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On lower estimates for linear forms involving certain transcendental numbers

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Let

$$\phi_\lambda(z) = \sum_{n=0}^{\infty} z^n/(\lambda+1) \ldots (\lambda+n) ,$$

where λ is rational and not an integer. The author investigates lower estimates for example for

$$\left|x_1'x_2' \ldots x_k'\left(x_1\phi_\lambda\left(\alpha_1\right) + \ldots + x_k\phi_\lambda\left(\alpha_k\right)\right)\right| ,$$

where the α_i are distinct rational numbers not 0, and where $x_1, \ldots, x_k$ are integers and $x_i' = \max\left(1, \left|x_i\right|\right)$.

1. Introduction

In 1965 Baker [1] obtained lower bounds for the expressions

$$A = \left|x_1x_2 \ldots x_k\left(x_1F_1 + \ldots + x_kF_k\right)\right| , \quad B = \left|yF_1-y_1\right| \ldots \left|yF_k-y_k\right| ,$$

where $F_i = e^{\alpha_i}$, α_i $(i = 1, 2, \ldots, k)$ are distinct rational numbers, and in B all $\alpha_i \neq 0$; x_i $(i = 1, 2, \ldots, k)$ are non-zero integers, and y_i $(i = 1, 2, \ldots, k)$, $y > 0$ are integers. He proved that there exist positive constants c_0, c_1 depending only on k, $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the inequalities

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$$A < x^{1-c_0(\log\log x)^{-\frac{1}{2}}}, \quad B < y^{-1-c_1(\log\log y)^{-\frac{1}{2}}},$$

where $x = \max\{|x_1|, |x_2|, \ldots, |x_k|\}$, are respectively satisfied only by a finite number of sets of non-zero integers $x_1, x_2, \ldots, x_k$, and only by a finite number of positive integers y. In a recent paper Mahler [4] improved these estimates by obtaining bounds containing no unknown constants.

In order to prove the above estimates Baker developed a new method in which he used certain ideas of Siegel [6], [7]. The aim of the present paper is to use this same method to obtain estimates analogous to those of Baker, but here F_i $(i = 1, 2, \ldots, k)$ are certain values of the function

$$\phi_\lambda(z) = \sum_{n=0}^{\infty} z^n/(\lambda+1) \ldots (\lambda+n) \tag{1}$$

with rational $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

We define

$$f_i(z) = \phi_\lambda(\alpha_i z) \quad (i = 1, 2, \ldots, k), \tag{2}$$

where α_i are distinct non-zero rational numbers. The following theorems will be proved.

THEOREM 1. *Let* $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *be a rational number, and let the numbers* $f_1(1), f_2(1), \ldots, f_k(1)$ *be defined by* (2), *where* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are distinct non-zero rational numbers. There then exists a constant* $c_0 = c_0(k, \lambda, \alpha_1, \ldots, \alpha_k) > 0$ *such that for any integer* y *the inequality*

$$\left|x_1'x_2' \ldots x_k'\left(x_1f_1(1) + \ldots + x_kf_k(1) + y\right)\right| < x^{-c_0(\log\log x)^{-\frac{1}{2}}}, \tag{3}$$

where x_i $(i = 1, 2, \ldots, k)$ *are integers,* $x_i' = \max\{1, |x_i|\}$ *and* $x = \max\{x_1', x_2', \ldots, x_k'\}$ *, can be satisfied only if* $x < c$ *,* $\log\log c = 2\left(c_0/20k\right)^4$.

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THEOREM 2. *Let* $\lambda, \alpha_1, \alpha_2, \ldots, \alpha_k$ *satisfy the hypotheses of Theorem* 1. *Then there exist constants* $c_1 = c_1(k, \lambda, \alpha_1, \ldots, \alpha_k) > 0$ *and* $\bar{c}(c_1) > 0$ *such that for any integers* $y_1, y_2, \ldots, y_k$ *the inequality*

$$(4) \qquad |yf_1(1)-y_1| \ldots |yf_k(1)-y_k| < y^{-1-c_1(\log\log y)^{-\frac{1}{2}}}$$

can be satisfied only if the positive integer y *is less than* $\bar{c}$.

Fel'dman [3] considered the function values $\phi_{\lambda_i}(\alpha)$, proving that if $\alpha \neq 0$ is a rational number and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are rational numbers other than negative integers satisfying $\lambda_i - \lambda_j \notin Z$ if $i \neq j$, then there exists a constant $c_0 = c_0(\alpha, \lambda_1, \ldots, \lambda_k) > 0$ such that, for all integers $x_1, x_2, \ldots, x_k$, y , $x_1^2 + x_2^2 + \ldots + x_k^2 > 0$,

$$\left|x_1\phi_{\lambda_1}(\alpha) + \ldots + x_k\phi_{\lambda_k}(\alpha) + y\right| > X^{-1-c_0(\log\log(X+2))^{-1}} ,$$

where $X = x_1'x_2' \ldots x_k'$, $x_i' = \max\{1, |x_i|\}$.

