ON PROJECTIVE MODULES AND AUTOMORPHISMS OF CENTRAL SEPARABLE ALGEBRAS

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This paper developed from, and complements, the paper by F. R. DeMeyer (see **6**).

In the first section of this paper we note a correspondence between projective modules of a central separable R-algebra A and the two-sided ideals of central separable algebras in the same class as A in the Brauer group of R. When R has the property that rank one projective R-modules are free, this correspondence yields a bijection between isomorphism types of indecomposable projective A-modules and the isomorphism types of algebras in the Brauer class of A which are the analogue of division algebra components in the field case. This bijection was remarked on without proof by DeMeyer in (**6**).

Pursuing the ideas behind this correspondence, we consider the situation for a separable order A in a central simple algebra $\mathbf{A}$ over an algebraic number field, and obtain, by means of results involving the reduced norm, a generalization of DeMeyer's remark except when the division algebra component of $\mathbf{A}$ is a totally definite quaternion algebra (Theorem 3.3). We cite examples to show that our approach to this generalization fails in the totally definite case.

In (**6**), DeMeyer showed that certain well-known properties held by central simple algebras over fields hold also for central separable algebras over semi-local rings with no non-trivial idempotents and over polynomial rings with coefficients in a field of characteristic zero. Using the reduced norm results, we measure how badly these properties fail for maximal orders in central simple algebras over algebraic number fields. We conclude by applying a result of L. Silver to study the outer automorphism group of such orders.

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Notation. Throughout, we hold to the convention that modules are finitely generated. Unadorned $\otimes$ denotes tensor product over R. If A is a ring, A° denotes the opposite ring of A.

We shall say that two central R-algebras A and A' are Brauer equivalent if $A \otimes \mathrm{End}_R(E) \cong A' \otimes \mathrm{End}_R(E')$ for some faithful projective R-modules E and E'. If A and A' are central separable algebras, this is the usual notion of equivalence in the formation of the Brauer group of R. We shall say that two

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R-algebras A and B are Morita equivalent if there exists a left A-right B-module P such that $\operatorname{Hom}_A(P, P)^\circ \cong B$ and $(\ ) \otimes_A P$ induces an isomorphism from the category of right A-modules to the category of right B-modules. A left A-right B-module P induces such an isomorphism if P is a projective left A-module, $\operatorname{Hom}_A(P, P)^\circ \cong B$ and the trace ideal, $\operatorname{tr}_A(P)$, of P, namely, the two-sided idempotent ideal of A which is the image of the map from $P \otimes_B \operatorname{Hom}_A(P, A)$ to A given by $x \otimes f \rightarrow f(x)$, is all of A. This is a consequence of the Morita theorems, expositions of which may be found in (**1**; **3**; or **4**). If A is an R-algebra, we denote by $P_R(A)$ the group of isomorphism classes of projective left A-modules P with $\operatorname{Hom}_A(P, P)^\circ \cong A$.

If R is an integrally closed domain with quotient field K, "A is a maximal order" will mean "A is a maximal order over R in a central simple K-algebra $\mathbf{A} = A \otimes K$." An ideal I of A is a (finitely generated) left A-submodule of $\mathbf{A}$ such that $I \otimes K \cong \mathbf{A}$. If A is a maximal order, we let $T(A)$ denote the group of two-sided ideals of A and let $J(A)$ denote the group of left A-isomorphism classes of the elements of $T(A)$. By the degree of a central simple K-algebra $\mathbf{A}$ we mean $(\mathbf{A}:K)^{\frac{1}{2}}$.

1. Projective modules. We first note some facts concerning Brauer and Morita equivalence.

LEMMA 1.1. (a) *If A and B are two R-algebras which are Brauer equivalent, then A and B are Morita equivalent. Conversely, if R is a Noetherian ring of finite Krull dimension, A and B are R-algebras which are Morita equivalent, and A is a central separable R-algebra, then A and B are Brauer equivalent.*

(b) *If R is a Dedekind ring with quotient field K and A and B are maximal R-orders in central simple K-algebras* $\mathbf{A}$ *and* $\mathbf{B}$, *then A is Morita equivalent to B if and only if* $\mathbf{A}$ *is Brauer equivalent to* $\mathbf{B}$.

Proof. Part (a) is proved in (**4**, pp. 45–46). The "if" part of part (b) follows from (**1**, Theorem 3.9); the "only if" part is obtained by noting that the Morita equivalence of A and B is preserved by forming the tensor product with K and using part (a).

