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ALMOST CONTINUOUS FUNCTIONS WITH CLOSED GRAPHS

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ABSTRACT. A function $f: X \to Y$ is almost continuous if for every $x \in X$ and for each open set $V \subset Y$ containing $f(x)$, $Cl(f^{-1}(V))$ is a neighborhood of x. Various conditions are given that guarantee that an almost continuous function is continuous. The main theorem states that if $f: X \to Y$ is almost continuous with a closed graph (closed in $X \times Y$) and X and Y are complete metric spaces, then f is continuous.

1. Introduction. In [1], Husain gave the following definition of an almost continuous function between topological spaces.

DEFINITION. The function $f: X \to Y$ is almost continuous at $x_0 \in X$ if and only if for each open $V \subset Y$ containing $f(x_0)$, $Cl(f^{-1}(V))$ is a neighborhood of x_0. If f is almost continuous at each point of X, then f is called almost continuous.

In a paper studying this concept [2], Lin and Lin asked the following question [2, pg. 185]:

"Let $f: X \to Y$ be a mapping from a Baire space X to a second countable space Y. If f is almost continuous and has a closed graph; that is, the set $\{(x, f(x)) \mid x \in X\}$ is closed in the product space $X \times Y$. Is f necessarily continuous?"

Rose [4] has answered this question negatively, but a more general question is suggested:

What hypotheses on the domain and range of a function guarantee that if it is almost continuous with a closed graph then it is continuous?

Long and McGehee gave one answer to this: if enough separation axioms hold, local compactness of the range is enough [3, Theorem 9].

In this paper this question is explored further. In Section 3 we prove that if the domain and range are both complete metric spaces, almost continuity with a closed graph implies continuity. However, it is not enough for the range to be a Baire space. In Section 4, an example is given of an almost continuous

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function with a closed graph from $\mathbb{R}$ to a Baire subspace of $\mathbb{R}$ which is nowhere continuous (this also answers Lin and Lin's question negatively). Along the way, various other facts about the relationship between almost continuous functions with closed graphs and continuous functions are proved.

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2. Points of continuity.

DEFINITION 2.1. If $f:X\to Y$ is a function between topological spaces, let $C(f)=\{x\in X: f \text{ is continuous at } x\}$.

THEOREM 2.1. *Suppose Y is a regular space and $f:X\to Y$ is almost continuous. If $C(f)$ is dense in X, then f is continuous (i.e. $C(f)=X$).*

Proof. Suppose there were a point $p\in X-C(f)$. Let O be an open subset of Y such that $f(p)\in O$ but $f^{-1}(O)$ is not a neighborhood of p. Let O_1 and O_2 be open subsets of Y such that $f(p)\in O_2$ and $Cl(O_2)\subset O_1$ and $Cl(O_1)\subset O$. Since f is almost continuous, there is an open set $U\subset X$ containing p such that $U\subset Cl(f^{-1}(O_2))$. There is a point $q\in U$ such that $f(q)\notin O$, since $U\not\subset f^{-1}(O)$. Then, again by the almost continuity of f, there is an open set $V\subset X$ containing q such that $V\subset Cl(f^{-1}(Y-Cl(O_1))$. Since $U\cap V$ contains q, there is a point $d\in U\cap V\cap C(f)$. Since f is continuous at d and $U\cap V\subset U\subset Cl(f^{-1}(O_2))$, $f(d)\in Cl(O_2)$. But since $d\in U\cap V\subset V\subset Cl(f^{-1}(Y-Cl(O_1)))$, it follows that $f(d)\in Cl(Y-Cl(O_1))\subset Y-O_1$. This cannot be since $Cl(O_2)\subset O_1$. Therefore, there cannot be a point in X—$C(f)$, i.e. f is continuous.