It should be noted that the arithmetic nature of the function values $\phi_{\lambda_i}(\alpha_j)$ has been considered in many papers. Šidlovskiĭ [5] has established the algebraic independence over Q of the mn numbers $\phi_{\lambda_i}(\alpha_j)$, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are rational numbers such that λ_i , $\lambda_i - \lambda_j$ $(i, j = 1, 2, \ldots, n;\ i \neq j)$ are not integers, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are distinct non-zero algebraic numbers.

In the present paper we follow Baker's method. First we shall establish certain lemmas analogous to those of [1], and we shall then prove the above theorems using deductions analogous to the corresponding proofs of [1].

2. Lemmas

We begin with a lemma which can be proved easily by means of a box argument (see [1], p. 36).

LEMMA 1. *Let* m, n *be positive integers with* $n > m$. *Suppose that* a_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ *are integers with absolute values at most* A . *Then there are integers* $x_1, x_2, \ldots, x_n$, *not all zero, with absolute values at most* $(nA)^{m/(n-m)} + 2$, *such that*

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \quad (i = 1, 2, \ldots, m) .$$

In the following let $c_2, c_3, \ldots$ denote positive constants which depend only on $k, \lambda, \alpha_1, \ldots, \alpha_k$. First we should aim at a result analogous to Baker's [1], Lemma 2.

LEMMA 2. *Let* $r_1, r_2, \ldots, r_k$ *be positive integers and let* $r = \max\{r_i\} > 2$, $r_0 = r$. *Then there are polynomials* $P_i(z)$ $(i = 0, 1, \ldots, k)$, *not all identically zero, with the following properties:*

1°. *for each* i , $P_i(z)$ *has degree at most* r , *a zero at* $z = 0$ *of order at least* $r - r_i$, *and integer coefficients with absolute values at most*

$$r_i!c_2^{r(\log r)^{\frac{1}{2}}} ;$$

2°. *the approximation form*

$$R(z) = P_0(z) + \sum_{i=1}^{k} P_i(z)f_i(z) = \sum_{h=0}^{\infty} \rho_h z^h \tag{5}$$

vanishes at $z = 0$ *of order at least*

$$m = r + r_1 + \ldots + r_k + k - \left[r(\log r)^{-\frac{1}{2}}\right] , \tag{6}$$

and, for each h ,

$$|\rho_h| < r!(h!)^{-1}c_3^{h+r(\log r)^{\frac{1}{2}}} . \tag{7}$$

Proof. Put $L = \max\{|\alpha_i|\}$. Further, let l denote the least common denominator of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and let L_h $(h = 0, 1, \ldots)$

denote the least common denominator of the numbers

$$\frac{j!}{(\lambda+1)(\lambda+2)\dots(\lambda+j)} \quad (j = 0, 1, \dots, h) .$$

We put $p_{ij} = 0$ for all integral values i, j other than the $n = r + r_1 + \dots + r_k + k + 1$ pairs given by $0 \le i \le k$, $r - r_i \le j \le r$. For these values i, j we define p_{ij} as integers, not all zero, satisfying the following system of m equations;

$$(8) \quad L_h l^h p_{0h} + \sum_{i=1}^{k} \sum_{j=0}^{h} \binom{h}{j} L_h l^h \alpha_i^{h-j} \frac{(h-j)!}{(\lambda+1)(\lambda+2)\dots(\lambda+h-j)} p_{ij} = 0$$

$$(h = 0, 1, \dots, m-1) .$$

Lemma 1 implies that such integers exist. Further, since

$$\max_{j=0,1,\dots,h} \left\{L_h, \left|\frac{L_h(h-j)!}{(\lambda+1)(\lambda+2)\dots(\lambda+h-j)}\right|\right\} < c_4^h \quad (h = 0, 1, \dots)$$

(see [7], pp. 56-58), we can take p_{ij} with absolute values at most

$$M = \left\{n\left(2c_4 lL\right)^m\right\}^{m/(n-m)} + 2 .$$

We may now prove that the polynomials

$$P_i(z) = r! \sum_{j=0}^{r} p_{ij}(j!)^{-1} z^j \quad (i = 0, 1, \dots, k)$$

satisfy the conditions of Lemma 2.

First consider 1°. We have $m < n < 2(k+1)r$ and $n - m > r(\log r)^{-\frac{1}{2}}$. Thus

$$(9) \qquad M < \left\{2(k+1)r\left(2c_4 lL\right)^{2(k+1)r}\right\}^{2(k+1)(\log r)^{\frac{1}{2}}} + 2 < c_5^{r(\log r)^{\frac{1}{2}}} .$$

By noting that $p_{ij} = 0$ for $j < r - r_i$ we obtain the upper bound

$$\frac{r!M}{(r-r_i)!} \le 2^r M(r_i!)$$

for the absolute values of the coefficients of $P_i(z)$. By (9) this gives part 1° of our lemma.