LEMMA 1.2. *If A is an R-algebra, then there is a one-to-one correspondence between isomorphism classes of projective left A-modules P with* $\operatorname{tr}_A(P) = A$ *and* $\bigcup_B P_R(B)$, *where B runs over all the isomorphism types of R-algebras, Morita equivalent to A.*

Proof. Suppose that A and B are Morita equivalent via the left A-right B-module P and let Q be a projective left A-module with $\operatorname{tr}_A(Q) = A$ and $\operatorname{Hom}_A(Q, Q)^\circ \cong B$. Then A and B are Morita equivalent via Q, so B and B are Morita equivalent via $\operatorname{Hom}_A(P, A) \otimes_A Q$, and in particular,

$$\operatorname{Hom}_B(\operatorname{Hom}_A(P, A) \otimes_A Q, \operatorname{Hom}_A(P, A) \otimes_A Q)^\circ \cong B.$$

Thus, $\mathrm{Hom}_A(P, A) \otimes_A (\ )$ induces a bijection between left isomorphism classes of projective left A-modules Q with $\mathrm{Hom}_A(Q, Q)^\circ \cong B$ and elements of $P_R(B)$.

Lemmas 1.1 and 1.2 can be combined to prove the final remark in (**6**), generalizing Corollary 3 of that paper.

PROPOSITION 1.3. *Let R be a Noetherian ring of finite Krull dimension, suppose that R has no idempotents other than* 0 *and* 1, *and assume that rank one projective R-modules are free. If A is a central separable R-algebra, then the number of left A-isomorphism classes of indecomposable projective left A-modules is equal to the number of isomorphism types of algebras without idempotents other than* 0 *and* 1 *in the class of A in the Brauer group of R.*

Proof. A central separable R-algebra A with R a Noetherian ring with no non-trivial idempotents has no non-trivial two-sided idempotent ideals, since R has none. This follows from the Krull intersection theorem (for R) and Theorem 3.2 of (**2**) (for A). Thus, Lemma 1.2 yields a one-to-one correspondence between all isomorphism classes of projective left A-modules and $\cup_B P_R(B)$, where B runs over all isomorphism classes of algebras, Morita equivalent to A. By Lemma 1.1, B then runs over all isomorphism classes of algebras Brauer equivalent to A. By (**2**, Theorem 3.5), the algebras B are then central separable R-algebras.

Now, Rosenberg and Zelinsky (**16**) have shown that if rank one projective R-modules are free, then $P_R(B) = (1)$. Since B has no idempotents other than zero and one if and only if M, a left A-right B-module giving a Morita equivalence of A and B, is indecomposable, the result follows.

For maximal orders we can obtain a result analogous to Proposition 1.2 via the following lemma.

LEMMA 1.4. *If R is a Dedekind ring and B is a maximal R-order, then* $J(B) = P_R(B)$.

Proof. We must show that any left B-module M with $\mathrm{Hom}_B(M, M)^\circ \cong B$ is left B-isomorphic to a two-sided ideal of B. Now, $M \otimes K \cong \mathbf{B}$ as left $B \otimes K$-modules; therefore, by embedding M in $\mathbf{B}$ via this map we can assume that M is a left B-ideal. Then $\mathrm{Hom}_B(M, M) \subset \mathit{Hom}_B(M, M) \otimes K \cong \mathbf{B}$; therefore, $\mathrm{Hom}_B(M, M)^\circ = B' = \{x \text{ in } \mathbf{B}: mx \text{ is in } M \text{ for all } m \text{ in } M\}$, the right order of M. Since $B \cong B'$ and this isomorphism extends to an inner automorphism of $\mathbf{B}$, we have that $B = t^{-1}Bt$ for some t in $\mathbf{B}$. Then, the right order of Mt is B; therefore, Mt is a two-sided ideal of B which is left B-isomorphic to M.

Conversely, if I is a two-sided ideal of B, then $\mathrm{Hom}_B(I, I)^\circ$, the right order of I, is equal to B by the results in (**7**).

PROPOSITION 1.5. *If R is a Dedekind ring and A is a maximal R-order, then the left isomorphism classes of projective left A-modules correspond with $\bigcup_B J(B)$, where B runs over all maximal orders in central simple algebras Brauer equivalent to* $\mathbf{A}$.

