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AN ASYMPTOTIC THEORY FOR JUMP DIFFUSION MODELS

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This paper presents an asymptotic theory for recurrent jump diffusion models with well-defined scale functions. The class of such models is broad, including general nonstationary as well as stationary jump diffusions with state-dependent jump sizes and intensities. The asymptotics for recurrent jump diffusion models with scale functions are largely comparable to the asymptotics for the corresponding diffusion models without jumps. For stationary jump diffusions, our asymptotics yield the usual law of large numbers and the standard central limit theory with normal limit distributions. The asymptotics for nonstationary jump diffusions, on the other hand, are nonstandard and the limit distributions are given as generalized diffusion processes.

1. INTRODUCTION

Though various jump diffusion models have been commonly used to model asset prices in theoretical and empirical finance and in financial economics, the asymptotic properties of jump diffusions are largely unknown except for some simple cases. Jump diffusions are Markov processes whose asymptotics have already been established under general conditions. Nevertheless, the existing asymptotic theory of general Markov processes is of limited use for the statistical inference of jump diffusion models. The existing asymptotics for general Markov processes are given in terms of their invariant measures, which are difficult to obtain, and virtually impossible to compute even numerically for general nonstationary Markov processes (see, e.g., Höpfner and Löcherbach, 2003).[1] Furthermore, the existing asymptotics are only available for positive recurrent processes and integrable transformations of null recurrent processes. This severely restricts the

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[1]The asymptotics developed in Bandi and Nguyen (2003) may be regarded as a special case of the asymptotics in Höpfner and Löcherbach (2003). However, the asymptotics of Bandi and Nguyen (2003) are less informative than those of Höpfner and Löcherbach (2003), which provide more explicit limits in terms of Mittag-Leffler processes.

applicability of the existing theory, since nonintegrable transformations appear frequently in the statistical analysis of financial time series, such as in the unit root test and the maximum likelihood estimation of parametric jump diffusion models.

This paper develops an asymptotic theory for the class of general recurrent jump diffusions that are reducible to local martingales by strictly increasing and twice continuously differentiable nonlinear transformations. Not all jump diffusions are in the class of jump diffusions that are reducible to local martingales. However, the class is still broad, including nonstationary as well as stationary diffusions with jumps driven by general compound Poisson processes of state dependent sizes and intensities. Pure diffusion models without jumps are all reducible to local martingales by their scale transformations, which are defined as the solutions of simple differential equations. As the paper shows, the scale functions of jump diffusion models can also be defined as the solutions of certain integro-differential equations, reducing general jump diffusions to local martingale jump diffusions. In the paper, we provide some sufficient conditions for the existence of scale functions for general jump diffusion models and demonstrate how they may be obtained numerically.

The most critical and innovative step in the development of our asymptotics is to represent the scale-transformed local martingale jump diffusions approximately as time changed Brownian motions.[2] The representation is novel, and expected to be generally useful for various asymptotic analyses of jump diffusion models. Our asymptotics for jump diffusion models are comparable to those for pure diffusion models developed in Jeong and Park (2013) and Kim and Park (2017).[3] This is rather surprising, given that the jump diffusion model is different from the pure diffusion models in some essential aspects.[4] For stationary jump diffusions, our asymptotics yield the usual law of large numbers and the standard central limit theory with normal limit distributions. On the other hand, the asymptotics for the nonstationary jump diffusions are nonstandard and the limit distributions are given as generalized diffusions. In general, we show that the asymptotics for the jump diffusion models reducible to local martingales are essentially the same as those for the corresponding pure diffusion models without jumps.

Our asymptotics for the jump diffusion models are fully and exclusively determined by their functional parameters: the drift and diffusion functions for the

[2]For pure diffusion models, it is well known that the scale-transformed local martingale diffusions can be represented exactly as time changed Brownian motions without any approximations, on which the asymptotics in Jeong and Park (2013) and Kim and Park (2017) heavily rely.

[3]In particular, our asymptotics rely heavily on Jeong and Park (2013), which can be downloaded from https://fis.yonsei.ac.kr/app/yonsei/member/download.do?attachNo=158.

[4]For the pure diffusion model, the scale function and speed measure are defined explicitly in terms of its infinitesimal parameters, which fully characterize its recurrence property and invariant distribution. In contrast, this is not the case for the jump diffusion model, and the recurrence property and invariant distribution of the jump diffusion model are largely unknown except for some simple and special cases. Furthermore, unlike the pure diffusion model, the jump diffusion model is *not* representable as a time changed Brownian motion.

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diffusive part and the jump size and intensity functions for the jump part. As a result, the limit distributions derived from our asymptotics can be computed directly from the functional parameters of the jump diffusion models. This is in sharp contrast with the limit distributions obtained from the existing asymptotics of general Markov processes, which involve the unknown invariant measures of underlying Markov processes. In particular, our approach allows us to numerically obtain the invariant measures of stationary and nonstationary jump diffusion models, which greatly facilitates the statistical inference in jump diffusion models. Furthermore, all of our asymptotics, including the asymptotics for integrable and nonintegrable functions of nonstationary jump diffusions as well as the standard asymptotics for stationary jump diffusions, are developed within a single framework.

In the development of our asymptotics, we consider the limits of the continuous time additive functionals and the martingale transforms of Brownian motion and compensated Poisson process as the time span T increases. We assume that a jump diffusion process X is observed continuously up to time T. This is simply for expositional convenience. Our asymptotics are also directly applicable when X is only observed discretely at the sampling interval Δ, as long as we set $\Delta \to 0$ sufficiently fast relative to $T \to \infty$. This is seen clearly in Aït-Sahalia and Park (2012, 2016), Jeong and Park (2013), and Kim and Park (2017). The primary motivation of our asymptotics is to effectively analyze the continuous time models with jump diffusions using high frequency observations collected over a long time period. In many practical applications requiring such analyses, Δ is quite small while T is only moderately large, making our asymptotics relevant and useful.

The rest of the paper is organized as follows: Section 2 presents a jump diffusion model and the preliminaries necessary to develop its asymptotics. Several examples of the jump diffusion model are also introduced with the required technical regularity conditions. Section 3 defines and analyzes the scale function and the speed density of the jump diffusion model. Section 4 develops the limit theory for general jump diffusion models, including the standard asymptotics under stationarity, and the invariance principle and the asymptotics for the additive functionals and the martingale transforms under nonstationarity. The limit theory is then applied to find more explicit asymptotics for jump diffusion models commonly used in practical applications. Section 5 demonstrates that we may obtain the scale function and speed density of the jump diffusion model numerically. Illustrative application is provided in Section 6, and concluding remarks follow in Section 7. Mathematical proofs are collected in the Appendix.

A word on notation. In the paper, we follow the notational convention that is widely used and considered to be standard in the literature on Markov processes. The linear functional notation is used for integrals, which means that the integral of a function $f : \mathbb{R} \to \mathbb{R}$ with respect to a measure m on $\mathbb{R}$ is denoted as $m(f)$ in place of $\int f dm$ or $\int f(x)m(dx)$. Moreover, the same notation m is used to signify the measure m itself and the density of m with respect to the Lebesgue measure.

Therefore, we have

$$m(f) = \int f(x)m(dx) = \int (mf)(x)dx$$

as a consequence. This notational convention is maintained throughout the paper, and should cause no confusion.

2. MODEL AND PRELIMINARIES

We consider the jump diffusion model given by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t, \tag{1}$$

where W is a standard Brownian motion and J is a compound Poisson process, whose mean, variance, and intensity are given, respectively, by $\nu(X_{t-})$, $\tau^2(X_{t-})$, and $\lambda(X_{t-})$, conditional on X_{t-}. More specifically, we let

$$dJ_t = \left[\nu(X_{t-}) + \tau(X_{t-})Z_t\right]dN_t\big(\lambda(X_{t-})\big), \tag{2}$$

where Z is a sequence of i.i.d. zero mean and unit variance random variables whose common density function is given by ϕ, and $N\big(\lambda(X)\big)$ is a Poisson process of intensity $\lambda(X)$, which we may also write as $N_t(\lambda(X_{t-})) = (N \circ \Lambda)_t$ with a Poisson process N of unit intensity and $\Lambda_t = \int_0^t \lambda(X_s)ds$.[5] It is assumed that $W, Z,$ and N are mutually independent. Typically, X is defined on the domain $\mathcal{D} = (-\infty, \infty)$ or $(0, \infty)$.

It is possible to consider a more general jump process given by

$$dJ_t = \varpi(X_{t-}, Z_t)dN_t\big(\lambda(X_{t-})\big) \tag{3}$$

for some $\varpi : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$. However, in this paper, we focus on the jump diffusion model given by (2), to effectively analyze the differing roles of the mean and the variance of the jumps in the asymptotics of jump diffusions. Nevertheless, some of our asymptotics are also still applicable for the jump diffusion model with a more general jump process in (3), as argued later. Although we set the jump size and intensity to be flexible and state dependent, we only consider the compound Poisson-type jump processes with finite jump activities. In particular, we do not allow for general Lévy jump processes with infinite activities.

Below are examples of jump diffusion models that have been previously considered in the literature.

Example 2.1. (OU process with jumps) The Ornstein–Uhlenbeck (OU) process with jumps is defined by

$$dX_t = (\alpha_1 + \alpha_2 X_t)dt + \beta dW_t + (\gamma + \delta Z_t)dN_t(\eta)$$

[5] See, for example, Jeanblanc, Yor, and Chesney (2009, p. 476). Here and elsewhere in the paper, we denote the Poisson process of unit intensity simply by N, instead of by $N(1)$.

with $\alpha_2 < 0$ and $\beta, \delta, \eta > 0$, in which X may be written more explicitly as

$$X_t = X_0 e^{\alpha_2 t} + \frac{\alpha_1}{\alpha_2}(e^{\alpha_2 t} - 1) + \beta \int_0^t e^{\alpha_2(t-u)} dW_u + \int_0^t e^{\alpha_2(t-u)}(\gamma + \delta Z_u) dN_u(\eta) \tag{4}$$

using Itô's lemma. It is well known that X becomes stationary, if and only if the condition $\int_{\mathbb{R}} \log(1+|z|)\phi(z)dz < \infty$ holds and X_0 follows the invariant distribution of X given by

$$-\frac{\alpha_1 + \gamma\eta}{\alpha_2} + \beta \int_0^\infty e^{\alpha_2 t} dW_t + \gamma \int_0^\infty e^{\alpha_2 t}\left[dN_t(\eta) - \eta dt\right] + \delta \int_0^\infty e^{\alpha_2 t} Z_t dN_t(\eta)$$

(see, e.g., Applebaum, 2009, Thm. 4.3.17).