Remark. The assumption that Y is regular cannot be dropped. For example, let X be the reals, with the usual topology augmented to make each rational singleton open (this space is metrizable) and Y the reals with the topology generated by the sets:

$(a, b)\cap(\mathbb{Q}\cup\{r\})$ where (a, b) is a usual open interval and $r\in\mathbb{R}$. *The identity map from X* to Y is almost continuous, and the set of points of continuity is $\mathbb{Q}$, which is dense in X.

It is pointed out in [3] that the restriction of an almost continuous function to a subset of the domain need not be almost continuous, but that the restriction to an open subset is almost continuous [3, Theorem 4]. Also if $f:X\to Y$ has a closed graph and $Z\subset X$ then $f\mid Z:Z\to Y$ has a closed graph. These observations and Theorem 2.1 prove the following

COROLLARY 2.1. *If Y is a regular space and if for every space X and for every almost continuous (though not necessarily surjective) function $f:X\to Y$ with a closed graph, $C(f)\neq\varnothing$, then every almost continuous function $f:X\to Y$ with a closed graph is continuous.*

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THEOREM 2.2. *Suppose $f:X \to Y$ is almost continuous with a closed graph and y is an isolated point of Y. Then $f^{-1}(y) \subset C(f)$.*

Proof. We actually prove that $f^{-1}(y)$ is both open and closed. It is closed since if x is a limit point of $f^{-1}(y)$ then (x, y) is a limit point of the graph of f, which we are assuming to be closed. Now, since $\{y\}$ is open in Y, and $Cl(f^{-1}(y)) = f^{-1}(y)$, $f^{-1}(y)$ is a neighborhood of each of its points, by the almost continuity of f, and thus is open.

COROLLARY. *If Y is regular, $f:X \to Y$ is almost continuous with a closed graph and the inverse image of the set of isolated points of Y is dense in X, then f is continuous.*

Despite the corollary, it is not sufficient that Y have a dense set of isolated points (see example 3, section 4). However, the following theorem does hold.

THEOREM 2.3. *If Y has only one non-isolated point and $f:X \to Y$ is almost continuous with a closed graph then f is continuous.*

Proof. Let p be the non-isolated point of Y. Because of Theorem 2.2 we only need show that if $f(x) = p$ and O is an open set containing p then p is in the interior of $f^{-1}(O)$. Since $Y - O$ consists only of isolated points, the proof of Theorem 2.2 shows $f^{-1}(Y - O)$ is open, thus $f^{-1}(O)$ is closed, so by almost continuity x is in the interior of $f^{-1}(O)$ (in fact, $f^{-1}(O)$ is open).

3. Conditions that imply continuity. Long and McGehee [3] give several conditions which guarantee that an almost continuous function is continuous, including the following theorem.

THEOREM 3.1. [3, Theorem 9]. *Let $f:X \to Y$ be almost continuous where Y is locally compact. If Y is either regular or Hausdorff and the graph of f is closed, then f is continuous.*

The hypotheses of this theorem can be modified.

THEOREM 3.2. *Let $f:X \to Y$ be almost continuous where Y is locally countably compact and regular, and X is a Fréchet space (i.e. if $p \in X$ is a limit point of a set $C \in X$, then there is a sequence of points from C converging to p). If the graph of f is closed then f is continuous.*

Proof. (This proof is due to W. Mahavier).

Suppose f were not continuous at $p \in X$. Let $O \subset Y$ be an open set contaning $f(p)$ such that $f^{-1}(O)$ is not a neighborhood of p. Let U be an open set containing $f(p)$ such that $Cl(U) \subset O$ and $Cl(U)$ is countably compact. By the almost continuity of f, there is an open set V containing p such that $V \subset Cl(f^{-1}(U))$. There is a point $q \in V$ s.t. $f(q) \notin O$. There must then be a sequence

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$(q_i)(i \in N)$ of points of $f^{-1}(U)$ converging to q. Since $Cl(U)$ is countably compact, there is a point $y \in Cl(U)$ such that (q, y) is a limit point of $\{(q_i, f(q_i)) : i \in N\}$. But, since $f(q) \notin Cl(U)$, $y \neq f(q)$ violating the hypothesis that the graph of f is closed.