Proof. Once we note that A has no non-trivial two-sided idempotent ideals (**12**), the proposition follows immediately from Lemmas 1.1, 1.2, and 1.4.

Remark. Using Propositions 1.2 and 1.5 one can obtain, by using results of Rosenberg and Zelinsky, and Silver, respectively, a relationship between the projective modules of a central separable algebra or a maximal order A and the groups of R-algebra automorphisms of algebras Brauer equivalent to A. For example, the former authors have shown, in the central separable case, that $P_R(A) \cong P(R)/O(A)$, where $P(R)$ is the projective class group of R and $O(A)$ is the group of R-algebra automorphisms of A modulo inner automorphisms. We shall return to this theme in § 4.

2. The reduced norm. The results of § 1 suggest the desirability of computing $P_R(A)$ in order to attempt to obtain more results similar to Proposition 1.3. It turns out that in many arithmetic cases there exist explicit computations of $P_R(A) = J(A)$ in terms of the arithmetic of R, using the reduced norm. In this section we outline these results and some examples, in preparation for the results of §§ 3 and 4.

Henceforth, R is a Dedekind ring with quotient field an algebraic number field. A is a maximal order. Let m denote the degree of $\mathbf{A}$.

Definition. The (reduced) norm $n(M)$ of an integral left ideal M of A is defined to be the mth root of the product of the invariant factors of M in A (viewed as R-modules) (**7**, Chapter VI, § 4; **14**, p. 214).

If M is a principal ideal, $M = Ax$, then $n(m) = n(x)$, the element norm of x, defined equivalently as $(-1)^{\deg f} f(0)$, where f is the "hauptpolynom" of x (**7**, p. 51), or as $\det T$, where T is the matrix of $x \otimes 1$ in the representation of $A \otimes_K L$ as matrices over L, where L is a splitting field of $\mathbf{A}$. The norm preserves multiplication of ideals when multiplication is defined. One extends n to all ideals by representing them as multiples of integral ideals by elements in K (see **7**).

Concerning the reduced norm, we have the following classical facts:

n maps the ideals of $\mathbf{A}$ onto $I(R)$, the group of all fractional ideals of R. In fact, n maps the set of left A-ideals onto $I(R)$, where A is any maximal order of $\mathbf{A}$ (see **11**);

n sends two-sided ideals of A onto the subgroup of $I(R)$ generated by p^{m/m_p}, where p runs through all finite prime ideals of R, and m_p is the p-index of $\mathbf{A}$ (m_p is the degree of the division algebra component of $A \otimes_K K_p$, where K_p is the completion of K at the prime p).

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If $\mathbf{u}$ is the set of infinite primes at which $\mathbf{A}$ is ramified (i.e., not split), n sends principal ideals to $R_{\mathbf{u}}$, the ray mod $\mathbf{u}$, the set of principal ideals of R which have a generator which is positive at all primes of $\mathbf{u}$ (see **8**, § 3, p. 198).

A crucial result of Eichler is the following: If $\mathbf{A}$ is not a totally definite quaternion algebra, i.e., if the degree of $\mathbf{A}$ is not 2, or $\mathbf{A}$ splits at some infinite prime of K, then the only ideals sent to $R_{\mathbf{u}}$ by n are principal (**10**, p. 482, *Satz* 1).

From these facts, one obtains the following information: n maps $J(A)$ onto $\prod_p (p^{m/m_p})/\prod_p (p^{m/m_p}) \cap R_{\mathbf{u}}$. If $\mathbf{A}$ is not a totally definite quaternion algebra, the map is an isomorphism.

The ideal class number $h(\mathbf{A})$ (the number of isomorphism classes of left A-ideals) is greater than or equal to $[I(R):R_{\mathbf{u}}]$, and equality holds if $\mathbf{A}$ is not a totally definite quaternion algebra (**8**).

The number $t(\mathbf{A})$ of isomorphism types of maximal orders in $\mathbf{A}$ is at least as large as $[I(R):\prod (p^{m/m_p}) \cdot R_{\mathbf{u}}]$, and equality holds if $\mathbf{A}$ is not a totally definite quaternion algebra (**17**).

From these computations one immediately obtains the following results.

PROPOSITION 2.1. *If* $\mathbf{A}$ *is not a totally definite quaternion algebra and* A *and* A' *are any two maximal orders in* A, *then* $J(A) \cong J(A')$.

