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Some abelian-by-nilpotent varieties of p-groups

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The work reported in this thesis can be conveniently divided into two sections; the first comprising Chapters 2 and 3, and the second Chapters 4 and 5.

The first section is devoted to metabelian varieties. Possibly the most well-known result in this area is due to Cohen [5] who has shown that $\operatorname{lat}(\underline{\underline{A}}\underline{\underline{A}})$ has minimum condition. Other authors have given descriptions of various sublattices of $\operatorname{lat}(\underline{\underline{A}}\underline{\underline{A}})$. Brooks ([1], [2], and [3]) has studied the subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ and has given a complete classification of the non-nilpotent join-irreducible varieties in $\operatorname{lat}\left(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}\right)$. He has also shown that $\operatorname{lat}\left(\underline{\underline{A}}_3\underline{\underline{A}}_9\right)$ is not distributive. However, as far as the classification of nilpotent join-irreducible varieties in $\operatorname{lat}\left(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}\right)$ is concerned little work has been done, and the problem appears very difficult. This thesis contains a contribution to the theory of the nilpotent subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha}$, for $\alpha \geq 1$, although it is not directly related to the classification problem.

The principal results of the first section can be summarized in the following way. For any variety $\underline{\underline{V}}$ let $d(\underline{\underline{V}})$ be the minimum value of k such that $\underline{\underline{V}}$ is generated by its free group of rank k . In Chapter 2 the value of $d\left(\underline{\underline{A}}_p\underline{\underline{A}}_{p^\alpha}\wedge\underline{\underline{N}}_c\right)$ is found for all primes p and $\alpha = 1$ and 2 .

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For $\alpha > 2$ bounds are found within which $d\left(\underline{\underline{A}}_p \underline{\underline{A}}_{p^\alpha} \wedge \underline{\underline{N}}_c\right)$ must lie and it is conjectured that the lower bound is actually attained.

Chapter 3 looks at the question of distributivity in $\operatorname{lat}\left(\underline{\underline{A}}_2 \underline{\underline{A}}_4\right)$. One of the first examples of a non-distributive variety lattice was given by Higman [6] and this raised the question of whether the subvariety lattice of a given variety is distributive or not. Bryce [4] has shown that $\operatorname{lat}(\underline{\underline{A}}\underline{\underline{A}})$ is in general not distributive, although many sublattices are distributive. For example, Kovács and Newman [7] have shown that $\operatorname{lat}\left(\underline{\underline{A}}_{p^\alpha} \underline{\underline{A}}_p\right)$ is distributive for all primes p and all positive integers α. In this chapter an improvement is made on the result of Brooks stated earlier, by showing that $\operatorname{lat}\left(\underline{\underline{A}}_2 \underline{\underline{A}}_4\right)$ is not distributive.

The second section considers varieties of groups that are abelian-by-nilpotent, and in particular, subvarieties of $\underline{\underline{A}}_p \underline{\underline{T}}_p \wedge \underline{\underline{T}}_p \underline{\underline{A}}_p$, where $\underline{\underline{T}}_p = \underline{\underline{B}}_p \wedge \underline{\underline{N}}_2$ for $p \neq 2$, and $\underline{\underline{T}}_2 = \underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2$. For p an odd prime it is shown that a proper subvariety of $\underline{\underline{A}}_p \underline{\underline{T}}_p \wedge \underline{\underline{T}}_p \underline{\underline{A}}_p$ is either nilpotent or is contained in $\left[\underline{\underline{A}}_p \underline{\underline{A}}_p, k\underline{\underline{E}}\right]$ for some integer k. For $p = 2$ the results are similar, although more complicated. The first step towards these results is to find a basis for $\underline{\underline{T}}_p\left(F_\infty\left(\underline{\underline{A}}_p \underline{\underline{T}}_p\right)\right)$ and this is done in Chapter 4. Using this basis a basis is found for $\underline{\underline{T}}_p\left(F_\infty\left(\underline{\underline{A}}_p \underline{\underline{T}}_p \wedge \underline{\underline{T}}_p \underline{\underline{A}}_p\right)\right)$. The rest of the proof of the above results consists almost entirely of commutator calculations and this is done in Chapter 5.

Throughout this thesis extensive use has been made of commutator calculus, so that it seemed worthwhile to use a special form suitable for use in abelian-by-nilpotent groups. This is done in Chapter 1, which also includes some well-known commutator identities and which is basic to both sections of this thesis. The commutator calculus described in Chapter 1 is based on that of Brooks [1], with changes made to accommodate more easily the abelian-by-nilpotent situation. It is also noted that throughout this thesis much inspiration has been gained from the work of Brooks. In Chapter 2 his basis theorem for the derived group of $F_\infty\left(\underline{\underline{A}}_m \underline{\underline{A}}_n\right)$ is relied on quite heavily. In Chapter 3 his methods are extended to find the

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example of non-distributivity in lat$(\underline{A}_2\underline{A}_4)$. The philosophy of Chapters 4 and 5 is basically that of Brooks, where his methods and terminology are used, although his results are not used as they deal mainly with metabelian groups.

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