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To prove 2° we note that

$$P_0(z) + \sum_{i=1}^{k} P_i(z)f_i(z) = r! \sum_{h=0}^{\infty} \sigma_h (h!)^{-1} z^h ,$$

where, for each h, $L_h l^h \sigma_h$ denotes the left-hand side of (8). We thus have (5) with $\rho_h = r!(h!)^{-1}\sigma_h$ satisfying (6). For $h \geq m$ we have, by (9),

$$|\sigma_h| < (2Lc_4)^h M(k+1)(h+1) < c_3^{h+r(\log r)^{\frac{1}{2}}}$$

This implies (7), and thus Lemma 2 is proved.

The function $\phi_\lambda(z)$ satisfies the differential equation

$$y' = \left(1 - \frac{\lambda}{z}\right)y + \frac{\lambda}{z} . \tag{10}$$

Thus the functions $f_0(z) \equiv 1$, $f_i(z) = \phi_\lambda(\alpha_i z)$ $(i = 1, 2, \ldots, k)$ satisfy the following homogeneous system of differential equations,

$$\begin{aligned} &y_0' = 0 , \\ &y_i' = \frac{\lambda}{z} y_0 + \left(\alpha_i - \frac{\lambda}{z}\right) y_i \quad (i = 1, 2, \ldots, k) . \end{aligned} \tag{11}$$

Let $y_0, y_1, \ldots, y_k$ be an arbitrary solution of (11) and let $P_0, P_1, \ldots, P_k$ be the polynomials given in Lemma 2. We denote

$$R_0^* = \sum_{i=0}^{k} Q_{i0} y_i , \quad Q_{i0} = P_i \quad (i = 0, 1, \ldots, k) ,$$

$$R_j^* = \frac{d^j}{dz^j} R_0^* = \sum_{i=0}^{k} Q_{ij} y_i \quad (j = 1, 2, \ldots) ,$$

where, by (11),

$$Q_{0j} = Q_{0,j-1}' + \frac{\lambda}{z} \sum_{i=1}^{k} Q_{i,j-1} , \quad Q_{i,j} = Q_{i,j-1}' + \left(\alpha_i - \frac{\lambda}{z}\right) Q_{i,j-1} \tag{12}$$

$$(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots) .$$

LEMMA 3. *Suppose that* $Q_{i0}(z) \not\equiv 0$ $(i = 1, 2, \ldots, h;\ 1 \leq h \leq k)$,

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and $Q_{h+1,0} = \ldots = Q_{k0} \equiv 0$. *Then the determinant*

$$\Delta_1(z) = \det\left(z^j Q_{ij}\right)_{i,j=0,1,\ldots,h} \not\equiv 0 \ .$$

Proof. We follow Siegel's deduction (see [7], p. 43). If $\Delta_1(z) \equiv 0$, then there exist $\mu + 1 \le h + 1$ polynomials $A_0, \ldots, A_\mu$ satisfying

$$A_0 Q_{i0} + A_1 z Q_{i1} + \ldots + A_\mu z^\mu Q_{i\mu} = 0 \quad (i = 0, 1, \ldots, h) \ ;$$

$$A_\mu \neq 0 \ .$$

This implies that

$$B_0 R_0^* + B_1 R_1^* + \ldots + B_\mu R_\mu^* = 0 \ ; \quad B_j = z^j A_j \quad (j = 0, 1, \ldots, \mu) \ ,$$

and, by the definition of R_j^* ,

$$B_\mu R_0^{*(\mu)} + \ldots + B_1 R_0^{*\prime} + B_0 R_0^* = 0 \ . \tag{13}$$

Thus each of the functions

$$R_{0,l}^* = \sum_{i=0}^{k} Q_{i0} y_{il} \quad (l = 0, 1, \ldots, h) \ ,$$

where

$$y_{i,0} = f_i(z) \ , \quad y_{i,l} = \delta_{il} z^{-\lambda} e^{\alpha_i z} \quad (i = 0, 1, \ldots, k; \ l = 1, 2, \ldots, h)$$

(here $\delta_{il} = 1$ if $i = l$, and $\delta_{il} = 0$ if $i \neq l$) satisfy the homogeneous linear differential equation (13) of order $\mu \le h$. This means that we have constants $C_0, \ldots, C_h$, not all zero, such that

$$\sum_{l=0}^{h} C_l R_{0,l}^* = 0 \ .$$

We now immediately obtain

$$C_0\left(Q_{00} + Q_{10} f_1(z) + \ldots + Q_{k0} f_k(z)\right) = -z^{-\lambda} \sum_{l=1}^{h} C_l Q_{l0} e^{\alpha_l z} \ .$$