COROLLARY 2.2. *If* $\mathbf{A}$ *is not a totally definite quaternion algebra, then* $h(\mathbf{A}) = t(\mathbf{A}) \cdot [J(A):1]$, *where* A *is any maximal order in* $\mathbf{A}$.

Corollary 2.2 also follows from Proposition 2.1, using Proposition 1.5. Proposition 2.1 also follows directly from Eichler's result: define a map from $T(A)$ to $T(A')$ by $M \rightarrow Q^{-1}MQ$, Q any left A-right A'-ideal (such as the conductor); Eichler's result implies that multiplication of ideals "gets along" with the taking of left ideal classes.

The exceptional cases in the computations above are genuine. For example, Swan (**19**) gives an example of a totally definite quaternion algebra $\mathbf{A}$ with $t(\mathbf{A}) = h(\mathbf{A}) = 2$ although $I(R) = R_{\mathbf{u}}$; the orders in his example are separable. Also, the following example shows that Proposition 2.1 is not generally valid, thus indicating a limitation on the usefulness of Proposition 1.5.

Let $\mathbf{A}$ be a totally definite quaternion algebra over the rationals. Then Eichler, in the proof of (**9**, *Satz* 2) shows that all maximal orders over the integers in $\mathbf{A}$ whose groups of units have order 4 are isomorphic, and all maximal orders over the integers whose groups of units have order 6 are similarly isomorphic. Now, for any two such maximal orders, A_2 and A_3, Eichler computes $h_2 = [J(A_2):1]$ and $h_3 = [J(A_3):1]$ in that proof to be as follows: $h_2 = 2^{u-1}$, where u is the number of odd primes dividing the discriminant d^2 of $\mathbf{A}$, unless no prime divisor of d is congruent to 1 modulo 4 (in which case $h_2 = 0$); $h_3 = 2^{v-1}$, where v is the number of primes different from 3 dividing d, unless no prime divisor of d is congruent to 1 modulo 3 (in which case $h_3 = 0$). If $\mathbf{A}$ is the quaternion algebra generated over the rationals by

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$1, i, j, k = ij = -ji$ with $i^2 = -1$, $j^2 = a$ with $a = -11 \cdot 23 \cdot 2$, for example, then $d = a$ (see **9**, p. 104); thus, we have that $h_2 = 2$, $h_3 = 4$. (An example such as this, of course, is not a separable algebra; whether Corollary 2.2 holds for all separable orders is unknown.)

3. Simple properties. Using the results of § 2, we consider the questions answered affirmatively in certain cases by DeMeyer (**6**), and obtain "en route" a generalization of Proposition 1.3. The assumptions on R, K, A of § 2 are retained in this section.

1. *Is there a unique (up-to isomorphism) indecomposable projective A-module?*

Let A be a maximal order in $\mathbf{A}$, a central simple K-algebra of degree m. Let $\mathbf{A}$ have division algebra component $\mathbf{D}$ of degree m_0. Then $\mathbf{A} = M_n(\mathbf{D})$ and $m = m_0 n$, by Wedderburn's theorem.

Let P be a projective left A-module. Then, as is seen by forming the tensor product with K, P has R-rank $mm_0 r$ for some positive integer r. Let D be a maximal order in the division ring component $\mathbf{D}$ of $\mathbf{A}$. By Lemma 1.1, D and A are Morita equivalent via some left D-right A-module Q. Then $Q \otimes_A P$ is a projective left D-module, of R-rank $m_0^2 r$. Now let $\mathbf{B} = M_r(\mathbf{D})$, let B be a maximal order in $\mathbf{B}$, and let Q' be a left B-right D-module giving a Morita equivalence of D and B. Then $Q' \otimes_D Q \otimes_A P$ is a left B-module of the same R-rank as B, namely, $m_0^2 r^2$, hence is isomorphic to a left ideal of B (by an argument similar to that of Lemma 1.4).

Thus, $Q' \otimes_D Q \otimes_A (\ )$ is an invertible map sending left A-isomorphism classes of projective left A-modules of the same rank as P, $mm_0 r$, onto the set of left B-ideals. From this we have (by the computations of § 2) the following theorem.

THEOREM 3.1. *With R, K, A, $\mathbf{A}$, $\mathbf{D}$, m, and m_0 as above, the number of left isomorphism classes of projective left A-modules of R-rank equal to $mm_0 r$ is equal to $I(R){:}R_{\mathbf{u}}$ if $r > 1$, and is equal to $h(\mathbf{D})$ if $r = 1$. We have that $h(\mathbf{D}) \geqq I(R){:}R_{\mathbf{u}}$ with equality if $\mathbf{D}$ is not a totally definite quaternion algebra.*

This result was obtained for $\mathbf{A}$ a division algebra by Swan in (**19**).

