

(ii) *There exists $r \in \mathbb{Z}_+$ such that $d(f^n x, f^n y) \to 0$ uniformly for all (x, y) in P_r;*

(iii) *There exist $r \in \mathbb{Z}_+$ and an increasing sequence $\{k_n\}$ of positive integers such that $d(f^{k_n} x, f^{k_n} y) \to 0$ uniformly for all (x, y) in P_r, and $d(f^n z, f^{n+1} z) \to 0$.*

Proof: To get (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) apply Theorem 1 taking the singleton $\{f^r z\}$ as F and $c = c_r$. □

Corollary 2. (Convergence of the Successive Approximation). *Let f be a self-map on a complete metric space X. Then the following statements are equivalent:*

(i) *There exists p in X such that $f^n x \to p$, for all x in X;*

(ii) *For all x, y in X, $\liminf\limits_{n \to \infty} d(f^n x, f^n y) = 0$ and, for any $z \in X$, there exists r in $\mathbb{Z}_+$ such that $d(f^n x, f^n y) \to 0$ uniformly for all x, y in $O(f^r z)$ with $d(x, y) \leqslant d(f^r z, f^{r+1} z)$;*

(iii) *For all x, y in X, $\liminf\limits_{n \to \infty} d(f^n x, f^n y) = 0$ and, for any $z \in X$, there exist r in $\mathbb{Z}_+$ and an increasing sequence $\{k_n\}$ of positive integers such that $d(f^{k_n} x, f^{k_n} y) \to 0$ uniformly for all x, y in $O(f^r z)$ with $d(x, y) \leqslant d(f^r z, f^{f+1} z)$, and $d(f^n z, f^{n+1} z) \to 0$.*

Proof: Apply Corollary 1 and a completeness argument. □

4. Two necessary and sufficient fixed point principles involving attractors

Theorem 2. *Let f be a continuous self-map on a complete metric space X and $p \in X$. Then p attracts each point of X if and only if one (and hence all) of the conditions (i)-(iii) of Corollary 2 holds.*

In particular, if for all x, y in X $d(f^n x, f^n y) \to 0$ and, for some $c > 0$, this convergence is uniform for all x, y with $d(x, y) \leqslant c$ then f has a contractive fixed point.

Proof: Apply Corollary 2 and a continuity argument. □

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Theorem 3. *Let f be a self-map on a complete metric space X and $p \in X$. Let c be a positive real such that the set $F = \mathrm{Fix}_c f$ is nonempty and let P be the set as in Theorem 1. Then the following statements are equivalent:*

(1°) *The singleton $\{p\}$ is an attractor for $\mathrm{Fix}_c f$ under f;*

(2°) *The sequences $\{f^n x\}$ converge to p uniformly for all x in $\mathrm{Fix}_c f$;*

(3°) $d_H\left(\{p\}, \overline{f^n(\mathrm{Fix}_c f)}\right) \to 0$;

(4°) *For $x, y \in \mathrm{Fix}_c f$, $\liminf_{n\to\infty} d(f^n x, f^n y) = 0$ and (*) $d(f^n x, f^n y) \to 0$ uniformly for all (x, y) in P;*

(5°) *For $x, y \in \mathrm{Fix}_c f$, $\liminf_{n\to\infty} d(f^n x, f^n y) = 0$, $d\left(f^n z, f^{n+1} z\right) \to 0$ uniformly for all z in $\mathrm{Fix}_c f$ and there exists an increasing sequence $\{k_n\}$ of positive integers such that $d\left(f^{k_n} x, f^{k_n} y\right) \to 0$ uniformly for all (x, y) in P.*

Moreover, each of conditions (1°)-(5°) implies that

(6°) $\{p\} = \bigcap_{n\in N} f^n(\mathrm{Fix}_c f)$.

The proof of Theorem 3 will be preceded by two lemmas on attractors.

Lemma 1. *Let f be a self-map on a metric space X, $p \in X$ and $B \subseteq X$. Then the following statements are equivalent:*

(i) *The singleton $\{p\}$ is an attractor for B under f;*

(ii) *$p = f(p)$ and the sequences $\{f^n x\}$ converge to p uniformly for all x in B;*

(iii) *$p = f(p)$ and $d_H\left(\{p\}, \overline{f^n(B)}\right) \to 0$.*

Proof: The equivalence (i) ⇔ (ii) was observed in [8], (ii) ⇔ (iii) follows immediately from the equality

$$d_H\left(\{p\}, \overline{f^n(B)}\right) = \sup_{x\in B} d(p, f^n x).$$

∎

Lemma 2. *Let f be a self-map on a metric space X and let A, B be nonempty subsets of X such that $A \subseteq B$. Then the following statements are equivalent:*

(i) *A is an attractor for B under f;*

(ii) *A is compact and f-invariant, and $d_H\left(A, \overline{f^n(B)}\right) \to 0$. Moveover, each of the above conditions implies that*

(iii) $A = \bigcap_{n\in N} f^n(B)$.