THEOREM 3.3. *Suppose X and Y are complete metric spaces and $f : X \to Y$ is almost continuous with a closed graph. Then f is continuous.*

Proof. Suppose f is not continuous at a point $p \in X$. We will inductively define a sequence $(p_i)(i \in N)$ of points of X, a sequence $(V_i)(i \in N)$ of open subsets of X and a sequence $(U_i)(i \in N)$ of open subsets of Y satisfying the following conditions.

(i) $p_i \in V_i$
(ii) $f(p_i) \in U_i$
(iii) $Cl(U_1) \cap Cl(U_2) = \varnothing$
(iv) If i and j are either both even or both odd and $i < j$ then $Cl(U_j) \subset U_i$
(v) If $i < j$ then $Cl(V_j) \subset V_i$
(vi) diam $(U_i) < 1/i$
(vii) diam $(V_i) < 1/i$
(viii) $V_i \subset Cl(f^{-1}(U_i))$

First let $p_1 = p$. There is an open set $U \subset Y$ containing $f(p)$ such that if V is a neighborhood of p then $f(V) \not\subset U$. Let U_1 be an open set containing $f(p)$ such that diam$(U_1) < 1$ and $Cl(U_1) \subset U$. By the almost continuity of f, there is an open set $V_1 \subset X$ containing p such that diam$(V_1) < 1$ and $V_1 \subset Cl(f^{-1}(U_1))$. There must be a point $p_2 \in V_1$ such that $f(p_2) \notin U$ (thus $f(p_2) \notin Cl(U_1)$). Let U_2 be an open set containing $f(p_2)$ such that diam$(U_2) < \frac{1}{2}$ and $Cl(U_2) \cap Cl(U_1) = \varnothing$. Again using almost continuity, let V_2 be an open set containing p_2 with diam$(V_2) < \frac{1}{2}$, $V_2 \subset Cl(f^{-1}(U_2))$ and $Cl(V_2) \subset V_1$. Suppose now we have defined V_i, U_i and p_i satisfying i–viii for all $i \le j$. Since $\varnothing \neq V_j \subset V_{j-1} \subset Cl(f^{-1}(U_{j-1}))$, there is a point $p_{j+1} \in V_j$ such that $f(p_{j+1}) \in U_{j-1}$. Let U_{j+1} be an open set containing $f(p_{j+1})$ such that $Cl(U_{j+1}) \subset U_{j-1}$ and diam$(U_{j+1}) < 1/(j+1)$. By the almost continuity of f, we can choose an open set V_{j+1} containing p_{j+1} such that $V_{j+1} \subset Cl(f^{-1}(U_{j+1}))$, diam$(V_{j+1}) < 1/j+1$ and $Cl(V_{j+1}) \subset V_j$. This completes the inductive definitions.

Since X is a complete metric space, there is an x such that $(p_i)(i \in N)$ converges to x. Also since Y is a complete metric space, there are points y and z such that $(f(p_{2i}))(i \in N)$ converges to y and $(f(p_{2i-1}))(i \in N)$ converges to z. Since $y \in Cl(U_1)$ and $z \in Cl(U_2)$, $y \neq z$. But the points (x, y) and (x, z) are both limit points of the graph of f, contradicting the fact that the graph of f is closed.

4. Some examples and a non-example.

THEOREM 4.1. *Suppose X is a Hausdorff space and D_1 and D_2 are disjoint dense subsets of X with $D_1 \cup D_2 = X$. Let Y be the topological sum of the*

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subspaces D_1 and D_2 and let $f: X \to Y$ be the identity map. Then f is almost continuous, the graph of f is closed but f is nowhere continuous (i.e. $C(f)=\emptyset$).