Example 2.2. (Lévy process with jumps) The Lévy process with jumps is defined as

$$dX_t = \alpha dt + \beta dW_t + (\gamma + \delta Z_t) dN_t(\eta)$$

with $\beta, \delta, \eta > 0$. The process X becomes recurrent if $\alpha + \gamma\eta = 0$. If we let $Y_t = \exp(X_t)$, it follows from Itô's lemma that

$$dY_t = \left(\alpha + \frac{\beta}{2}\right) Y_t dt + \beta Y_t dW_t + Y_{t-}(e^{(\gamma+\delta Z_t)} - 1) dN_t(\eta).$$

This jump diffusion model was considered in Merton (1976).

Example 2.3. (Affine model with jumps) The affine model with jumps is given by

$$dX_t = (\alpha_1 + \alpha_2 X_t)dt + \sqrt{\beta_1 + \beta_2 |X_t|}dW_t + (\gamma + \delta Z_t) dN_t\big(\eta_1 + \eta_2 |X_{t-}|\big)$$

with $\alpha_2 < 0$, $\beta_1, \beta_2, \delta, \eta_1 > 0$ and $\eta_2 \geq 0$. Duffie, Pan, and Singleton (2000) used this process earlier. Due to Theorems 2.1 and 2.2 of Zhang (2011), X admits a stationary distribution if either (i) $-\alpha_2 > \gamma\eta_2$ and $\mathbb{E}|Z_t|^\varepsilon < \infty$ for some $\varepsilon > 0$, or (ii) $\eta_2 = 0$ and $\mathbb{E}\log(1+|Z_t|) < \infty$. Moreover, Lemma 2.1 of Zhang (2011) shows that X defined with the restriction $\alpha_2 = \eta_2 = 0$ is a unique weak solution that is càdlàg, nonexplosive, and satisfies the Feller property.

Example 2.4. (AQ model with jumps) The affine-quadratic (AQ) model with jumps is given by

$$dX_t = (\alpha_1 + \alpha_2 X_t)dt + (\beta_1 + \beta_2 |X_t|)dW_t + (\gamma + \delta Z_t) dN_t(\eta)$$

with $\alpha_2 < 0$ and $\beta_1, \beta_2, \delta, \eta > 0$. Theorem 1(2) of Wee (1999) shows that the transition of X with normally distributed Z converges weakly as $t \to \infty$ to a proper distribution independent of the initial value, and that X has a unique invariant distribution. Consequently, X becomes stationary if started from its invariant distribution, which in most cases we may expect to be identical to its limit distribution.

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Example 2.5. (GHK model with jumps) The generalized Höpfner–Kutoyant (GHK) jump diffusion model is defined as

$$dX_t = \alpha_1 X_t(\alpha_2 + X_t^2)^{\beta_1 - 1}dt + \beta_2(\beta_3 + X_t^2)^{\beta_1/2}dW_t + (\gamma + \delta Z_t)dN_t(\eta)$$

for $\beta_1, \beta_2, \delta, \eta > 0$, $\beta_3 \geq 0$ and $2\alpha_1 > -\beta_2^2$. Kim and Park (2017) introduce the GHK model without jumps and show that it generates diffusions with many distinctive asymptotic properties depending upon their parameter values. The GHK model extends the model used earlier by Höpfner and Kutoyants (2003). This model accommodates a flexible class of jump diffusions, which are not covered by our previous examples.

For the jump diffusion X defined in (1), we assume throughout the paper that it is Harris recurrent, and that it satisfies the following assumption.[6]

Assumption 2.1. (a) $\sigma^2(x), \tau^2(x), \lambda(x) > 0$ for all $x \in \mathcal{D}$.

(b) $\mu(x)$, $\sigma^2(x)$, $\nu(x)$, $\tau^2(x)$ and $\lambda(x)$ are piecewise infinitely differentiable for all $x \in \mathcal{D}$ and regularly varying at the boundaries of $\mathcal{D}$.

(c) $|\nu|$ and $|\tau|$ are regularly varying with indices κ_ν and κ_τ, respectively, such that $\kappa_\nu < 1$ and $\kappa_\tau < 1$ at the boundaries of $\mathcal{D}$.

Assumption 2.1 is not very stringent. In (a), there are the standard regularity conditions that we expect to hold for all jump diffusion models used in practical applications. The differentiability of functional parameters and their boundary regularity conditions in (b) are also routinely imposed in the study of diffusion and jump diffusion models. In contrast, (c) is more important and is crucial for our asymptotics. It requires that both $|\nu|$ and $|\tau|$ are majorized asymptotically by any linear function. This is necessary to regulate the effect of jumps and to maintain the distributional invariance in our asymptotics of jump diffusion models. If (c) fails, the asymptotics of jump diffusion models may become irregular and model dependent.

For various classes of functions f, g_1 and g_2 defined on $\mathcal{D}$, we will obtain the asymptotics of

$$F_T = \int_0^T f(X_t)dt, \quad G_T^1 = \int_0^T g_1(X_t)dW_t,$$

$$G_T^2 = \int_0^T g_2(X_{t-})\big[\upsilon(Z_t)dN_t\big(\lambda(X_{t-})\big) - \phi(\upsilon)\lambda(X_t)dt\big]$$

as $T \to \infty$, where υ is given arbitrarily as a function and other notations are defined earlier. For expositional convenience, we assume that X is continuously observable here. However, as discussed below, our asymptotics also hold for the corresponding moments of discrete samples from X.

[6] The reader is referred to, for example, Höpfner and Löcherbach (2003) for the precise meaning of Harris recurrence, and to Menaldi and Robin (1999) for the recurrence properties of jump diffusion models.

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To be more precise, let X be observed discretely at intervals of length Δ up to time $T = n\Delta$, and define $F_{T,\Delta} = \Delta \sum_{i=1}^{n} f(X_{i\Delta})$, $G^1_{T,\Delta} = \sum_{i=1}^{n} g_1(X_{(i-1)\Delta})(W_{i\Delta} - W_{(i-1)\Delta})$ and $G^2_{T,\Delta} = \sum_{i=1}^{n} g_2(X_{(i-1)\Delta})[J_{i\Delta}(\upsilon) - J_{(i-1)\Delta}(\upsilon)]$, where $J(\upsilon)$ is a jump process defined as

$$dJ_t(\upsilon) = \upsilon(Z_t) dN_t\big(\lambda(X_{t-})\big) - \phi(\upsilon)\lambda(X_t)dt,$$

that is, a compensated compound Poisson process with intensity $\lambda(X)$ and compounding distribution given by $\upsilon(Z)$. Then $F_{T,\Delta} = F_T\big(1 + o_p(1)\big)$, $G^1_{T,\Delta} = G^1_T\big(1 + o_p(1)\big)$, and $G^2_{T,\Delta} = G^2_T\big(1 + o_p(1)\big)$ for Δ sufficiently small relative to T, and it follows that $F_{T,\delta}$, $G^1_{T,\Delta}$, and $G^2_{T,\Delta}$ have the same limits as F_T, G^1_T, and G^2_T, respectively, if we let $\Delta \to 0$ sufficiently fast relative to $T \to \infty$. This can be shown rigorously as in Aït-Sahalia and Park (2012, 2016), Jeong and Park (2013), and Kim and Park (2017).

3. SCALE FUNCTION AND SPEED DENSITY

To derive our asymptotics, we need to introduce the scale function and the speed density for the jump diffusion model. The scale function for the jump diffusion model is motivated and defined similarly as in the pure diffusion model. However, the notion of the speed density for the jump diffusion model is novel, and plays an important role in our asymptotics.

3.1. Scale Function

For the jump diffusion X introduced in (1), we define its scale function s as a solution to the integro-differential equation

$$\left(\mu s^{\bullet} + \frac{1}{2}\sigma^2 s^{\bullet\bullet}\right)(x) = -\lambda(x)\int_{\mathbb{R}} \Big(s\big[x + \nu(x) + \tau(x)z\big] - s(x)\Big)\phi(z)dz \tag{5}$$

whenever it exists, wherein we denote by $s^{\bullet}$ and $s^{\bullet\bullet}$ the first and second derivatives of s, respectively. If we transform X with this scale function, then the scale transformed process $s(X)$ becomes a local martingale. To show this, we apply Itô's lemma to deduce

$$\begin{aligned} ds(X_t) &= \left(\mu s^{\bullet} + \frac{1}{2}\sigma^2 s^{\bullet\bullet}\right)(X_t)dt + (\sigma s^{\bullet})(X_t)dW_t \\ &\quad + \Big(s\big[X_{t-} + \nu(X_{t-}) + \tau(X_{t-})Z_t\big] - s(X_{t-})\Big) dN_t\big(\lambda(X_{t-})\big), \end{aligned} \tag{6}$$

and note it follows from (5) that

$$\mathbb{E}_{t-}\Big(s\big[X_{t-} + \nu(X_{t-}) + \tau(X_{t-})Z_t\big] - s(X_{t-})\Big) dN_t\big(\lambda(X_{t-})\big) = -\left(\mu s^{\bullet} + \frac{1}{2}\sigma^2 s^{\bullet\bullet}\right)(X_t)dt,$$

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where $\mathbb{E}_{t-}$ denotes the conditional expectation at time $t-$.[7] Therefore, $s(X)$ becomes a local martingale. If $\mu = -\nu\lambda$, then $s(x) = x$ and X is said to be in natural scale. In what follows, the first derivative $s^{\bullet}$ of the scale function s of X is referred to as the scale density of X.