2. *Is there a unique (up-to isomorphism) algebra Brauer equivalent to A containing no idempotents other than* 0 *and* 1*?*

PROPOSITION 3.2. *If A is a separable order, the number of isomorphism classes of such algebras is equal to $t(\mathbf{D})$, where $\mathbf{D}$ is the division algebra component of $\mathbf{A}$.*

Proof. If D is any algebra Brauer equivalent to A, then $\mathbf{D}$ is Brauer equivalent to $\mathbf{A}$. Thus, D has no idempotents other than 0 and 1 if and only if D is an order in $\mathbf{D}$, the division algebra component of $\mathbf{A}$. (Note that by (**2**, Theorem 3.5), D is separable, since A is.) Now, if D is a maximal order in $\mathbf{D}$, then since $\mathbf{A} = \mathbf{D} \otimes_K \operatorname{End}_K(E)$, $A' = D \otimes_R \operatorname{End}_R(E')$ is an order in $\mathbf{A}$, where E' is some free R-module of the same R-rank as the K-dimension of E.

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A' and D are Morita equivalent by Lemma 1.1; therefore, A' is a maximal order in $\mathbf{A}$ by (**12**). Thus, A' is a separable R-algebra since A is (**12**, Proposition 7.3). Since the map from the Brauer group of R to the Brauer group of K given by $A \to A \otimes K$ is a monomorphism by (**2**, Proposition 7.2), A and A' are in the same class in the Brauer group of R since they are orders in the same central simple algebra. Thus, D and A are Brauer equivalent.

The number $t(\mathbf{D})$ was described in § 2. Regarding that description, it is worth noting that A is separable over R if and only if all $m_p = 1$ (see **7**, p. 108, and **15**, § 5).

From the last three results (Corollary 2.2, Theorem 3.1, and Proposition 3.2), we now immediately have the generalization of Proposition 1.3 above, namely, the following theorem.

THEOREM 3.3. *If A is a separable order in $\mathbf{A}$ and the division ring component $\mathbf{D}$ of $\mathbf{A}$ is not a totally definite quaternion algebra, then the number of isomorphism types of indecomposable projective left A-modules is equal to the product of $[J(A):1]$ and the number of isomorphism types of algebras with no idempotents other than 0 and 1, Brauer equivalent to A.*

We now continue with the questions considered in (**6**).

3. *If B is a separable R-subalgebra of A with no non-trivial central idempotents and $f: B \to A$ is an R-algebra monomorphism, does f extend to an automorphism of A?*

This question has a negative answer whenever one has the situation: A and A' are non-isomorphic separable orders in $\mathbf{A}$, and there exist free R-modules E and E' such that

$$A'' = \mathrm{Hom}_R(E, E) \otimes A \cong \mathrm{Hom}_R(E', E') \otimes A'.$$

For then, viewing $\mathrm{Hom}_R(E', E')$ and A' as embedded in A'' via that isomorphism, the obvious map from $\mathrm{Hom}_R(E, E)$ to $\mathrm{Hom}_R(E', E')$ cannot extend to an automorphism of A'', or else (using **2**, Theorem 3.5) it would yield an isomorphism of A and A'. This was pointed out and applied to Swan's example in (**5**).

In fact, in our situation, one obtains a counterexample whenever there exist two non-isomorphic separable orders A and A' in an algebra $\mathbf{A}$. For then, by (**2**, Theorem 7.2), A and A' are in the same class in the Brauer group of R. There is an epimorphism from $\bar{B}(R)$ to $B(R)$, where $\bar{B}(R)$ is the group constructed like the Brauer group $B(R)$ with the equivalence relation being A equivalent to A' if and only if there exist *free* R-modules, E and E', such that $A \otimes \mathrm{Hom}_R(E, E) \cong A' \otimes \mathrm{Hom}_R(E', E')$. Hoobler points out (**13**, pp. 34, 57) that if R has a torsion ideal class group, as in our case, this map is an isomorphism. Thus, in our situation, any non-isomorphic separable orders A and A' in $\mathbf{A}$ are equivalent in the more restricted sense, thus yield a counterexample to the extension of the isomorphism problem.