Here the left-hand side of this equation and $\sum_{l=1}^{h} C_l Q_{l0} e^{\alpha_l z}$ are entire functions, and since $\lambda \notin \mathbb{Z}$, we get

$$C_0\left(Q_{00} + Q_{10}f_1(z) + \ldots + Q_{k0}f_k(z)\right) = 0 \ ; \quad \sum_{l=1}^{h} C_l Q_{l0} e^{\alpha_l z} = 0 \ .$$

The functions $f_1(z), \ldots, f_k(z)$ are algebraically independent over $C(z)$ (see [5]), and so $C_0 = 0$. Further $\alpha_i \neq \alpha_j$ if $i \neq j$, and thus the functions $e^{\alpha_i z}$ $(i = 1, 2, \ldots, h)$ are linearly independent over $C(z)$. This means that $C_l Q_{l0} = 0$ $(l = 1, 2, \ldots, h)$. Our assumption $Q_{l0} \not\equiv 0$ $(l = 1, 2, \ldots, h)$ implies $C_l = 0$ for all these l. This contradiction means that $\Delta_1(z) \not\equiv 0$, thus proving our lemma.

We now denote

$$R_j(z) = z^j \frac{d^j}{dz^j}\left(\sum_{i=0}^{k} P_i(z)f_i(z)\right) \quad (j = 0, 1, \ldots) \ , \tag{14}$$

obtaining

$$R_j(z) = \sum_{i=0}^{k} P_{ij}(z)f_i(z) \quad (j = 0, 1, \ldots) \ , \tag{15}$$

where the polynomials P_{ij} are given by

$$P_{ij}(z) = z^j Q_{ij}(z) \quad (i = 0, 1, \ldots, k;\ j = 0, 1, \ldots) \ . \tag{16}$$

LEMMA 4. *Let the hypotheses of Lemma 2 be true, and let* $P_i(z)$ $(i = 0, 1, \ldots, k)$ *be the polynomials given there. Let*

$$s = \left[r(\log r)^{-\frac{1}{2}}\right] + k(k-1)/2 \ , \tag{17}$$

and suppose that $r_i > 2s$ *for all* i. *Let the polynomials* $P_{ij}(z)$ $(i = 0, 1, \ldots, k;\ j = 0, 1, \ldots)$ *be defined inductively by the equations* (12) *and* (16). *Then the determinant*

$$\Delta(z) = \det\left(P_{ij}(z)\right)_{i,j=0,1,\ldots,k} \not\equiv 0 \ ,$$

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and cannot have a zero at $z = \alpha \neq 0$ *of order greater than* s .

Proof. First we prove that $P_i(z) \not\equiv 0$ $(i = 1, 2, \ldots, k)$. The argument here is similar to that of Baker ([1], pp. 619-620). We suppose that exactly h of the polynomials $P_i(z)$ $(i = 1, 2, \ldots, k)$ do not vanish identically. Without loss of generality we may assume that these are $P_1(z), \ldots, P_h(z)$ (clearly $h \geq 1$). Let

$$\Delta_1(z) = \det\left(P_{ij}(z)\right)_{i,j=0,1,\ldots,h} .$$

From Lemma 3 it follows that $\Delta_1(z) \not\equiv 0$. Thus $\Delta_1(z)$ is a polynomial of degree at most

$$d = (h+1)r + h(h+1)/2 .$$

On the other hand

$$\Delta_1(z) = \begin{vmatrix} R_0(z) & R_1(z) & \ldots & R_h(z) \\ P_{10}(z) & P_{11}(z) & \ldots & P_{1h}(z) \\ . & . & \ldots & . \\ P_{h0}(z) & P_{h1}(z) & \ldots & P_{hh}(z) \end{vmatrix} ,$$

and thus $\Delta_1(z)$ has a zero at $z = 0$ of order at least

$$d_0 = m + \sum_{i=1}^{h} \left(r-r_i\right) = (h+1)r + k - \left[r(\log r)^{-\frac{1}{2}}\right] + \sum_{i=h+1}^{k} r_i .$$

Since $r_i > 2s$, we obtain $d < d_0$ if $h < k$. Hence $h = k$. Thus Lemma 3 implies that $\Delta(z) \not\equiv 0$.

The polynomial $\Delta(z)$ is of degree at most

$$d_1 = (k+1)r + k(k+1)/2 .$$

As before, we find immediately that $\Delta(z)$ has a zero at $z = 0$ of order at least

$$d_2 = m + \sum_{i=1}^{k} \left(r-r_i\right) = (k+1)r + k - \left[r(\log r)^{-\frac{1}{2}}\right] .$$

Thus $d_1 - d_2 \leq s$, which proves our lemma.