Throughout the paper, the following conditions are assumed.

Assumption 3.1. The scale function s exists, and it is strictly increasing from $-\infty$ to ∞ and it has the derivative $s^{\bullet}$ which is asymptotically monotone and regularly varying with an index κ such that $\kappa > -1$ at the boundaries of $\mathcal{D}$.

For any continuous diffusion X without jumps, the scale function s always exists and $s^{\bullet}$ satisfies all conditions in Assumption 3.1 as long as $\kappa > -1$ under the conditions in Assumption 2.1. Furthermore, s diverges to $\pm\infty$ at boundaries of $\mathcal{D}$, if and only if X is recurrent.

Unfortunately, for jump diffusion X, the scale function s may not exist, and the divergence of s at the boundaries of $\mathcal{D}$ is only necessary and generally not sufficient to ensure the recurrence property. Nevertheless, Assumption 3.1 appears to hold for a large class of jump diffusion models. To show this, we further characterize the solution of the integro-differential equation (5) below.

LEMMA 3.1. *The function* $f = s^{\bullet\bullet}/s^{\bullet}$ *is a fixed point of the integral operator A defined as*

$$(Af)(x) = -\frac{2\mu}{\sigma^2}(x) - \frac{2\tau\lambda}{\sigma^2}(x)\int_{\mathbb{R}} \exp\left(\int_x^{x+\nu(x)+\tau(x)z} f(y)dy\right)\varphi(z)dz,$$

that is, $f = Af$.

Therefore, we can expect that the scale function s exists, whenever the operator A defined in Lemma 3.1 is a contraction. In the next lemma, we provide a sufficient condition, which ensures that A is a contraction and that the scale function exists. In what follows, we let $\Phi(x) = \int_{-\infty}^{x}\phi(z)dz$ and $\varphi(x) = 1\{x \geq 0\} - \Phi(x)$.

PROPOSITION 3.2. *Let* $|\nu(x)| \leq \bar{\nu}$, $\tau(x) \leq \bar{\tau}$, $(\lambda/\sigma^2)(x) \leq \bar{\lambda}_\sigma$, *and* $|(\mu/\sigma^2)(x)| \leq \bar{\mu}_\sigma$ *for all* $x \in \mathcal{D}$ *with some constants* $\bar{\mu}_\sigma, \bar{\lambda}_\sigma, \bar{\nu}, \bar{\tau} > 0$. *Moreover, define*

$$P_{c,\bar{\nu},\bar{\tau}} = \int_{\mathbb{R}} \Pi_{c,\bar{\nu},\bar{\tau}}(z)dz,$$
$$Q_{c,\bar{\nu},\bar{\tau}} = \int_{\mathbb{R}} \left| P_{c,\bar{\nu},\bar{\tau}} 1\{y \geq 0\} - \int_{-\infty}^{y} \Pi_{c,\bar{\nu},\bar{\tau}}(z)dz \right| dy,$$

where $\Pi_{c,\bar{\nu},\bar{\tau}}(z) = \exp[c(\bar{\nu} + \bar{\tau}|z|)]|\varphi(z)|$ *for some* $c > 0$, *and assume that there exists* $0 < c < \infty$ *such that* $2\bar{\mu}_\sigma + 2\bar{\tau}\bar{\lambda}_\sigma P_{c,\bar{\nu},\bar{\tau}} \leq c$ *and* $2\bar{\tau}^2\bar{\lambda}_\sigma Q_{c,\bar{\nu},\bar{\tau}} < 1$. *Then there exists a strictly increasing solution s to the integro-differential equation (5).*

[7] See, for example, Jeanblanc et al. (2009, Sect. 8.8.4 and Prop. 8.8.6.1) for the computation of this type of expectation.

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Remark 3.1. When ϕ is the standard normal density function and $\nu = 0$, the conditions in Proposition 3.2 hold if[8]

$$K(\bar{\tau}, \bar{\mu}_\sigma) \leq \frac{1 + 2\bar{\tau}(\bar{\mu}_\sigma + \bar{\tau}\bar{\lambda}_\sigma)}{\bar{\tau}^2\bar{\lambda}_\sigma},$$
$$K(\bar{\tau}, \bar{\mu}_\sigma) < \frac{1 + 4\bar{\tau}\bar{\mu}_\sigma - \bar{\tau}^2\left(\bar{\lambda}_\sigma - 4\bar{\mu}_\sigma^2 + \bar{\lambda}_\sigma\sqrt{2/\pi}\right) - 2\bar{\tau}^3\bar{\lambda}_\sigma\bar{\mu}_\sigma\sqrt{2/\pi}}{4\bar{\tau}^3\bar{\lambda}_\sigma\bar{\mu}_\sigma(1 + \bar{\tau}\bar{\mu}_\sigma)},$$

where

$$K(\bar{\tau}, \bar{\mu}_\sigma) = 2\exp\left(\frac{(2\bar{\tau}\bar{\mu}_\sigma + 1)^2}{2}\right)\Phi\left(2\bar{\tau}\bar{\mu}_\sigma + 1\right)$$

and Φ is the standard normal distribution function. Note that, for any given value of $\bar{\mu}_\sigma$, there will always be a $\bar{\tau}$ and $\bar{\lambda}_\sigma$ small enough to satisfy the conditions introduced here. Therefore, for the jump diffusions with normal-compound Poisson jumps, the scale functions exist whenever $\bar{\tau}$ and $\bar{\lambda}_\sigma$ are sufficiently small.

Remark 3.2. The boundedness conditions $|\nu(x)| \leq \bar{\nu}$, $\tau(x) \leq \bar{\tau}$, $(\lambda/\sigma^2)(x) \leq \bar{\lambda}_\sigma$ and $|(\mu/\sigma^2)(x)| \leq \bar{\mu}_\sigma$ required in Proposition 3.2 are satisfied by a large class of jump diffusion models. The conditions are satisfied by the Lévy jump diffusion model in Example 2.2, the affine jump diffusion model in Example 2.3, the AQ jump diffusion model in Example 2.4, and the GHK jump diffusion model in Example 2.5, that is, all jump diffusion models provided as examples in the previous section, except for the OU jump diffusion model.

Proposition 3.2 only provides a set of sufficient conditions for the existence of the scale function. Needless to say, a jump diffusion that does not satisfy the conditions there may also have the scale function. In some special cases, the scale function may be obtained explicitly, as shown below.

Example 3.1. Let X be given by

$$dX_t = \alpha X_t dt + \beta X_t dW_t + X_{t-}(e^{(\gamma+\delta Z_t)} - 1)dN_t(\eta)$$

with parameter restriction $\gamma\eta = -\alpha + \beta/2$ as well as $\beta > 0$. Clearly, X may be redefined as

$$dX_t = \alpha X_t dt + \beta X_t dW_t + \left(\gamma^* X_{t-} + \delta^* X_{t-} Z_t^*\right)dN_t \tag{7}$$

with $\gamma^* = \mathbb{E}e^{(\gamma+\delta Z_t)} - 1$, $\delta^{*2} = \mathbb{E}\left(e^{(\gamma+\delta Z_t)} - 1 - \gamma^*\right)^2$, and $Z^* = \left(e^{(\gamma+\delta Z)} - 1 - \gamma^*\right)/\delta^*$, conformably as our representation of jump diffusion models. For the jump diffusion model (7), the scale function is given by $s(x) = \log(x)$. In fact, it follows immediately from Itô's lemma that we have $d\log(X_t) = (\alpha - \beta/2)dt + \beta dW_t + (\gamma + \delta Z_t)dN_t(\eta)$, which becomes a local martingale jump diffusion under parameter restriction $\gamma\eta = -\alpha + \beta/2$. Moreover, it is easy to see that $s(x) = \log(x)$

[8] Though quite lengthy, the derivation of this sufficient condition is rather straightforward. It will be provided upon request.

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satisfies the integro-differential equation (5) with any distribution for Z^* defined from arbitrary Z as long as $\mathbb{E}Z_t = 0$ and $\mathbb{E}Z_t^2 = 1$.

In some cases, our asymptotics rely only on the limit of the scale function $s(x)$ as x approaches the boundaries of $\mathcal{D}$. Therefore, it suffices to find a limit version of the scale function, which becomes asymptotically equivalent to the scale function as x approaches the boundaries of $\mathcal{D}$. An asymptotically equivalent version of the scale function may be obtained by applying the so-called method of dominant balance to the integro-differential equation (5).[9] To employ the method of dominant balance, we need to introduce a moment condition on Z.

Assumption 3.2. Let $\phi(\iota^k) < \infty$ for some k such that

$$k > \max\left(2(\kappa+2), \frac{|\kappa_+ - \kappa_-| - 1 + (\kappa_\nu \vee \kappa_\tau)}{1-\kappa_\tau}\right),$$

where κ_+ and κ_- are the regularly varying indices of $s^\bullet$ at the right and the left boundaries of $\mathcal{D}$ with $\kappa = \kappa_+ \vee \kappa_-$, and κ_ν and κ_τ are the regularly varying indices of ν and τ at the boundaries of $\mathcal{D}$.

If Z has a bounded support, the condition in 3.2 is trivially satisfied.

LEMMA 3.3. *The asymptote of the solution $s^\bullet$ to the integro-differential equation (5) obtained by the method of dominant balance is given as*

$$s^\bullet(x) \sim \exp\left(-\int_w^x \frac{2(\mu+\nu\lambda)}{\sigma^2+(\nu^2+\tau^2)\lambda}(u)du\right) \tag{8}$$

for any $w \in \mathcal{D}$ as x approaches the boundaries of $\mathcal{D}$.