Proof. To see that f is almost continuous, let $x \in X$ and O be an open set in Y containing $f(x)$. Suppose $x \in D_i$. Then there is an open set $V \subset X$ containing x such that $V \cap D_i \subset O$. Thus $Cl(f^{-1}(O)) \supset Cl(f^{-1}(V \cap D_i)) \supset V$, since D_i is dense in X.

To see that the graph of f is closed, suppose (p, q) is a limit point of the graph of f where $q \in D_i$. If $p \neq q$, then there are disjoint open sets $U \subset X$ and $V \subset X$ with $p \in U$ and $q \in V$. But then $U \times (V \cap D_i)$ misses the graph of f. So $p = q$, i.e. (p, q) is in the graph of f.

If $x \in D_i$ then $f^{-1}(D_i) = D_i$ is not a neighborhood of x since D_2 is dense in X. So f is nowhere continuous. Theorem 4.1 provides some interesting examples, as well as suggesting some possible spaces that, as we will show later, cannot be constructed.

Example 1. Theorem 3.1 and Theorem 3.3 show that if $f: \mathbb{R} \to Y$ is almost continuous with a closed graph and if Y is either locally compact or complete metric, then f must be continuous. However, we can use Theorem 4.1 to get a Baire space Y and an almost continuous function with closed graph $f: \mathbb{R} \to$ that is nowhere continuous. We need the following fact about the reals.

Proposition. *If S is a dense G_δ in the reals and O is an open set, then $S \cap O$ has cardinality c, where c is the cardinality of the reals.*

Using this we can construct, by transfinite induction, two disjoint, dense subspaces D_1 and D_2 of $\mathbb{R}$ each of which is a Baire space. We will identify c with an initial ordinal and consider it as the set of previous ordinals.

Let $\mathscr{C}$ be the collection of all dense G_δ subsets of $\mathbb{R}$. The cardinality of $\mathscr{C}$ is c, so we can well-order $\mathscr{C}$ as $\{S_\alpha : \alpha < c\}$. For each $\alpha < c$, we will inductively define sets $D(\alpha, 1)$ and $D(\alpha, 2)$ to be countable dense subsets of S_α (and thus dense in $\mathbb{R}$) in such a way that $D(\alpha, 1) \cap D(\beta, 2) = \emptyset$ for every $\alpha, \beta < c$. Suppose we have defined $D(\beta, 1)$ and $D(\beta, 2)$ for every $\beta < \alpha$. Since the cardinality of $\bigcup D(\beta, 1) \cup D(\beta, 2)(\beta < \alpha)$ is less than c, the proposition guarantees we can pick $D(\alpha, 1)$ and $D(\alpha, 2)$ to be disjoint countable dense subsets of $S_\alpha - (\bigcup D(\beta, 1) \cup D(\beta, 2)(\beta < \alpha))$. This inductively defines the sets we want.

Now let $D_1 = \bigcup D(\alpha, 1)(\alpha < c)$ and let $D_2 = \mathbb{R} - D_1$ (thus $D(\alpha, 2) \subset D_2$ for every $\alpha < c$). Notice that D_1 and D_2 are dense in $\mathbb{R}$.

Suppose $(O_1)(i \in N)$ is a sequence of dense open subsets of D_1. Then, for each i, there is a dense open subset V_i of $\mathbb{R}$ such that $O_i = V_i \cap D_1$. Thus $\bigcap O_i(i \in N) = \bigcap V_i(i \in N) \cap D_1$. For some $\alpha < c, \bigcap V_i(i \in N) = S_\alpha$. For that α, $D(\alpha, 1) \subset \bigcap O_i(i \in N)$, so $\bigcap O_i(i \in N)$ is dense in D_1, showing that D_1 is a Baire space. Likewise, D_2 is a Baire space. The sets D_1 and D_2 fit the conditions of Theorem 4.1, where $X = \mathbb{R}$, and the space Y thus constructed is a Baire space.

