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SIMULTANEOUS APPROXIMATION OF e^t AND $\wp(t)$

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Abstract

Let t be any complex number different from the poles of a Weierstrass elliptic function $\wp(z)$, having algebraic invariants. Then we estimate from below the sum

$$|e^t - \alpha| + |\wp(t) - \beta|,$$

where α and β are algebraic numbers. The estimate is given in terms of the heights of α and β and the degree of the field $\mathbf{Q}(\alpha, \beta)$, where $\mathbf{Q}$ is the field of rationals.

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1. Introduction

Let $\wp(z)$ be the Weierstrass elliptic function with the invariants g_2 and g_3 algebraic. Let t be any complex number different from the poles of $\wp(z)$. Then it is known that at least one of the numbers

$$e^t, \wp(t)$$

is transcendental. In this paper we approximate these numbers simultaneously by algebraic numbers.

2. Notations

Let $\mathbf{Q}$ be the field of rational numbers. By the height of an algebraic number α, denoted by $h(\alpha)$, we mean the maximum of the absolute values of the coefficients of the minimal polynomial of α. It is then easy to derive that

$$|\alpha^{(i)}| \leqslant h(\alpha) + 1,$$

where $\alpha^{(i)}$ denotes the various conjugates of α. By size of α we mean the maximum of the absolute values of the conjugates of α. We also use the symbol $\wp(z, \lambda, t)$ for $d^t/dz^t \wp^\lambda(z)$. $c_1, c_2, \ldots$ denote absolute constants which are effectively computable.

3. Statement of the theorem

Let t be any complex number different from the poles of $\wp(z)$. Let α and β be algebraic numbers of heights at most h_1 and h_2 respectively. Assume also that α is not a root of unity. Let $K = \mathbf{Q}(\alpha, \beta)$ be of degree D over $\mathbf{Q}$. Let h_1 and $h_2 \geqslant e^e$. Then

$$|e^t - \alpha| + |\wp(t) - \beta| > \exp\left\{-cD^7 \log h_1 (\log h_2)^2 \log^4(D \log h_1 \log h_2) \log^{-4}(D \log h_2)\right\}$$

where c is a large constant effectively computable, independent of D, h_1 and h_2.

4. Proof

Let $|e^t - \alpha| = \varepsilon_1$; $|\wp(t) - \beta| = \varepsilon_2$ and $\varepsilon_0 = \max(\varepsilon_1, \varepsilon_2)$. Suppose

$$\varepsilon_0 \leqslant \exp\left\{-x^7 D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\right\} \tag{1}$$

where $B = xD \log h_1 \log h_2$ and $E = D^{1/2} (\log h_2)^{1/2}$. Then we shall get a contradiction which proves the theorem. x is a large number and restrictions on it will appear as we go along.

The proof runs through three steps. Firstly, we construct an auxiliary function which is small at certain lattice points up to certain order using the standard Siegel's lemma. Secondly, using Lemma 2 of [2] we increase the order. The final contradiction is got by using Theorem 1 of [1].

Let e_1, e_2, e_3 be the zeros of $4x^3 - g_2 x - g_3 = 0$, where g_2 and g_3 are the invariants of $p(z)$. If $p(t) \notin \{e_1, e_2, e_3\}$ we note that when (1) holds, for large x, there exists β' with $\beta = \wp(\beta')$ such that

$$|t - \beta'| \leqslant c_1 |\wp(t) - \beta|.$$

If $p(t) \in \{e_1, e_2, e_3\}$, for large z, $p(t) = \beta$. But $p(\beta') = \beta$ and hence the above inequality holds trivially by taking $\beta' = t$ on the left hand side. Thus if $|t - \beta'| = \varepsilon_2'$ and $\varepsilon_0' = \max(\varepsilon_1, \varepsilon_2')$ then it is enough to show that

$$\varepsilon_0' \leqslant \exp\left\{-x^7 D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\right\} \tag{2}$$

leads to a contradiction.