In our subsequent discussions, the function defined in (8) and its anti-derivative are referred to as the asymptotic scale density and the asymptotic scale function, respectively, to distinguish them from the exact scale density and the true scale function given by the integro-differential equation (5). Though always obtainable, the asymptotic scale density and the asymptotic scale function are meaningfully defined only when the exact scale density and the exact scale function exist.[10]

The asymptotic scale density of the jump diffusion model in (8) is indeed well expected from the scale density of the diffusion model, which is given by $\exp\left(-\int_w^x \left(2\mu/\sigma^2\right)(u)du\right)$. For the diffusion model, $\mu(x)$ and $\sigma^2(x)$ in the scale density represent the conditional mean and variance of the infinitesimal increment in X at $X = x$ for $x \in \mathcal{D}$, respectively. Analogously, $\left(\mu+\nu\lambda\right)(x)$ and $\left(\sigma^2 + (\nu^2+\tau^2)\lambda\right)(x)$ represent the conditional mean and variance of the infinitesimal

[9]The method of dominant balance is a well-known and widely used method to determine the asymptotic behavior of solutions to a differential equation or an integro-differential equation. Though we may normally expect the method of dominant balance to find an asymptotically equivalent version of the solution to a differential equation or an integro-differential equation, it may fail in some pathological cases (see Lin and Segel, 1974, pp. 188–189).

[10]Clearly, the method of dominant balance only works for the integro-differential equation (5) when it has a solution.

increment in X at $X = x$ for $x \in \mathcal{D}$, respectively. As expected, if we set $\lambda = 0$, the asymptotic scale density of the jump diffusion model reduces to the scale density of the diffusion model.

Example 3.2. We may easily obtain the asymptotic scale functions for the jump diffusions introduced earlier in Section 2. For all of them, we have $\mathcal{D} = \mathbb{R}$. The asymptotic scale densities of the OU process with jumps in Example 2.1 under $\alpha_2 < 0$ and the affine model with jumps in Example 2.3 under $\alpha_2 < 0$ and $\alpha_2 < \gamma\eta_2 < -\alpha_2$ both exponentially increase as $x \to \pm\infty$. On the other hand, for the Lévy process with jumps in Example 2.2, the asymptotic scale density is 1 under $\alpha + \gamma\eta = 0$. Moreover, the asymptotic scale densities of the AQ and GHK models with jumps in Examples 2.4 and 2.5 are given, respectively, by

$$c(x)|x|^{-2\alpha_2/\beta_2^2} \quad \text{and} \quad c(x)|x|^{-2\alpha_1/\beta_2^2}$$

as $x \to \pm\infty$, where $c(x) = a1\{x \geq 0\} + b1\{x < 0\}$ for some constants $a, b \geq 0$ depending upon their parameter values.

3.2. Speed Density

For a local martingale jump diffusion given by

$$dX_t = -(\nu\lambda)(X_t)dt + \sigma(X_t)dW_t + \left[\nu(X_{t-}) + \tau(X_{t-})Z_t\right]dN_t\big(\lambda(X_{t-})\big), \tag{9}$$

we define its speed density as

$$m(x) = \left(\frac{1}{\sigma^2 + \left(\nu^2 + \tau^2\right)\lambda}\right)(x) \tag{10}$$

for $x \in \mathcal{D}$. Following the usual convention, m is also used to denote the measure defined by the density m with respect to the Lebesgue measure.

It follows from our definition that

$$d\langle X\rangle_t = \frac{1}{m(X_t)}dt,$$

where $\langle X\rangle$ is the conditional quadratic variation of X in (9). This is analogous to the definition of the speed density m for the local martingale diffusion $dX_t = \sigma(X_t)dW_t$, which is given by $m(x) = 1/\sigma^2(x)$ and yields $d[X]_t = d\langle X\rangle_t = \left[1/m(X_t)\right]dt$. For a local martingale jump diffusion X, $\langle X\rangle$ becomes a compensator for $[X]$ and we have, in particular, $\mathbb{E}_{t-}d[X]_t = d\langle X\rangle_t = \left[1/m(X_t)\right]dt$ with $\mathbb{E}_{t-}$ denoting the conditional expectation at time $t-$ (see, e.g., Protter, 2005 for more discussions). Note that even instantaneous futures are unknown for jump diffusions, whereas continuous diffusions are perfectly predictable over any infinitesimal time interval. As is well known, any continuous local martingale diffusion X can be represented as a time changed Brownian motion, and its speed density indicates how fast to read X at each spatial point $x \in \mathbb{R}$ to make it a Brownian motion. Likewise, we show in the paper that the speed density of a local martingale jump diffusion X represents

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the *expected* speed at which X should be read at each spatial point $x \in \mathbb{R}$ to make it an *approximate* Brownian motion asymptotically.

Our definition of the speed density for local martingale jump diffusions introduced above may be extended to general jump diffusions reducible to local martingales by their scale transformations. If X is the general jump diffusion in (1) and it has a well-defined scale function s, the speed density m_s for its scale transformation $X^s = s(X)$ in (6) is given by

$$m_s(x) = \left(\frac{1}{\sigma_s^2 + \omega_s^2 \lambda_s}\right)(x), \tag{11}$$

where $\sigma_s = (\sigma s^\bullet) \circ s^{-1}$, $\lambda_s = \lambda \circ s^{-1}$ and $\omega_s^2 = \omega^2 \circ s^{-1}$ with

$$\omega^2(x) = \int_{\mathbb{R}} \Big(s\big[x + \nu(x) + \tau(x)z\big] - s(x)\Big)^2 \phi(z)dz \tag{12}$$

for $x \in \mathbb{R}$. If X is a local martingale jump diffusion in (9), then s becomes identity and ω^2 in (12) reduces to $\omega^2 = \nu^2 + \tau^2$. In this case, the speed density m_s in (11) reduces to the speed density m in (10). The speed density m_s of X^s introduced in (11) leads us to define the speed density m of X as

$$m(x) = \left(\frac{1}{\sigma^2 s^\bullet + (\omega^2 \lambda)/s^\bullet}\right)(x) \tag{13}$$

for $x \in \mathcal{D}$.[11] Note that, for the speed densities m and m_s, respectively, of X and $X^s = s(X)$ defined in (11) and (13), we have $m(f) = m_s(f_s)$ for all m-integrable f with $f_s = f \circ s^{-1}$.

As we may expect, the speed density defined in (13) represents the invariant measure of any jump diffusion X satisfying our assumptions, particularly a jump diffusion reducible to a local martingale by its scale transformation. The invariant measure of a recurrent Lévy process is the Lebesgue measure, as shown in, for example, Applebaum (2009, Exer. 6.7.7), and this corresponds to the speed density obtained from (13). More generally, it can be easily seen by comparing the existing asymptotics for general Markov processes with our asymptotics specifically developed for jump diffusions. The former is given in terms of the invariant measures of underlying Markov processes, whereas the latter is fully characterized by the speed densities of jump diffusions (see, e.g., Höpfner and Löcherbach, 2003, Thm. 3.1 and the theorems in the next section). As a result, the speed density m can be used to find an essential recurrence property for any Harris recurrent jump diffusion X reducible to a local martingale by its scale function: X becomes positive recurrent if m is integrable, and null recurrent if m is nonintegrable. This is precisely the same as in the continuous diffusion model without jumps.

[11] For the jump diffusion models specified in (3), the speed densities m_s and m are similarly defined with $\omega^2(x)$ replaced by $\int_{\mathbb{R}} \big(s\big[x + \varpi(x,z)\big] - s(x)\big)^2 \phi(z)dz$.

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Once the scale density $s^{\bullet}$ is given, the speed density m may be readily found from (13). The speed densities may be obtained from either the exact scale function or the asymptotic scale function, and the resulting speed densities are referred to as the exact speed density or the asymptotic speed density, respectively. Of course, the latter is much easier to find than the former. Typically, the integrability of m is determined entirely by its behavior at the boundaries of $\mathcal{D}$. Therefore, we only need its asymptotic speed density to find whether a given jump diffusion is positive recurrent or null recurrent.

Example 3.3. For the jump diffusions introduced in Section 2, all with $\mathcal{D} = \mathbb{R}$, their asymptotic speed densities may be easily derived from the asymptotic scale densities obtained in Example 3.2. The asymptotic speed densities of the OU process with jumps in Example 2.1 under $\alpha_2 < 0$ and the affine model with jumps in Example 2.3 under $\alpha_2 < 0$ and $\alpha_2 < \gamma\eta_2 < -\alpha_2$ both decrease exponentially as $x \to \pm\infty$, which implies that they are integrable. For the Lévy process with jumps in Example 2.2, the asymptotic speed density is constant under $\alpha + \gamma\eta = 0$, and therefore, it is not integrable. On the other hand, the asymptotic speed densities of the AQ and the GHK models with jumps in Examples 2.4 and 2.5 are given, respectively, by

$$c(x)|x|^{\alpha_2/\beta_2^2-2} \quad \text{and} \quad c(x)|x|^{\alpha_1/\beta_2^2-2\beta_1}$$

as $x \to \pm\infty$, where $c(x) = a1\{x \geq 0\} + b1\{x < 0\}$ for some constants $a, b \geq 0$ as defined in Example 3.2. Note that the asymptotic speed density of the AQ model with jumps is always integrable under $\alpha_2 < 0$.

For our asymptotics developed later in the paper, the following conditions are employed.

Assumption 3.3. The speed density m_s of $X^s = s(X)$ in (11) is either integrable or regularly varying such that $m_s(\lambda x)/|\lambda|^r \to \bar{m}_s(x)$ as $\lambda \to \infty$, where

$$\bar{m}_s(x) = a|x|^r 1\{x \geq 0\} + b|x|^r 1\{x < 0\} \qquad \textbf{(14)}$$

for some $r > -1$ and $a, b \geq 0$ with $a + b > 0$.