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LEMMA 5. *Let the hypotheses of Lemma 4 be valid. Then there are integers* $0 \le J(0) < J(1) < \ldots < J(k) \le k + s$ *such that*

$$D = \det\left(P_{i,J(j)}(1)\right)_{i,j=0,1,\ldots,k} \neq 0 .$$

Proof. Let $J(j)$ $(j = 0, 1, \ldots, k)$ be any integers satisfying $0 \le J(0) < J(1) < \ldots < J(k)$. We denote

$$D\left(z;\, J(0),\, J(1),\, \ldots,\, J(k)\right) = \begin{vmatrix} P_{0,J(0)} & P_{0,J(1)} & \cdots & P_{0,J(k)} \\ P_{1,J(0)} & P_{1,J(1)} & \cdots & P_{1,J(k)} \\ \cdot & \cdot & \cdots & \cdot \\ P_{k,J(0)} & P_{k,J(1)} & \cdots & P_{k,J(k)} \end{vmatrix} .$$

From equations (12) and (16) it follows that

$$zP'_{0j} = jP_{0j} + P_{0,j+1} - \lambda \sum_{i=1}^{k} P_{ij} ,$$

$$zP'_{ij} = jP_{ij} + P_{i,j+1} - \left(\alpha_i z - \lambda\right)P_{ij} \quad (i = 1, 2, \ldots, k;\ j = 0, 1, \ldots) .$$

Let D_{ij} denote the complement of D corresponding to the element $P_{i,J(j)}$. We then obtain

$$\begin{aligned} &zD'\left(z;\, J(0),\, J(1),\, \ldots,\, J(k)\right) \\ &\quad = \sum_{j=0}^{k} \Big\{J(j)D\left(z;\, J(0),\, J(1),\, \ldots,\, J(k)\right) \\ &\qquad + D\left(z;\, J(0),\, \ldots,\, J(j-1),\, J(j)+1,\, J(j+1),\, \ldots,\, J(k)\right) - \lambda \sum_{i=1}^{k} P_{i,J(j)}D_{0j} \\ &\qquad - \sum_{i=1}^{k} \left(\alpha_i z-\lambda\right)P_{i,J(j)}D_{ij}\Big\} \\ &\quad = D\left(z;\, J(0),\, J(1),\, \ldots,\, J(k)\right)\left(J(0) + \sum_{j=1}^{k} \left(J(j)-\alpha_j z+\lambda\right)\right) \\ &\qquad + \sum_{j=0}^{k} D\left(z;\, J(0),\, \ldots,\, J(j-1),\, J(j)+1,\, J(j+1),\, \ldots,\, J(k)\right) . \end{aligned}$$

Thus, if our lemma were not true, then for all $J(k) \le k + s - \tau$,

$$D^{(\tau)}\left(1;\, J(0),\, J(1),\, \ldots,\, J(k)\right) = 0 .$$

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On the other hand, by Lemma 4, there exists $\tau \le s$ such that

$$D^{(\tau)}(1;\ 0,\ 1,\ \ldots,\ k) \neq 0\ .$$

Hence $k > k + s - \tau$, which is impossible. Thus there exist the suffixes $J(j)$ $(j = 0, 1, \ldots, k)$ such that Lemma 5 holds.

Next we prove our final lemma, which is for use in the proof of Theorem 1.

LEMMA 6. *Let the hypothesis of Lemma 4 be valid. Then we can find* $(k+1)^2$ *integers* q_{ij} $(i, j = 0, 1, \ldots, k)$ *satisfying the following properties:*

1°. $\det(q_{ij}) \neq 0$;

2°. *for each pair* i, j *we have*

$$|q_{ij}| < r_i!\,c_6^{r(\log r)^{\frac{1}{2}}}\ ; \tag{18}$$

3°. *the inequality*

$$\left|\sum_{i=0}^{k} q_{ij}f_i(1)\right| < c_7^{r(\log r)^{\frac{1}{2}}} \prod_{i=1}^{k} (r_i!)^{-1} \tag{19}$$

holds for each $j = 0, 1, \ldots, k$.

Proof. Let l be the least common denominator of the numbers $\lambda, \alpha_1, \ldots, \alpha_k$, and put $L = \max\{1, |\lambda|, |\alpha_1|, \ldots, |\alpha_k|\}$. We shall prove that the integers

$$q_{ij} = l^{k+s}P_{i,J(j)}(1)\quad (i, j = 0, 1, \ldots, k)\ ,$$

where $J(j)$ $(j = 0, 1, \ldots, k)$ are given in Lemma 5, have the required properties.

We see immediately by Lemma 5 that 1° holds.

To prove 2° we note, by Lemma 2, (12) and (16), that the coefficients $P_{i,J(j)}$ have absolute values of at most

$$(r+KL)^{k+s}(r_i!)c_2^{r(\log r)^{\frac{1}{2}}} < r_i!\,c_8^{r(\log r)^{\frac{1}{2}}}\ ,\quad K = \max\{2, k\}\ .$$

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We see easily that this implies (18).