Clearly, whenever B is a separable order and $t(\mathbf{B}) > 1$, B may be used in an example of this type.

4. Automorphisms. We now use the results of § 2 to study the automorphisms of a maximal order. The assumptions concerning R, A, K, etc., in § 2 remain in effect in this section.

In (**16**), Rosenberg and Zelinsky derived, for central separable R-algebras, the exact sequence

$$1 \to O(A) \to C(R) \to P_R(A) \to 1, \tag{1}$$

where $O(A)$ is the group of R-algebra automorphisms of A modulo inner automorphisms, $C(R)$ is the ideal class group of R, and $P_R(A)$ is as defined above. They showed that if $A \cong \mathrm{Hom}_R(E, E)$ with the R-rank of E equal to m, then the sequence becomes

$$1 \to O(A) \to C(R) \to C(R), \tag{2}$$

where the last map sends the class of I to the class of I^m. The sequence (1) was generalized by Silver (**18**) to tame orders. The special case of his generalization which we use here is for maximal orders: if A is a maximal order, then the following sequence is exact:

$$1 \to O(A) \to C(A) \to J(A) \to 1, \tag{3}$$

where $C(A)$ is the group of $A \otimes A^{\circ}$-isomorphism classes of two-sided ideals of A, the epimorphism is the obvious one, and the monomorphism assigns to any automorphism σ of A the ideal At, where t is an element in $\mathbf{A}$ with $\sigma(x) = txt^{-1}$ for all x in A; see the proof of Lemma 1.4.

The group $C(A)$ is defined by

$$1 \to P(R) \to T(A) \to C(A) \to 1, \tag{4}$$

where $P(R)$ is the group of principal ideals of R, the maps in (4) being $(x) \to Ax$, $I \to \mathrm{cl}(I)$ (see **18**, Chapter I, Lemma 3.3). Now, $T(A)$ is the direct product of infinite cyclic groups generated by $N(p)$, where $N(p)$ is the two-sided prime ideal of A lying over p, as p runs over all finite prime ideals of R (see **12**). We have that $pA = N(p)^{m_p}$, where m_p is the p-index of A (see **7**, p. 114, *Satz* 5). We thus have the exact sequence

$$1 \to I(R) \to T(A) \to \textstyle\prod_p \mathbf{Z}/m_p\mathbf{Z} \to 1. \tag{5}$$

The monomorphisms in (5), (4) and in the exact sequence which defines $C(R)$ form an obvious commutative diagram, from which we obtain immediately that $C(A)/C(R) = \prod_p \mathbf{Z}/m_p\mathbf{Z}$. Out of this discussion, our final results follow immediately.

THEOREM 4.1. *If A is a maximal order in $\mathbf{A}$ and $\mathbf{A}$ is not a totally definite quaternion algebra, then*

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(a) $[O(A):1] = \prod_p m_p \cdot h(R)/(h(\mathbf{A})/t(\mathbf{A}))$;
(b) *The following sequence is exact:*

$$1 \to O(A) \to C(A) \xrightarrow{b} \prod_p (p^{m/m_p}) \Big/ \prod_p (p^{m/m_p}) \cap R_{\mathbf{u}} \to 1, \tag{6}$$

where m is the index of $\mathbf{A}$, *m_p the p-index, and $b(\mathrm{cl}(pA))$ = the coset of p^m.*

Proof. The first statement of the theorem is immediate from the sequence (3) together with the calculation of $C(A)$ and Corollary 2.2.

In sequence (6), the monomorphism is the same as in the sequence (3). The epimorphism b sends the class of a two-sided ideal I of A to the coset of $n(I)$, where n is the reduced norm. In particular, the class of pA is sent by b to the coset of p^m since n is multiplicative and $n(N(p)) = p^{m/m_p}$ (see **7**). The sequence (6) is clearly exact, since (3) is.

When A is also separable, $C(A) \cong C(R)$ via: class of I in $C(R)$ goes to class of IA. All $m_p = 1$; thus, (6) becomes

$$1 \to O(A) \to C(R) \to \prod_p (p^m) \Big/ \prod_p (p^m) \cap R_{\mathbf{u}} \to 1,$$

where the epimorphism sends the class of I to the coset of $I^m = n(IA)$. In particular, when $A \cong \mathrm{Hom}_R(E, E)$, $R_{\mathbf{u}} = P(R)$, and the sequence (6) reduces to the sequence (2) above.

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