If m_s is regularly varying with index $r > -1$, we may let $m_s(\lambda)$ denote either $m_s(\lambda)$ or $m_s(-\lambda)$ depending upon whether ∞ or $-\infty$ is the dominating boundary and have $m_s(\lambda x) \sim m_s(\lambda)\bar{m}_s(x)$ as $\lambda \to \pm\infty$ with $\bar{m}_s$ given by (14), in which case either a or b becomes unity. The reader is referred to, for example, Kim and Park (2017) for more details. In Assumption 3.3, we effectively assume that $m_s(\lambda) \sim |\lambda|^r$, which excludes the possibility of $m_s(\lambda)$ having a slowly varying component in the limit as $\lambda \to \infty$. This can be restrictive. However, this assumption is not essential and is only made to more clearly and explicitly present our subsequent

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asymptotics.[12] For expositional brevity, we will simply say that m_s is regularly varying with index $r > -1$ if it satisfies Assumption 3.3.

In case m_s is not integrable and regularly varying, our asymptotics involve its index $r > -1$.[13] To find it, we only need to know any asymptotic version of the speed density m_s of $X^s = s(X)$ on $\mathbb{R}$ defined in (11), for which we have the following.

PROPOSITION 3.4. *We have* $\omega_s^2(x) \sim [(\nu^2 + \tau^2)s^{\bullet 2}] \circ s^{-1}(x)$ *as* $x \to \pm\infty$*, where* s *and* $s^{\bullet}$ *are any asymptotic versions of the scale function and the density, respectively.*

Therefore, the index r of regular variation for m_s is easily found, once we obtain any asymptotic versions of σ_s^2 and λ_s.

Example 3.4. Under the required conditions, the OU process with jumps and the affine model with jumps in Examples 2.1 and 2.3 become stationary and they have integrable speed densities. The Lévy process with jumps in Example 2.2 is already in natural scale under the recurrence condition and it has a constant speed density. Let $c(x)$ be defined as in Examples 3.2 and 3.3. For the AQ model with jumps in Example 2.4, under the given condition, we have $m_s(x) \sim c(x)|x|^{-2}$ as $x \to \pm\infty$, which follows directly from (11) using the results in Example 3.2 and Proposition 3.4. Finally, the GHK model with jumps in Example 2.5 yields $m_s(x) \sim c(x)|x|^{(4\alpha_1/\beta_2^2 - 2\beta_1)/(1-2\alpha_1/\beta_2^2)}$ as $x \to \pm\infty$. Therefore, it is regularly varying with index $r > -1$ if $(4\alpha_1/\beta_2^2 - 2\beta_1)/(1-2\alpha_1/\beta_2^2) > -1$. On the other hand, it is integrable if $(4\alpha_1/\beta_2^2 - 2\beta_1)/(1-2\alpha_1/\beta_2^2) < -1$.

4. ASYMPTOTIC THEORY

In this section, we establish the asymptotics of jump diffusion models reducible to local martingales by their scale transformations under positive and null recurrences.

4.1. Asymptotics Under Positive Recurrence

For the positive recurrent jump diffusions reducible to local martingales by their scale transformations, we have the following limit theorems.

THEOREM 4.1. *Let* X *be positive recurrent and define* $\pi(x) = m(x)/m(\mathcal{D})$*. If* f*,* $g_1 g_1'$ *and* $\lambda g_2 g_2'$ *are* π*-integrable, then we have*

[12] For instance, the normalizing sequences in our subsequent asymptotics are defined as explicit functions of the sample span T depending on r, which would be impossible without our assumption here.

[13] Recall that m_s is integrable if and only if m is integrable.

$$\frac{1}{T}\int_0^T f(X_t)dt \to_{a.s.} \pi(f), \quad \frac{1}{\sqrt{T}}\int_0^T g_1(X_t)dW_t \to_d \mathbb{N}_1\big(0,\pi(g_1g_1')\big),$$
$$\frac{1}{\sqrt{T}}\int_0^T g_2(X_{t-})\big[\upsilon(Z_t)dN_t\big(\lambda(X_{t-})\big)-\phi(\upsilon)\lambda(X_t)dt\big] \to_d \mathbb{N}_2\big(0,\phi(\upsilon^2)\pi(\lambda g_2g_2')\big)$$

jointly as $T\to\infty$, where $\mathbb{N}_1$ and $\mathbb{N}_2$ denote two independent multivariate normal distributions and $\upsilon:\mathbb{R}\to\mathbb{R}$ is any function such that $\phi(\upsilon^4)<\infty$.

Theorem 4.1 establishes the asymptotics for the additive functionals and the martingale transforms of positive recurrent jump diffusions, which yield the usual law of large numbers and the standard central limit theory with normal limit distributions. This is well expected. In fact, our asymptotics here are known to hold for all positive recurrent Markov processes satisfying the Feller property, as long as the required integrability conditions are satisfied. The reader is referred to, for example, Maruyama and Tanaka (1959) and Küchler and Sørensen (1999) for more details.

Though they are already well known, our asymptotics here are meaningful for at least two reasons. First, we show that they can be derived in parallel with the asymptotics for nonstationary jump diffusions within a single unified framework. Second and more importantly, our asymptotics show that the speed density m defined in the previous section indeed defines the invariant measure. The invariant measures of jump diffusion models are generally not available in any closed-forms, except in some special cases. For instance, the invariant distribution of the OU jump diffusion in Example 2.1 is already well known to be a self-decomposable distribution (see, e.g., Applebaum, 2009, Thm. 4.3.17). Moreover, Theorem 2.4 of Zhang (2011) characterizes the invariant distribution of the affine jump diffusion in Example 2.3 in terms of its Fourier transformation. Of course, it is possible to simulate the invariant distributions of any jump diffusions under stationarity. See Panloup (2008) for the conditions needed to obtain the invariant distribution of a stationary jump diffusion by simulating its sample paths.

4.2. Asymptotics Under Null Recurrence

For the null recurrent jump diffusions reducible to local martingales by their scale transformations, we establish an invariance principle and use it to derive their additive functional and martingale transform asymptotics.

4.2.1. *Invariance Principle.* In our subsequent discussions, we let the jump diffusion X be in natural scale, by redefining if necessary $s(X)$ as X, and say that it is regular with index $r>-1$ if its speed density m is regularly varying with index $r>-1$ and satisfies Assumption 3.3. Moreover, we denote by $\mathcal{C}[0,1]$ the space of continuous functions defined on $[0,1]$.

PROPOSITION 4.2. *Let X be regular with index $r > -1$, and define X^T on $[0,1]$ for each T by $X_t^T = T^{-1/(r+2)} X_{Tt}$. Then we have*

$$X^T \to_d X^\circ \tag{15}$$

in $\mathcal{C}[0,1]$ as $T \to \infty$, where X° is defined as $X^\circ = B \circ \bar{A}$ from the standard Brownian motion B with local time L and

$$\bar{A}_t = \inf \left\{ u \middle| \int_{\mathbb{R}} \bar{m}(x) L(u,x) dx > t \right\}$$

for $0 \le t \le 1$.

The invariance principle established in Proposition 4.2 for the null recurrent jump diffusions is essentially the same as that of the null recurrent continuous diffusions without jumps previously obtained in Jeong and Park (2013). The limit process X° in (15) is a generalized diffusion process called the skew Bessel process.
To derive our limit theorems, the following assumption is introduced.

Assumption 4.1. We assume that (a) there exists $p > 1$ such that $\mathbb{E}|X_1^T|^p$ is bounded uniformly in T, and (b) λ is regularly varying with an index greater than or equal to $-\min(r+2, p)$ at $\pm\infty$.

Assumption 4.1 does not appear to be stringent. In fact, the limit process X° obtained in Proposition 4.2 has finite moments to the infinite order, and Jeong and Park (2013) show that all the moments of X^T are uniformly bounded for null recurrent continuous diffusions without jumps.
Under these additional assumptions, we obtain the following proposition, which further decomposes and characterizes the limit process X°. This decomposition of X° plays an important role in characterizing the limit distributions in Theorem 4.4(b).

PROPOSITION 4.3. *Let X be regular with index $r > -1$, and let Assumption 4.1 hold. Then, for W^T, N^T, and Z^T defined by*

$$W_t^T = \frac{1}{\sqrt{T}} W_{Tt},$$
$$N_t^T = \frac{1}{\sqrt{T}} \int_0^{Tt} \left[\frac{1}{\lambda^{1/2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \lambda^{1/2}(X_u) du \right],$$
$$Z_t^T = \frac{1}{\sqrt{T}} \int_0^{Tt} \frac{Z_u}{\lambda^{1/2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big),$$

we have

$$W^T \to_d W^\circ, \qquad N^T \to_d N^\circ, \qquad Z^T \to_d Z^\circ$$

in $\mathcal{C}[0,1]$ jointly with $X^T \to_d X^\circ$ in (15) as $T \to \infty$, where W°, N° and Z° are mutually independent standard Brownian motions. Furthermore, X° is given in terms of W°, N° and Z° as

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$$\bar{m}^{1/2}(X_t^\circ)dX_t^\circ = \left(\sqrt{p_c}1\{X_t^\circ \geq 0\} + \sqrt{q_c}1\{X_t^\circ < 0\}\right)dW_t^\circ + \left(\sqrt{1-p_c}1\{X_t^\circ \geq 0\} + \sqrt{1-q_c}1\{X_t^\circ < 0\}\right)dJ_t^\circ \quad \textbf{(16)}$$

with

$$dJ_t^\circ = \left(\sqrt{p_z}1\{X_t^\circ \geq 0\} + \sqrt{q_z}1\{X_t^\circ < 0\}\right)dN_t^\circ + \left(\sqrt{1-p_z}1\{X_t^\circ \geq 0\} + \sqrt{1-q_z}1\{X_t^\circ < 0\}\right)dZ_t^\circ,$$

where p_c, q_c, and p_z, q_z are constants defined by

$$\frac{\sigma^2}{\sigma^2 + (\nu^2 + \tau^2)\lambda}(x) \to p_c, q_c, \qquad \frac{\nu^2}{\nu^2 + \tau^2}(x) \to p_z, q_z$$

as x approaches the right and the left boundaries of $\mathcal{D}$, respectively.