If B is compact and $f(B) \subseteq B$ then the conditions (i)-(iii) are equivalent.

PROOF: (i) $\Rightarrow$ (ii). Since A is f-invariant and $A \subseteq B$, we get that $A \subseteq f^n(B)$, for any $n \in \mathbb{N}$. Thus $d_H\left(A, \overline{f^n(B)}\right) = \sup_{x \in f^n(B)} d(x, A)$. Given $\varepsilon > 0$, define $A_\varepsilon := \bigcup_{x \in A} K(x, \varepsilon)$. By (i), there exists n_0 in $\mathbb{N}$ such that $f^n(B) \subseteq A_\varepsilon$, for all $n \geqslant n_0$. That means given $n \geqslant n_0$ and $x \in f^n(B)$, $d(x, A) < \varepsilon$. Thus $d_H\left(A, \overline{f^n(B)}\right) \leqslant \varepsilon$, so (ii) holds.

(ii) $\Rightarrow$ (i). Take any open set G containing A. Since A is compact, there exists $\varepsilon > 0$ such that $A_\varepsilon \subseteq G$, where A_ε is as in the proof of (i) $\Rightarrow$ (ii). By (ii), there exists n_0 in $\mathbb{N}$ such that, for $n \geqslant n_0$, $d_H\left(A, \overline{f^n(B)}\right) < \varepsilon$. Thus $f^n(B) \subseteq A_\varepsilon \subseteq G$, so (i) holds.

(ii) $\Rightarrow$ (iii). Suppose there exists $x \in \bigcap_{n \in N} f^n(B) \setminus A$. Then, for any $n \in \mathbb{N}$, $d_H\left(A, \overline{f^n(B)}\right) \geqslant d(x, A) > 0$ which contradicts (ii). Since simultaneously $A \subseteq \bigcap_{n \in N} f^n(B)$, we get that (iii) holds.

Now assume that B is compact and $f(B) \subseteq B$. Then $\bigcap_{n \in N} f^n(B)$ is nonempty compact and f-invariant. We leave it to the reader to verify that (iii) implies then that $d_H(A, f^n(B)) \to 0$, so (ii) holds. ∎

PROOF OF THEOREM 3: The conditions (1°), (2°) and (3°) are equivalent by Lemma 1. To prove the equivalence of (2°), (4°) and (5°) apply Theorem 1 and use the condition $\liminf_{n \to \infty} d(f^n x, f^n y) = 0$, for all x, y in X. That each of conditions (1°)-(5°) implies (6°), follows from Lemma 2. ∎

REMARK. Simple examples show that if X is not compact then (6°) need not imply any of the conditions (1°)-(5°).

COROLLARY 3. *For a nonexpansive self-map f on a compact metric space, the conditions (1°)-(6°) of Theorem 3 are equivalent.*

PROOF: Observe that in this case $\mathrm{Fix}_c f$ is compact and f-invariant, and apply Theorem 3 and Lemma 2. ∎

COROLLARY 4. *Under the assumptions of Theorem 3, the condition (*) implies that $\mathrm{Fix}\, f$ is nonempty closed, $d_H\left(\mathrm{Fix}\, f, \overline{f^n(\mathrm{Fix}_c f)}\right) \to 0$ and $\mathrm{Fix}\, f = \bigcap_{n \in N} f^n(\mathrm{Fix}_c f)$. Hence $\mathrm{Fix}\, f$ attracts $\mathrm{Fix}_c f$ under f, if X is compact.*

PROOF: Apply Theorem 1 and Lemma 2. ∎

5. FINAL REMARKS

The theorems of Section 4 subsume a body of fixed point theorems. In particular, Theorem 2 easily yields Theorem 1.2 in [9] and it improves Theorem 3 in [1], which

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holds also for $\Phi 1$ functions defined there. Theorem 3 can also be used then to deduce that a fixed point attracts a set of c-fixed points. Corollary 4 easily yields a recent result of Hicks (Theorem 3 in [**3**]).

It is worth underlining that for many contractive mappings their fixed point attracts a set of c-fixed points for some c in $\mathbb{R}_+$ or even for all c in $\mathbb{R}_+$. In particular, if f is a continuous self-map on a complete metric space X satisfying the condition

$$\text{(HR)} \qquad d(fx,\, f^2x) \leqslant \alpha\, d(x,\, fx), \text{ for some } 0 \leqslant \alpha < 1 \text{ and all } x \text{ in } X$$

then f has a fixed point p; if such a point is unique then it attracts a set of c-fixed points for all c in $\mathbb{R}_+$. To see that use the inequality

$$d(f^n x,\, p) \leqslant (\alpha^n/(1-\alpha))d(x,\, fx), \text{ for all } x \text{ in } X,$$

which can be deduced from (HR). For examples of mappings satisfying (HR) see [**4**] and [**11**]. Theorem 3 can be also applied to mappings satisfying some generalised (HR) condition introduced in [**3**].

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Institute of Mathematics
Technical University of Lódź
ul. Zwirki 36
90-924 Lódź
Poland