First step. Let ξ be a primitive element of K. Consider

$$F(z) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{i=0}^{D-1} p(\lambda_1, \lambda_2, i)\xi^i e^{\lambda_1 z}\wp^{\lambda_2}(z) \tag{3}$$

where

$$L_1 = \left[x^4 D^3 \log h_2 (\log B)^2 \log^{-3} E\right] \tag{4}$$

and

$$L_2 = \left[x^3 D^2 \log h_1 (\log B) \log^{-2} E\right]. \tag{5}$$

Here and in the sequel $[x]$ denotes the integral part of x. $p(\lambda_1, \lambda_2, i)$ are unknowns to be determined soon. Now

$$F^{(k)}(st) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{i=0}^{D-1} p(\lambda_1, \lambda_2, i)\sigma^i \sum_{\mu=0}^{k} \binom{k}{\mu} \lambda_1^{\mu} e^{\lambda_1 st} \wp(st, \lambda_2, k-\mu).$$

Define

$$F_s(z) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{i=0}^{D-1} p(\lambda_1, \lambda_2, i)\xi^i e^{\lambda_1 z}\wp^{\lambda_2}(z + s\varepsilon)$$

where $\varepsilon = \beta' - \log\alpha$. Here we take a fixed branch of the logarithm, say the principal. Then

$$F_s^{(t)}(s\log\alpha) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{i=0}^{D-1} p(\lambda_1, \lambda_2, i)\xi^i \sum_{\mu=0}^{t} \binom{t}{\mu} \lambda_1^{\mu} \alpha^{\lambda_1 s} \wp(s\beta', \lambda_2, t-\mu).$$

Take

$$F_s^{(t)}(s\log\alpha) = 0 \quad \text{for } 1 \leqslant s \leqslant S \text{ and } 0 \leqslant t \leqslant T-1, \tag{6}$$

where

$$S = \left[xD \log B \log^{-1} E\right] \quad \text{and} \tag{7}$$

$$T = \left[x^5 D^4 \log h_1 \log h_2 (\log B)^2 \log^{-4} E\right]. \tag{8}$$

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It is known from Lemma 5.1 of [4] that for every complex number u, there exist polynomials $\Phi_u, \Phi_u^* \in Z[x_1,\dots,X_5]$, their heights and degrees bounded by an absolute constant such that

$$\wp(z+u) = (\Phi_u^*/\Phi_u)(\wp(z+\omega_1/2), \wp'(z+\omega_1/2), \wp(u), \wp'(u), \wp(\omega_1/2))$$

where $\Phi_u(\wp(\omega_1/2), \wp'(\omega_1/2), \wp(u), \wp'(u), \wp(\omega_1/2)) \neq 0$. From Lemma 6.1 of [3], for every rational integer s, there exist coprime polynomials Ψ_s, Ψ_s^* of degree at most s^2 such that

$$\wp(sz) = \frac{\Psi_s^*}{\Psi_s}(\wp(z)).$$

The coefficients of Ψ_s^*, Ψ_s are themselves polynomials in $g_2/4$, g_3 with a degree at most s^2 and rational integer coefficients, not larger than $c_2^{s^2}$. Define

$$\phi_{s,u}(z) = \Phi_u^{s^2}(\wp(z+\omega_1/2), \wp'(z+\omega_1/2), \wp(u), \wp'(u), \wp(\omega_1/2))\Psi_s(\wp(z+u))$$

and

$$\psi_{s,u}(z) = \phi_{s,u}(z)\wp(sz+su).$$

Then (6) becomes

$$(9)\quad F_s^{(t)}(s\log\alpha) = \sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2}\sum_{i=0}^{D-1} p(\lambda_1,\lambda_2,i)\xi^i\frac{d^t}{dz^t}\left(e^{\lambda_1(z+s\log\alpha)}\phi_{s,\beta'}^{-\lambda_2}\psi_{s,\beta'^2}^{\lambda_2}\right)\Big|_{z=0}$$
$$= 0 \quad \text{for } 1 \leqslant s \leqslant S;\ 0 \leqslant t \leqslant T-1.$$

But (9) is equivalent to solving the following system of equations.

$$(10)\quad f_{t,s} = \sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2}\sum_{i=0}^{D-1} p(\lambda_1,\lambda_2,i)\xi^i\frac{d^t}{dz^t}\left(e^{\lambda_1(z+s\log\alpha)}\psi_{s,\beta'}^{\lambda_2}\right)\Big|_{z=0}$$
$$= 0 \quad \text{for } 1 \leqslant s \leqslant S;\ 0 \leqslant t \leqslant T-1.$$

From Lemma 5.2 of [4], the expression

$$\frac{d^t}{dz^t}\psi_{s,\beta'}^{\lambda_2}\Big|_{z=0}$$

can be written as a polynomial in $\wp(\beta')$ and $\wp'(\beta')$ of degree $c_3\lambda_2 s^2$ in each of the variables. The coefficients belong to $\mathbf{Q}(\wp'(\omega_1/2), \wp''(\omega_1/2))$ and their size is bounded by $\exp\{c_4(\lambda_2 s^2 + t\log t)\}$. They have a common denominator of the form m^n where m depends on $\wp(z)$ and $n \leqslant c_4(\lambda_2 s^2 + t\log t)$. Thus size of the coefficients in the system of linear equations in (10) is bounded by

$$T^{c_5 T}(2L_1 Q)^T h_1^{c_5 L_1 S} h_2^{c_5 L_2 S^2}(ht\xi+1)^D$$
$$\leqslant \exp\left\{c_6 x^5 D^4 \log h_1 \log h_2 (\log B)^3 \log^{-4} E\right\}.$$