Proposition 4.3 shows that the limit process X° of X is defined as a generalized diffusion driven by three mutually independent Brownian motions W°, N°, and Z°. Note that W° is the limit of the Brownian motion driving the diffusive part of X, while N° and Z° are the Brownian motions representing the limit behaviors of the mean and volatility components, respectively, of the jump part in X. The limit fractions p_c and q_c denote the proportion of X° generated by the diffusive limit Brownian motion W°, and the limit fractions p_z and q_z designate the proportions of the limit Brownian motion of the jump part J° contributed by its mean component N°.

In general, X° is dependent on all of the three Brownian motions W°, N°, and Z°. However, there are special cases in which X° becomes independent of some subsets of the Brownian motions. If, for instance, $p_c, q_c = 1$, then X° is driven only by W°, and it becomes independent of N° and Z°. This happens if $\sigma^2 \gg (\nu^2 + \tau^2)\lambda$ at the boundaries of $\mathcal{D}$ and the diffusive part of X asymptotically dominates its jump part. Likewise, if $\sigma^2 \ll (\nu^2 + \tau^2)\lambda$ at the boundaries of $\mathcal{D}$ and the jump part of X dominates its diffusive part asymptotically, then $p_c, q_c = 0$ and X° is driven entirely by N° and Z° and becomes independent of W°. On the other hand, depending upon whether the mean or the volatility component of jumps dominates, that is, $\nu^2 \gg \tau^2$ or $\nu^2 \ll \tau^2$ at the boundaries of $\mathcal{D}$, then $p_z, q_z = 1$ or $p_z, q_z = 0$, and therefore, X° becomes independent of Z° or N°.

Example 4.1. As a simple illustration, we consider the jump diffusion in natural scale given by

$$dX_t = -\nu(X_t)dt + \sigma(X_t)dW_t + [\nu(X_{t-}) + \tau(X_{t-})Z_t]dN_t.$$

If $\sigma^2 \sim \nu^2 \sim \tau^2 \sim 1$ at boundaries $\pm\infty$ of $\mathcal{D} = \mathbb{R}$, then

$$X_t^T = \frac{1}{\sqrt{T}}\int_0^{Tt} \sigma(X_t)dW_t + \frac{1}{\sqrt{T}}\left[\int_0^{Tt} \nu(X_{u-})(dN_u - du) + \int_0^{Tt} \tau(X_{u-})Z_u dN_u\right] \to_d X_t^\circ = W_t^\circ + N_t^\circ + Z_t^\circ$$

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as $T \to \infty$. However, if $\sigma^2 \sim 1 \gg \nu^2, \tau^2$ at $\pm\infty$, then $X^\circ = W^\circ$ and X° is independent of N° and Z°. Likewise, if $\sigma^2 \sim \nu^2 \sim 1 \gg \tau^2$ at $\pm\infty$, then $X^\circ = W^\circ + N^\circ$ and X° is independent of Z°. Finally, if $\sigma^2 \sim \tau^2 \sim 1 \gg \nu^2$ at $\pm\infty$, then $X^\circ = W^\circ + Z^\circ$ and X° is independent of N°.

Remark 4.1. For the models with more general jumps in (3), Proposition 4.3 is expected to hold with the same limits, where the asymptotes p_c, q_c, and p_z, q_z are defined similarly with $\nu(x)$ and $\tau^2(x)$ replaced by

$$\int_{\mathbb{R}} \varpi(x,z)\phi(z)dz, \qquad \int_{\mathbb{R}} \varpi^2(x,z)\phi(z)dz - \left(\int_{\mathbb{R}} \varpi(x,z)\phi(z)dz\right)^2,$$

respectively.

4.2.2. *Asymptotics for Additive Functional and Martingale Transforms.* Clearly, we may write $f(X) = f_s(X^s)$ with $X^s = s(X)$ and $f_s = f \circ s^{-1}$ for any function on $\mathcal{D}$. Therefore, without loss of generality, we assume in what follows that X is already in natural scale.

DEFINITION 4.1. *We say that f is m-asymptotically homogeneous if f is not m-integrable, and*

$$f(\lambda x) = \kappa(f,\lambda)h(f,x) + \delta(f,\lambda,x)$$

with

$$|\delta(f,\lambda,x)| \le a(f,\lambda)p(f,x) + b(f,\lambda)q(f,\lambda x)$$

as $\lambda \to \infty$, where (i) $h(f,\cdot)$, $p(f,\cdot)$, and $q(f,\cdot)$ are locally bounded on $\mathbb{R}\backslash\{0\}$, locally integrable in measures m and $\bar{m}$, (ii) $\kappa(f,\lambda)$ is nonsingular for all large λ, (iii) $q(f,\cdot)$ is vanishing at infinity, and (iv)

$$\limsup_{k\to\infty} \left\|\kappa(f,\lambda)^{-1}a(f,\lambda)\right\| = 0, \quad \limsup_{k\to\infty} \left\|\kappa(f,\lambda)^{-1}b(f,\lambda)\right\| < \infty.$$

We call $\kappa(f,\cdot)$ and $h(f,\cdot)$ the asymptotic order and the limit homogeneous function of f, respectively. If (i)′ $h(f,\cdot)$, $p(f,\cdot)$, and $q(f,\cdot)$ are locally square integrable in measures m and $\bar{m}$ in place of (i), then f is said to be m-square asymptotically homogeneous.

Our main asymptotics are given in the following theorem.

THEOREM 4.4. *Let X be regular with index $r > -1$, and let Assumption 4.1 hold. Moreover, let $\upsilon : \mathbb{R} \to \mathbb{R}$ be any function such that $\phi(\upsilon^4) < \infty$.*

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(a) *If f is m-integrable and g_1 and $\sqrt{\lambda}g_2$ are m-square integrable, then*

$$\frac{1}{T^{1/(r+2)}}\int_0^T f(X_t)dt \to_d Km(f)E^{1/(r+2)},$$

$$\frac{1}{\sqrt{T^{1/(r+2)}}}\int_0^T g_1(X_t)dW_t \to_d \sqrt{K}m(g_1g_1')^{1/2}B^1\circ E^{1/(r+2)},$$

$$\frac{1}{\sqrt{T^{1/(r+2)}}}\int_0^T g_2(X_{t-})\big[\upsilon(Z_t)dN_t\big(\lambda(X_{t-})\big)-\phi(\upsilon)\lambda(X_t)dt\big]$$
$$\to_d \sqrt{K\phi(\upsilon^2)}m\big(\lambda g_2g_2'\big)^{1/2}B^2\circ E^{1/(r+2)},$$

jointly as $T\to\infty$, where $E^{1/(r+2)}$ is the Mittag-Leffler process with index $1/(r+2)$ at time 1, and B^1 and B^2 are mutually independent standard vector Brownian motions independent of $E^{1/(r+2)}$, and

$$K=\frac{\Gamma((r+1)/(r+2))}{\Gamma((r+3)/(r+2))}\frac{(r+2)^{2/(r+2)}}{\big(a^{1/(r+2)}+b^{1/(r+2)}\big)},$$

where a and b are from (14) and Γ is the gamma function.

(b) *Let f be m-asymptotically homogeneous, and let g_1 and $\sqrt{\lambda}g_2$ be m-square asymptotically homogeneous, with their asymptotic orders smaller than $|x|^p$ at the boundaries of $\mathbb{R}$. Then*

$$\frac{1}{T}\kappa\big(f,T^{1/(r+2)}\big)^{-1}\int_0^T f(X_t)dt\to_d\int_0^1 h\big(f,X_t^\circ\big)\,dt,$$

$$\frac{1}{\sqrt{T}}\kappa\big(g_1,T^{1/(r+2)}\big)^{-1}\int_0^T g_1(X_t)dW_t\to_d\int_0^1 h\big(g_1,X_t^\circ\big)\,dW_t^\circ,$$

$$\frac{1}{\sqrt{T}}\kappa\big(\sqrt{\lambda}g_2,T^{1/(r+2)}\big)^{-1}\int_0^T g_2(X_{t-})\big[\upsilon(Z_t)dN_t\big(\lambda(X_{t-})\big)-\phi(\upsilon)\lambda(X_t)dt\big]$$
$$\to_d\sqrt{\phi(\upsilon^2)}\int_0^1 h\big(\sqrt{\lambda}g_2,X_t^\circ\big)dV_t^\circ$$

jointly as $T\to\infty$ in the notations defined in Definition 4.1 and Proposition 4.2, where V° is a standard Brownian motion defined jointly as a multidimensional Brownian motion with W°,N°, and Z° introduced in Proposition 4.3 such that $\mathbb{E}W_t^\circ V_t^\circ=0$, $\mathbb{E}N_t^\circ V_t^\circ=t\phi(\upsilon)/\sqrt{\phi(\upsilon^2)}$ and $\mathbb{E}Z_t^\circ V_t^\circ=t\phi(\iota\upsilon)/\sqrt{\phi(\upsilon^2)}$.

Remark 4.2. (a) Our asymptotics in Theorem 4.4(a) may be viewed as special cases of more general asymptotics in Theorem 3.1 of Höpfner and Löcherbach (2003). However, in our asymptotics, the invariant measure m is explicitly given in terms of the functional parameters $\mu,\sigma,\nu,\tau,\lambda$, and ϕ of the jump diffusion model. In contrast, the invariant measure m is not specified in their asymptotics, and it is generally impossible to compute the actual limit distributions derived from their asymptotics. Note that the invariant measure of a null recurrent jump diffusion cannot be obtained by the usual simulation method.

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(b) In Theorem 4.4(b), we have non-degenerate limit distributions only if the limit homogeneous functions of f, g_1 and g_2 have supports that have nonempty intersections with the support of $\bar{m}$ defined in Assumption 3.3.[14] The asymptotics for such degenerate cases can also be readily developed as in Kim and Park (2017), though the details are not reported in the paper.