From our definitions of q_{ij} and R_j it follows that

$$\left|\sum_{i=0}^{k} q_{ij} f_i(1)\right| = l^{k+s} |R_{J(j)}(1)| .$$

Further, by (14),

$$R_{J(j)}(z) = z^{J(j)} \frac{d^{J(j)}}{dz^{J(j)}} \left(\sum_{i=0}^{k} P_i(z) f_i(z)\right) = \sum_{h=m}^{\infty} \frac{h!}{(h-J(j))!} \rho_h z^h ,$$

and here $\rho_h = r!(h!)^{-1}\sigma_h$, where σ_h is defined in the proof of Lemma 2. There it is also proved that

$$|\sigma_h| < c_3^{h+r(\log r)^{\frac{1}{2}}} \quad (h = 0, 1, \ldots) .$$

Using these facts and the inequality $J(j) \le k + s$, we obtain the following relations

$$\begin{aligned}
\left|\sum_{i=0}^{k} q_{ij} f_i(1)\right| &= l^{k+s} \left|\sum_{h=m}^{\infty} \left((h-J(j))!\right)^{-1} r! \sigma_h\right| \\
&< l^{k+s} (r!) c_3^{r(\log r)^{\frac{1}{2}}} \left|\sum_{h=m}^{\infty} \left((h-J(j))!\right)^{-1} c_3^h\right| \\
&< l^{k+s} (r!) c_3^{r(\log r)^{\frac{1}{2}}} c_3^m \left((m-J(j))!\right)^{-1} e^{c_3} \\
&< c_9^r (r!) c_3^{r(\log r)^{\frac{1}{2}}} \left((r + r_1 + \ldots + r_k - 2s)!\right)^{-1} \\
&< c_9^r (r!) c_3^{r(\log r)^{\frac{1}{2}}} \left((k+1)r\right)^{2s} \prod_{i=0}^{k} (r_i!)^{-1} \\
&< c_7^{r(\log r)^{\frac{1}{2}}} \prod_{i=1}^{k} (r_i!)^{-1} .
\end{aligned}$$

This completes the proof of our lemma.

3. Proof of Theorem 1

We define positive constants a, b , and c by setting

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(20) $a = (4kc_6c_7)^{16k^2}$, $b = 20k \log a$, $\log\log c = 2(\log a)^4$,

where c_6 and c_7 are constants appearing in Lemma 6. Here we assume, as we may without loss of generality, that c_6 and c_7 are greater than 1 . To prove Theorem 1 we suppose that (3), where $c_0 = b$, is valid for some $x_1, x_2, \ldots, x_k, y$, and prove that this implies

$$x = \max\{x_1', x_2', \ldots, x_k'\} < c .$$

Let us assume, against this, that $x \geq c$.

We define the function f of the positive integer r by putting

(21) $$f(r) = r! a^{-r(\log r)^{\frac{1}{2}}} .$$

Since (see [4], p. 73)

$$r! = \sqrt{2\pi r}\, r^r e^{-r+g(r)} , \quad 0 < g(r) < \frac{1}{12r} ,$$

we have, for $r \geq 2$,

(22) $\log r - (\log r)^{\frac{1}{2}} \log a - 1 < \frac{\log f(r)}{r} < \log r - (\log r)^{\frac{1}{2}} \log a$.

From this it follows that there exists a positive integer r satisfying

(23) $$\log r > (\log a)^2 , \quad f(r-1) \leq x < f(r) .$$

This yields

(24) $$(r-1)! \leq a^{r(\log r)^{\frac{1}{2}}} x < r! .$$

Further we define the integers $r_1, r_2, \ldots, r_k$ by the inequalities

(25) $$(r_i-1)! \leq a^{r(\log r)^{\frac{1}{2}}} x_i' < r_i! \quad (i = 1, 2, \ldots, k) .$$

Clearly, we have $r = \max\{r_1, r_2, \ldots, r_k\}$. We may now proceed by proving that these integers r_i satisfy all the other hypotheses of Lemma 4. By (20) and (23),

$$r(\log r)^{-\frac{1}{2}} > (\log r)^{\frac{1}{2}} > \log a > 16k^2 ,$$

and thus $2s < 3r(\log r)^{-\frac{1}{2}}$. By using this inequality we obtain, if $r_i \le 2s$,

$$\log r_i! \le r_i \log r_i \le 2s \log 2s$$

$$< 3r(\log r)^{-\frac{1}{2}} \log\left(3r(\log r)^{-\frac{1}{2}}\right) < 3r(\log r)^{\frac{1}{2}} .$$

This implies

$$r_i! < e^{3r(\log r)^{\frac{1}{2}}} ,$$

which is impossible by (25). Thus we have $r_i > 2s$ for all $i = 1, 2, \ldots, k$.