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EXAMPLE 2. The idea behind Theorem 4.1 can be modified to get an almost continuous function with a closed graph that is not continuous, but where the range has a dense set of isolated points (see the corollary to Theorem 2.2). Let X be the plane, with the usual topology expanded to include the set $\{(x, y)\}$ whenever $y \neq 0$ and $\mathbb{R} \times \{0\}$. Let Y_1 be $\mathbb{Q} \times \mathbb{R}$ with the usual topology expanded to include $\{(x, y)\}$ whenever $y \neq 0$ and let Y_2 be $(\mathbb{R} - \mathbb{Q}) \times \mathbb{R}$, again with the usual topology expanded to include $\{(x, y)\}$ whenever $y \neq 0$. If Y is the topological sum of Y_1 and Y_2, the identity map from X to Y has the desired properties.

NON-EXAMPLE 1. We cannot use Theorem 4.1 and have both D_1 and D_2 be complete metric spaces.

Proof. Suppose X is Hausforff and we thought we had disjoint dense subspaces D_1 and D_2, each of which was a complete metric space (note: there is no assumption about X being metric). Let d_1 be a complete metric on D_1 and d_2 be a complete metric on D_2. At the risk of ambiguity, we will use this notation: for $x \in D_i$, $B(x, \varepsilon) = \{y \in D_i : d_i(x, y) < \varepsilon\}$. Also, if $S \subset D_i$, the closure of S in D_i will be denoted $Cl_i(S)$.

Pick $p_1 \in D_1$ and let O_1 be an open subset of X such that $O_1 \cap D_1 = B(p_1, 1)$. Since D_2 is dense in X we can choose $p_2 \in O_1 \cap D_2$, and let O_2 be an open subset of X such that $O_2 \subset O_1$, and $O_2 \cap D_2 \subset B(p_2, \frac{1}{2})$. Suppose now that for each $i \leq 2n$ we have defined an open set O_i and a point $p_i \in O_i$ such that

(i) if $i < j \leq 2n$ then $O_i \subset O_j$

(ii) if i is odd then $O_i \cap D_1 \subset B(p_i, 1/i)$

(iii) if i is even then $O_i \cap D_2 \subset B(p_i, 1/i)$

(iv) if i and j are both odd and $i < j \leq 2n$ then $Cl_1(O_j \cap D_1) \subset O_i \cap D_1$

(v) if i and j are both even and $i < j \leq 2n$ then $Cl_2(O_j \cap D_2) \subset O_i \cap D_2$

Pick $p_{2n+1} \in O_{2n} \cap D_1$. Let O_{2n+1} be an open set such that $O_{2n+1} \subset O_i$, $O_{2n+1} \cap D_1 \subset B(p_{2n+1}, 1/(2n+1))$ and $Cl_1(O_{2n+1} \cap D_1) \subset O_{2n-1} \cap D_1$. Similarly pick p_{2n+2} and O_{2n+2}.

Since D_1 and D_2 are complete metric spaces, there is a point $p \in D_1$ such that $\{p\} = \bigcap (O_i \cap D_1)(i \text{ odd})$ and a point $q \in D_2$ such that $\{q\} = \bigcap (O_i \cap D_2)(i \text{ even})$. Let U and V be disjoint open subsets of X containing p and q respectively. It is evident from the construction that $q \in O_i$ for every $i \in N$, thus, for each i, $O_i \cap V \cap D_1 \neq \varnothing$. Also since $O_i \cap V \cap D_1 \subset O_i \cap D_1 \subset B(p_i, 1/i)$ be i odd, $\bigcap Cl_1(O_i \cap V \cap D_i)(i \text{ odd})$ cannot be empty. But (taking all intersections over odd values of i), $\bigcap Cl_1(O_i \cap V \cap D_1) \subset Cl_1(O_i \cap D_1) - U = \bigcap O_i \cap D_1 - U = \{p\} - U = \varnothing$. So D_1 and D_2 cannot both be complete metric spaces.

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