The distributions of $B^1 \circ E^{1/(r+2)}$ and $B^2 \circ E^{1/(r+2)}$ appearing in the martingale transform asymptotics in Theorem 4.4(a) are mixed normal. However, the distributions of

$$P = \int_0^1 h\left(g_1, X_t^\circ\right) dW_t^\circ \quad \text{and} \quad Q = \int_0^1 h\left(\sqrt{\lambda} g_2, X_t^\circ\right) dV_t^\circ$$

representing the martingale transform asymptotics in Theorem 4.4(b) are generally not normal mixtures, and reduce to be mixed normal only when X° is independent of W° and V°, respectively. As shown in Proposition 4.3, X° is driven by three Brownian motions W°, N°, and Z°. Therefore, X° and W° become independent if and only if X° is driven only by N° and Z°, which requires $\sigma^2 \ll (\nu^2 + \tau^2)\lambda$ at the boundaries of $\mathcal{D}$ so that $p_c, q_c = 0$, and in this case, the distribution of P becomes mixed normal. For the independence of X° and V°, on the other hand, we may consider three cases. First, if X° is driven entirely by W°, which requires $\sigma^2 \gg (\nu^2 + \tau^2)\lambda$ at the boundaries of $\mathcal{D}$ so that $p_c, q_c = 1$, X° becomes independent of V°, since V° is independent of W°. Second, if $\nu^2 \gg \tau^2$ at the boundaries of $\mathcal{D}$ so that $p_z, q_z = 1$ and X° is driven by W° and N°, then X° becomes independent of V° as long as $\phi(\upsilon) = 0$ so that N° is independent of V°, and if $\nu^2 \ll \tau^2$ at the boundaries of $\mathcal{D}$ so that $p_z, q_z = 0$ and X° is driven by W° and Z°, then X° becomes independent of V° as long as $\phi(\iota\upsilon) = 0$ so that Z° is independent of V°. Finally, if $\phi(\upsilon) = 0$ and $\phi(\iota\upsilon) = 0$, then X° and V° will always become independent regardless of the values of p_c, q_c, and p_z, q_z.

Example 4.2. Let X be the Lévy process with jumps in Example 2.2 with $\gamma > 0$ and the recurrence condition $\alpha + \gamma\eta = 0$. Clearly, X is in natural scale, and its speed measure is given by the Lebesgue measure. In this simple case, we may directly obtain

$$X_t^\circ = \beta W_t^\circ + \gamma\sqrt{\eta} N_t^\circ + \delta\sqrt{\eta} Z_t^\circ,$$

which also follows from Propositions 4.2 and 4.3 as a special case. For this model, we may explicitly obtain the distributions of P and Q with $g_1 = g_2 = \iota$. In fact, it follows that

$$P = \beta \int_0^1 W_t^\circ dW_t^\circ + \sqrt{\eta} \int_0^1 \left(\gamma N_t^\circ + \delta Z_t^\circ\right) dW_t^\circ,$$

where the distribution of the first term is essentially non-normal, though the distribution of the second term is mixed normal. On the other hand, we have

[14] Since X is assumed to be in natural scale, we denote $\bar{m}_s$ from Assumption 3.3 as simply $\bar{m}$ here.

$$Q = \sqrt{\eta}\beta \int_0^1 W_t^\circ dV_t^\circ + \eta \int_0^1 \left(\gamma N_t^\circ + \delta Z_t^\circ\right) dV_t^\circ,$$

where the distribution of the first term is mixed normal, but the distribution of the second term becomes mixed normal only when $\phi(\upsilon) = 0$ and $\phi(\iota\upsilon) = 0$.

5. NUMERICAL COMPUTATION

For a wide class of jump diffusion models, the exact scale function may be obtained numerically. If the integral operator A introduced in Lemma 3.1 is a contraction and has a fixed point, we may numerically compute the scale function s by iteratively solving for a fixed point of A until convergence. It is natural to use the asymptotic scale function in Lemma 3.3 obtained by the method of dominant balance as an initial function needed to start iterations. To demonstrate the numerical computation of the exact scale function, we consider two classes of jump diffusion models, the affine jump models and the GHK jump models introduced, respectively, in Examples 2.3 and 2.5, with parameter values obtained from the real data. The affine jump model is fitted with the term spread of interest rates, defined by the 10-year treasury constant maturity rate minus the 2-year treasury constant maturity rate (from January 1985 to June 2015). The GHK jump model is fitted with the logs of the USD/GBP exchange rates (from January 1974 to June 2015) and the USD/EUR exchange rates (from January 1999 to June 2015). For the former, we consider both normal and uniform jumps to see the effect of the jump distribution on the scale function. For the latter, jumps are generated as normals.

In sum, we have four fitted models, labeled Models I–IV. Models I and II are the affine jump models with $(\alpha_1, \alpha_2, \beta_1, \beta_2, \delta, \eta_1) = (0.01, -0.1, 0.14, 0.1, 0.08, 16)$ and $\gamma = \eta_2 = 0$, respectively, with normally and uniformly distributed Z, Models III and IV are the GHK jump models with $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \delta, \eta) = (-0.009, 0.02, 0.02, 0.08, 0.0002, 0.01, 20)$ and $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \delta, \eta) = (-0.005, 0.004, 0.07, 0.11, 0.002, 0.01, 15)$, respectively, with $\gamma = 0$ and normally distributed jumps. From our asymptotic theory, we may readily show that the jump diffusions generated from Models I to III are positive recurrent, while those from Model IV are null recurrent.[15]

The exact scale functions of Models I–IV are numerically computed and presented, together with their asymptotic scale functions introduced in (8), in Figure 1. The exact and asymptotic scale functions become identical at the boundaries of $\mathcal{D} = \mathbb{R}$, which shows that the method of dominant balance works for all of the models considered here. In fact, the exact and asymptotic scale functions are quite close to each other over the entire domain, as well as at the boundaries of, $\mathcal{D} = \mathbb{R}$. Our iterative procedure, started at the asymptotic scale function, converges rather quickly and almost instantly finds the exact scale function in all models.

[15] This implies, in particular, that the log of the USD/GBP exchange rates are stationary, while the log of the USD/EUR exchange rates are nonstationary.

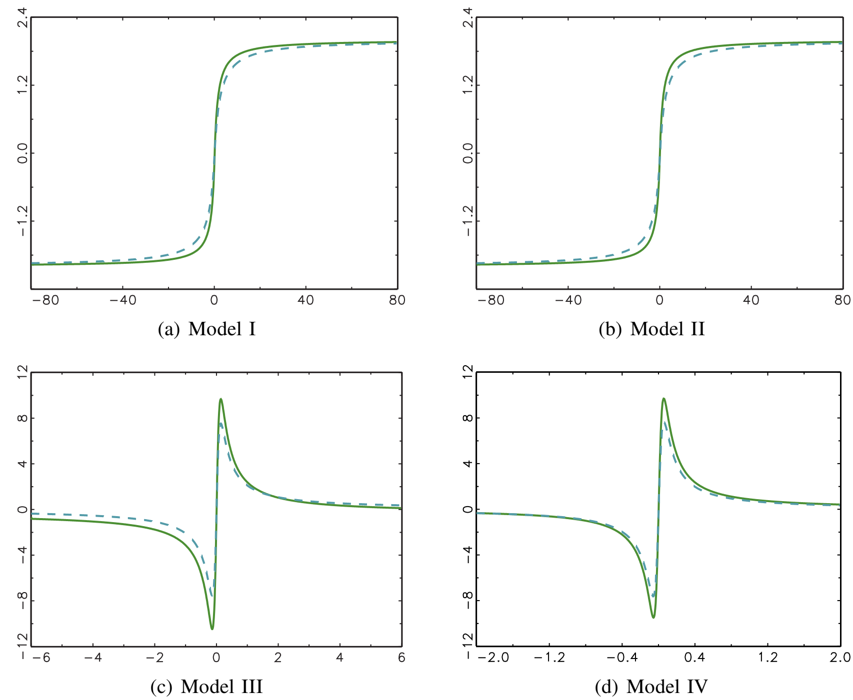

FIGURE 1. Exact and asymptotic log-scale densities.
Note: The exact and asymptotic log-scale densities defined as $(\log s)^{\bullet} = s^{\bullet\bullet}/s^{\bullet}$ are presented, respectively, by the solid and the dotted lines in Models I—IV.

Once the exact and asymptotic scale functions are computed, we may use them and obtain the exact and asymptotic speed densities defined in (13). The exact and asymptotic speed densities are defined with the exact and asymptotic scale densities, respectively. Figure 2 presents the exact and asymptotic speed densities for Models I–IV. It is easy to see that Models I–III are positive recurrent, whereas Model IV is null recurrent. All the speed densities are normalized so that they integrate up to unity.[16] The exact and asymptotic speed densities are the most distinctive around the origin and, as expected, they tend to converge at ordinates away from the origin.

For any given functions f, g_1, and g_2 on $\mathcal{D}$, we may use the computed speed densities to obtain $\pi(f), \pi(g_1 g_1')$, and $\pi(\lambda g_2 g_2')$, and $m(f), m(g_1 g_1')$, and $m(\lambda g_2 g_2')$, which are needed to fully specify the limit distributions in Theorem 4.1 and Part (a) of Theorem 4.4, respectively, for the positive recurrent and null recurrent jump diffusion models. This is in contrast to the corresponding asymptotics for

[16]We also normalize the speed densities for Model IV, although the normalization is only meaningful for Models I–III.