The number of unknowns $p(\lambda_1, \lambda_2, i)$ is equal to

$$(L_1+1)(L_2+1)D \geqslant c_7 x^7 D^6 \log h_1 \log h_2 (\log B)^3 \log^{-5} E,$$

while the number of equations over K is

$$ST \leqslant c_8 x^6 D^5 \log h_1 \log h_2 (\log B)^3 \log^{-5} E.$$

Thus if $x > c_7^{-1} c_8$, then there exist rational integers $p(\lambda_1, \lambda_2, i)$, not all zero such that

$$\begin{aligned} P &= \max |p(\lambda_1, \lambda_2, i)| \\ &\leqslant \exp\left\{c_9 x^5 D^4 \log h_1 \log h_2 (\log B)^3 \log^{-4} E\right\}. \end{aligned} \tag{11}$$

Second step. Let $T' = [x^2 T]$.

Claim. $f_{k,s} = 0$ for $1 \leqslant s \leqslant S$; $0 \leqslant k \leqslant T' - 1$. If not, choose the least k such that $f_{k,s} \neq 0$ for some s.

Lower bound for $|f_{k,s}|$. $|f_{k,s}|$ is an algebraic number of degree at most $c_{10} D$ and size at most $\exp\{c_{11} x^5 D^4 \log h_1 \log h_2 (\log B)^3 \log^{-4} E\}$. Hence

$$|f_{k,s}| > \exp\left\{-c_{12} x^5 D^5 \log h_1 \log h_2 (\log B)^3 \log^{-4} E\right\}. \tag{12}$$

Upper bound for $|f_{k,s}|$. For $0 \leqslant k \leqslant T-1$,

$$\begin{aligned} |F^{(k)}(st)| &= |F^{(k)}(st) - F_s^{(k)}(s \log \alpha)| \\ &= \left| \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{i=0}^{D} p(\lambda_1, \lambda_2, i) \xi^i \sum_{\mu=0}^{k} \binom{k}{\mu} \lambda_1^{\mu} \right. \\ &\qquad \left. \times \left\{ e^{\lambda_1 st} p(st, \lambda_2, k-\mu) - \alpha^{\lambda_1 s} \wp(s\beta', \lambda_2, k-\mu) \right\} \right|. \end{aligned} \tag{13}$$

Let $\wp(st, \lambda_2, k-\mu) - \wp(s\beta', \lambda_2, k-\mu) = \varepsilon_3$. We can show easily by the integral formula for the left hand side of the above (see page 93, [3]) that

$$|\varepsilon_3| < \exp\left\{-c_{13} x^7 D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\right\}.$$

Thus from (13) it follows that for $0 \leqslant k \leqslant T-1$,

$$\begin{aligned} |F^{(k)}(st)| &= |F^{(k)}(st) - F_s^{(k)}(s \log \alpha)| \\ &< \exp\left\{-c_{14} x^7 D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\right\}. \end{aligned} \tag{14}$$

Now let $G(z) = (\sigma(z))^{2L_2} F(z)$, where $\sigma(z)$ is the Weierstrass sigma function associated to $\wp(z)$. Put

$$g(z) = (\sigma(z))^{2L_2}.$$

Then $G(z)$ is entire and

(15) $$G^{(k)}(st) = \sum_{\tau=0}^{k} \binom{k}{\tau} g^{(\tau)}(st) F^{(k-\tau)}(st).$$

Since $|g^{(\tau)}(st)| \leqslant \tau!\, c_{15}^{(L_2S^2+\tau)}$, it is clear from (14) and (15) that for $0 \leqslant k \leqslant T-1$,

(16) $$|G^{(k)}(st)| \leqslant \exp\{-c_{16}x^7D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\}.$$

Now using Lemma 2 of [2], we see that

(17) $$\max_{|z| \leqslant c_{17}S} |G(z)| \leqslant \max_{|z| \leqslant c_{17}SE} |G(z)| \left(\frac{c_{18}}{E}\right)^{ST} + ST(c_{18})^{ST} \max_{\substack{1 \leqslant s \leqslant S \\ 0 \leqslant k \leqslant T-1}} |G^{(k)}(st)|$$

where $E = D^{1/2} (\log h_2)^{1/2}$. By the choice of S, T and by (16) it follows easily that