From (3) and (25) we obtain, by denoting

$$L = y + x_1f_1(1) + x_2f_2(1) + \ldots + x_kf_k(1) ,$$

$$(26) \quad |L| < x^{-b(\log\log x)^{-\frac{1}{2}}} \prod_{i=1}^{k} \frac{1}{x_i} \le x^{-b(\log\log x)^{-\frac{1}{2}}} a^{kr(\log r)^{\frac{1}{2}}} \prod_{i=1}^{k} r_i(r_i!)^{-1}$$

$$\le a^{-r(\log r)^{\frac{1}{2}}}\left(\prod_{i=1}^{k} (r_i!)^{-1}\right) x^{-b(\log\log x)^{-\frac{1}{2}}} a^{(k+1)r(\log r)^{\frac{1}{2}}} r^k .$$

We have

$$c \le x < r! < r^r < e^{r^2} .$$

Using (20) we obtain

$$\log r > (\log \log x)/2 \ge (\log a)^4 > 16^4 .$$

By (21) and (22), this gives

$$\log f(r-1) > (r-1)\{\log(r-1)-(\log(r-1))^{\frac{1}{2}} \log a-1\}$$

$$> (r-1)\{\log(r-1)-(\log r)^{\frac{3}{4}} - 1\} > (r \log r)/2 .$$

Now it follows from (23) that

$$(r \log r)/2 < \log x < r \log r .$$

Hence

$$x^{b(\log\log x)^{-\frac{1}{2}}} = \exp\left(b \log x(\log\log x)^{-\frac{1}{2}}\right)$$
$$> \exp\left(br \log r/4(\log r)^{\frac{1}{2}}\right) = \exp\left(5kr(\log r)^{\frac{1}{2}}\log a\right)$$
$$> a^{4kr(\log r)^{\frac{1}{2}}} r^k ,$$

and this with (26) gives an inequality

$$|L| < a^{-r(\log r)^{\frac{1}{2}}} \prod_{i=1}^{k} (r_i!)^{-1} . \tag{27}$$

Since the hypotheses of Lemma 6 hold, we can now use linearly independent forms

$$L_j = \sum_{i=0}^{k} q_{ij} f_i(1) \quad (j = 0, 1, \ldots, k)$$

obtained by this lemma. We can select k forms, say $L_1, L_2, \ldots, L_k$ that together with L are linearly independent. We have

$$\begin{vmatrix} y & q_{01} & \cdots & q_{0k} \\ x_1 & q_{11} & \cdots & q_{1k} \\ \vdots & & & \vdots \\ x_k & q_{k1} & \cdots & q_{kk} \end{vmatrix} = \begin{vmatrix} L & L_1 & \cdots & L_k \\ x_1 & q_{11} & \cdots & q_{1k} \\ \vdots & & & \vdots \\ x_k & q_{k1} & \cdots & q_{kk} \end{vmatrix}$$

and since the left-hand side of this equation is a non-zero integer, we obtain, by (18),

$$1 \le \sum_{i=1}^{k} |L_i| (k-1)! \sum_{j=1}^{k} |x_j| \left(\prod_{\substack{l=1 \\ l \ne j}}^{k} \left(r_l! c_6^{r(\log r)^{\frac{1}{2}}} \right) \right)$$
$$+ |L| k! \prod_{i=1}^{k} \left(r_i! c_6^{r(\log r)^{\frac{1}{2}}} \right) .$$

From the inequalities (19), (25), and (27) it follows that

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$$1 \le (k+1)!\left\{\left(c_6^{k-1}c_7a^{-1}\right)^{r(\log r)^{\frac{1}{2}}} + \left(c_6^k a^{-1}\right)^{r(\log r)^{\frac{1}{2}}}\right\} .$$

From the definition of a it follows that this is impossible. This contradiction proves our Theorem 1.

4. Proof of Theorem 2

Let a, b, and c be the numbers given in the preceeding section. Let $\alpha = 2kb$, $\beta = 4kc$, and, further, let γ be given by the equation $\log\log\gamma = (b\beta)^2$. We shall prove that if $c_1 = \alpha$, then (4) has no solution $y > \gamma$. Assume, against this, that there exist integers $y > \gamma$, $y_1, y_2, \ldots, y_k$ such that

$$y^{1+\alpha(\log\log y)^{-\frac{1}{2}}}|yf_1(1)-y_1| \ldots |yf_k(1)-y_k| < 1 .$$

We shall prove that this leads to a contradiction.

For this purpose we denote

$$w = y|yf_1(1)-y_1| \ldots |yf_k(1)-y_k| , \tag{28}$$

$$t_i = w^{1/k}|yf_i(1)-y_i|^{-1} \quad (i = 1, 2, \ldots, k) . \tag{29}$$

Without loss of generality we may assume that

$$t_1 \ge t_2 \ge \ldots \ge t_k > 0 .$$

Since

$$t_1t_2 \ldots t_k = y ,$$

we find the smallest integer $K \le k$ for which

$$t_{K+1}t_{K+2} \ldots t_k \le 1 .$$

Consider now the following system of $K + 1$ linear inequalities

$$\begin{aligned} &|x_i| \le t_i \quad (i = 1, 2, \ldots, K-1) ; \quad |x_K| \le t_Kt_{K+1} \ldots t_k \le t_K ; \\ &|x_1y_1 + \ldots + x_Ky_K + Xy| < 1 \end{aligned} \tag{30}$$

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for $x_1, x_2, \ldots, x_K, X$. By Minkoswki's Theorem on linear forms (see [2], p. 151) there exist integers $x_1, x_2, \ldots, x_K, X$, not all zero, satisfying these inequalities. From the last inequality we get

(31) $$x_1y_1 + \ldots + x_Ky_K + Xy = 0 .$$

Thus we have non-zero integers in the set $\{x_1, x_2, \ldots, x_K\}$. Let these be $x_{i(1)}, x_{i(2)}, \ldots, x_{i(l)}$. Clearly, $1 \le l \le K$. Further, from (30) it follows that

(32) $$\left|x_{i(1)}x_{i(2)} \ldots x_{i(l)}\right| \le t_1t_2 \ldots t_k = y .$$

By (31),

$$0 = x_1y_1 + \ldots + x_Ky_K + Xy = \left(x_1f_1(1) + \ldots + x_Kf_K(1) + X\right)y - \sum_{i=1}^{K} x_i\left(yf_i(1)-y_i\right) ,$$

which implies

$$\left(x_1f_1(1) + \ldots + x_Kf_K(1) + X\right)y = \sum_{i=1}^{K} x_i\left(yf_i(1)-y_i\right) .$$

By (29) and (30),

$$|x_i| \le t_i , \quad \left|yf_i(1)-y_i\right| = w^{1/k}t_i^{-1} \quad (i = 1, 2, \ldots, K) .$$

Hence

(33) $$\left|x_1f_1(1) + \ldots + x_Kf_K(1) + X\right| \le Kw^{1/k}y^{-1} .$$

We define $x_{K+1} = \ldots = x_k = 0$, and denote as before

$$x = \max\{x_i'\} , \quad x_i' = \max\{1, |x_i|\} \quad (i = 1, 2, \ldots, k) .$$

Then we obtain, by (32),

(34) $$x \le x_1'x_2' \ldots x_k' = \left|x_{i(1)}x_{i(2)} \ldots x_{i(l)}\right| \le y .$$

By (28), (33), (34), and our original hypothesis we obtain

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$$(35)\quad |x_1'x_2' \ldots x_k'(x_1f_1(1) + \ldots + x_kf_k(1) + X)|$$

$$\le kw^{1/k} < ky^{-2b(\log\log y)^{-\frac{1}{2}}}$$

We now define rational integers v_i by putting $v_i = 2[c]x_i$ $(i = 1, 2, \ldots, k)$. Again let

$$v = \max\{v_i'\}\ ,\quad v_i' = \max\{1, |v_i|\}\quad (i = 1, 2, \ldots, k) .$$

We then have, by (35),

$$(36)\quad |v_1'v_2' \ldots v_k'(v_1f_1(1) + \ldots + v_kf_k(1) + V)|$$

$$< k(2c)^{k+1}y^{-2b(\log\log y)^{-\frac{1}{2}}} ,$$

where $V = 2[c]X$.

Since $y > \gamma$, where $\log\log\gamma = (b\beta)^2$, we obtain

$$y^{-b(\log\log y)^{-\frac{1}{2}}} = \exp\left(-b(\log\log y)^{-\frac{1}{2}}\log y\right) < \exp\left(-b(\log\log y)^{\frac{1}{2}}\right)$$

$$< \exp\left(-b^2\beta\right) < \exp(-20k\beta) < \beta^{-20k} .$$

The use of (34) gives

$$c < v \le 2cx \le 2cy < \beta y .$$

This implies

$$y^{-b(\log\log y)^{-\frac{1}{2}}} < \beta(\beta y)^{-b(\log\log y)^{-\frac{1}{2}}} < \beta(\beta y)^{-b\left(\log\log(\beta y)\right)^{-\frac{1}{2}}}$$

$$< \beta v^{-b(\log\log v)^{-\frac{1}{2}}} .$$

By these estimates and (36) we obtain the following inequality

$$|v_1'v_2' \ldots v_k'(v_1f_1(1) + \ldots + v_kf_k(1) + V)| < v^{-b(\log\log v)^{-\frac{1}{2}}} .$$

Since $v > c$, this is impossible by the previous section. Thus we have established the truth of Theorem 2.

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