(18) $$\max_{|z| \leqslant c_{17}S} |G(z)| \leqslant \exp\{-c_{19}x^6D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\}.$$

Using Cauchy's inequality, $|G^{(k)}(st)|$, for $0 \leqslant k \leqslant T' - 1$, also has the same estimate as in (18). For $\tau \neq 0$, from (14)

$$\left|\binom{k}{\tau} g^{(\tau)}(st) F^{(k-\tau)}(st)\right| \leqslant \exp\{-c_{20}x^7D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\}.$$

But left hand side of (15) has the same estimate as (18). Hence

(19) $$|g(st) F^{(k)}(st)| \leqslant \exp\{-c_{21}x^6D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\}.$$

It is known that $|g(st)| > c_{22}^{-L_2S^2}$. See Lemma 7.1 of [3]. Hence $|F^{(k)}(st)|$ and therefore from (14) $|F_s^{(k)}(s \log \alpha)|$ has the same estimate as in (19), But

$$f_{k,s} = \sum_{\tau=0}^{k} \binom{k}{\tau} \frac{d^\tau}{dz^\tau} \phi_{s,\beta'}^{\lambda_2} \Big|_{z=0} F_s^{(k-\tau)}(s \log \alpha).$$

But by the choice of k,

$$f_{k,s} = \phi_{s,\beta'}^{\lambda_2}\big|_{z=0} F_s^{(k)}(s \log \alpha).$$

Thus

(20) $$|f_{k,s}| \leqslant \exp\{-c_{23}x^6D^7 \log h_1 (\log h_2)^2 (\log B)^4 \log^{-4} E\}.$$

From (12) and (20) it is clear that if $x > c_{23}^{-1}c_{12}$ then there is a contradiction to the lower and upper estimates of $|f_{k,s}|$. Hence the claim.

Third step. Let $P(X, Y) = \sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2} p(\lambda_1, \lambda_2)X^{\lambda_1}Y^{\lambda_2}$, where $p(\lambda_1, \lambda_2) = \sum_{i=0}^{D-1} p(\lambda_1, \lambda_2, i)\xi^i$. Suppose $P \not\equiv 0$. Let

$$N_s = \operatorname{Ord} P(e^z, \wp(z + s\varepsilon)).$$

Now we use the theorem due to Brownawell and Masser. See Theorem 1 of [1]. Thus

$$\sum_{s=1}^{S} N_s \leqslant c_{24}\{(L_1 + 1)(L_2 + 1) + (L_2 + 1)S\}$$
$$\leqslant c_{25}x^7D^5 \log h_1 \log h_2(\log B)^3 \log^{-5} E.$$

But by Second Step,

$$\sum_{s=1}^{S} N_s \geqslant c_{26}ST' \geqslant c_{27}x^8D^5 \log h_1 \log h_2(\log B)^3 \log^{-5} E.$$

Hence if $x > c_{27}^{-1}c_{25}$, there is a contradiction. So $P \equiv 0$. But by the algebraic independence of the functions e^z and $\wp(z)$ this means that $p(\lambda_1, \lambda_2) = 0$ for every (λ_1, λ_2). This by First Step, implies that $1, \xi, \ldots, \xi^{D-1}$ are linearly dependent. But $\deg \xi = D$. Hence we arrive at the final contradiction for the assumption on $|\varepsilon_0'|$ which proves the theorem.

5. Remarks

REMARK 1. When heights of α and β of the theorem are of the same magnitude h, we get

$$|e^t - \alpha| + |\wp(t) - \beta| > \exp\{-cD^7(\log h)^3\}.$$

Thus when heights of the approximating algebraic numbers are of different magnitude, we get sharper results with regard to each of the heights as exhibited in the theorem.

REMARK 2. When u is an algebraic point of $\wp(z)$, we get from the theorem that e^u has transcendence type at most $8 + \varepsilon$, $\varepsilon > 0$.

REMARK 3. By using the above method (see [1] for more details) many other numbers involving values of the elliptic function can be approximated simultaneously. As an example, we can deal with algebraic points of an elliptic function. For these results, see [5].

Addendum

There is an announcement of a quantitative, one variable general Schneider-Lang result in D. W. Masser, some recent results in transcendence theory, *Astérisque* **61** (1979), 145–154. The method of proof is more involved and the dependence on the degree less satisfying, due to greater generality.

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