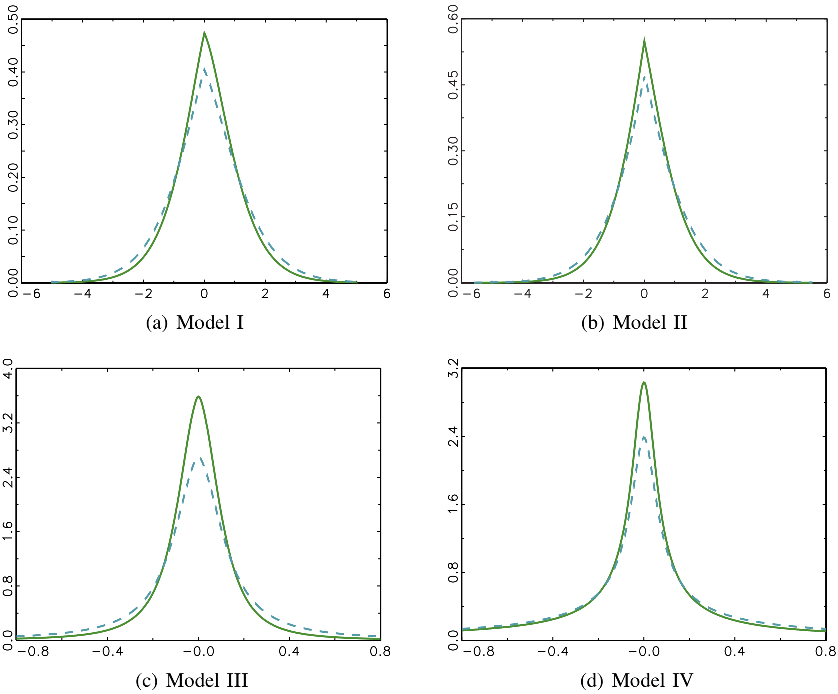

FIGURE 2. Exact and asymptotic invariant densities.
Note: The invariant densities obtained from the exact and asymptotic scale functions are presented, respectively, by the solid and dotted lines for Models I–IV. Normalizations are made so that their integrals are unity. For Model IV, the invariant density is not integrable and, strictly speaking, our normalization is not meaningful.

the general Markov processes available in the literature, from which the limit distributions are not obtainable explicitly for the jump diffusion models considered here. Our asymptotics are therefore more directly useful for statistical inference in these and other jump diffusion models.

6. ILLUSTRATIVE APPLICATION

To show how to apply our asymptotic theory, we consider a variance ratio-type test in continuous time. In this section, we suppose that a sample of size n is collected from a jump diffusion X at interval Δ over time T, that is, $X_\Delta, X_{2\Delta}, \ldots, X_{n\Delta}$ with $T = n\Delta$ and initial value X_0. We define a statistic

$$Q_{n,\Delta} = \frac{\sum_{i=1}^n (X_{i\Delta} - X_0)^2}{n\sum_{i=1}^n (X_{i\Delta} - X_{(i-1)\Delta})^2}.$$

Clearly, $Q_{n,\Delta}$ may be viewed as a high frequency version of the variance ratio statistic used to test for random walk and martingale hypothesis in economics and finance. Note that $Q_{n,\Delta}$ compares the variances of X over the time intervals $[0, i\Delta]$ and $[(i-1)\Delta, i\Delta]$.

Let X be regular and null recurrent with index $r > -1$. We assume that $s(x) \sim ax$ and $m_s(x) \sim b|x|^r$ for some $a, b > 0$ as x approaches $\pm\infty$, and that $s^\bullet$ is bounded away from zero and $1/m_s$ is m_s-asymptotically homogeneous. If $\Delta \to 0$ fast enough relative to $T \to \infty$, we have

$$\Delta \sum_{i=1}^{n} X_{i\Delta}^2 \approx \int_0^T X_t^2 dt + o_p(1) \quad \text{and} \quad \sum_{i=1}^{n} (X_{i\Delta} - X_{(i-1)\Delta})^2 \approx [X]_T.$$

The proofs of the continuous time approximations here and the precise conditions required for their validity are provided in the Appendix. Furthermore, it follows from Theorem 4.4 (b) that

$$\frac{1}{T^{1+2/(r+2)}} \int_0^T X_t^2 dt = \frac{1}{T^{1+2/(r+2)}} \int_0^T \left(s^{-1}(X_t^s)\right)^2 dt \to_d \frac{1}{a^2} \int_0^1 X_t^{\circ 2} dt \tag{17}$$

and

$$\begin{aligned}
\frac{1}{T^{2/(r+2)}}[X]_T &= \frac{1}{T^{2/(r+2)}} \int_0^T \sigma^2(X_t)dt + \frac{1}{T^{2/(r+2)}} \int_0^T \left(\nu(X_{t-}) + \tau(X_{t-})Z_t\right)^2 dN_t\left(\lambda(X_{t-})\right) \\
&= \frac{1}{T^{2/(r+2)}} \int_0^T \sigma^2(X_t)dt + \frac{1}{T^{2/(r+2)}} \int_0^T \left(\nu^2(X_{t-}) + \tau^2(X_{t-})Z_t^2\right) dN_t\left(\lambda(X_{t-})\right) + o_p(1) \\
&= \frac{1}{T^{2/(r+2)}} \int_0^T \left[\sigma^2 + (\nu^2 + \tau^2)\lambda\right](X_t)dt \\
&\quad + \frac{1}{T^{2/(r+2)}} \int_0^T \Big[\left(\nu^2(X_{t-}) + \tau^2(X_{t-})Z_t^2\right) dN_t\left(\lambda(X_{t-})\right) - \left((\nu^2 + \tau^2)\lambda\right)(X_t)dt\Big] + o_p(1) \\
&= \frac{1}{T^{2/(r+2)}} \int_0^T \left[\sigma^2 + (\nu^2 + \tau^2)\lambda\right](X_t)dt + o_p(1) \\
&= \frac{1}{T^{2/(r+2)}} \int_0^T \frac{1}{(s^{\bullet 2} \circ s^{-1})(X_t^s)} \frac{1}{m_s(X_t^s)} dt + o_p(1) \to_d \frac{1}{a^2 b} \int_0^1 |X_t^\circ|^{-r} dt
\end{aligned} \tag{18}$$

as $T \to \infty$ and $\Delta \to 0$. Note that $f = (s^{-1})^2$ is m_s-asymptotically homogeneous with $\kappa(f, \lambda) = \lambda^2$ and $h(f, x) = x^2/a^2$ for (17), and that

$$\begin{aligned}
1/m_s = \sigma_s^2 + \omega_s^2 \lambda_s &\sim (\sigma s^\bullet)^2 \circ s^{-1} + \left[(\nu^2 + \tau^2)s^{\bullet 2} \circ s^{-1}\right](\lambda \circ s^{-1}) \\
&= \left(\left[\sigma^2 + (\nu^2 + \tau^2)\lambda\right] s^{\bullet 2}\right) \circ s^{-1}
\end{aligned}$$

and $f = 1/[(s^{\bullet 2} \circ s^{-1})m_s]$ is m_s-asymptotically homogeneous with $\kappa(f, \lambda) = \lambda^{-r}$ and $h(f, x) = |x|^{-r}/(a^2 b)$ for (18). The limit process X° is a generalized diffusion given by $dX_t^\circ = b^{-1/2}|X_t^\circ|^{-r/2} dW_t$.

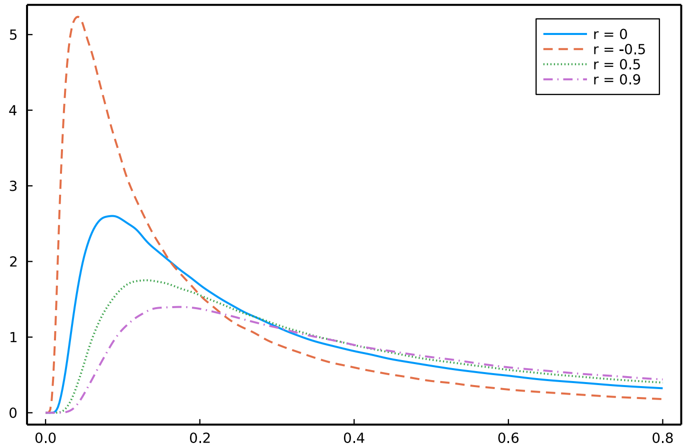

FIGURE 3. Limit distributions of $Q_{n,\Delta}$ under the null and alternatives.

Let $X^{\circ} = b^{-1/(r+2)} X^*$ so that X^* is the generalized diffusion given by $dX_t^* = |X_t^*|^{-r/2} dW_t$. Then we may deduce from (17) and (18) that

$$Q_{n,\Delta} \to_d \frac{\int_0^1 X_t^{*2} dt}{\int_0^1 |X_t^*|^{-r} dt} \tag{19}$$

as $T \to \infty$ and $\Delta \to 0$. If, in particular, $r = 0$, the generalized diffusion X^* reduces to Brownian motion, and we have

$$Q_{n,\Delta} \to_d \int_0^1 W_t^2 dt$$

as $T \to \infty$ and $\Delta \to 0$. The limit distribution of $Q_{n,\Delta}$ are provided in Figure 3 for some selected values of r.

If X is positive recurrent and has finite second moment, we may easily show that $TQ_{n,\Delta}$ converges almost surely to a well-defined positive random variable as $T \to \infty$ and $\Delta \to 0$. Of course, this implies that $Q_{n,\Delta} \to_p 0$ as $T \to \infty$ and $\Delta \to 0$, and the statistic $Q_{n,\Delta}$ can be used to test for random walk and martingale hypothesis in continuous time framework. If X is a mean zero Ornstein–Uhlenbeck process given by $dX_t = -\kappa X_t dt + \sigma dW_t$ with $\kappa > 0$ and if we set $X_0 = 0$ in the definition of $Q_{n,\Delta}$, then we have $TQ_{n,\Delta} \to_p 1/2\kappa$. The limit of $TQ_{n,\Delta}$ is solely determined by the mean reversion parameter κ.

We apply our variance ratio test to the logs of the daily USD/GBP exchange rates (from January 1974 to June 2015) and the daily CBOE volatility index (from January 1990 to June 2015). The values of the statistic $Q_{n,\Delta}$ are given by 0.296 and 0.0047 for the exchange rate and volatility index, respectively. If we take the limit Brownian motion as the null hypothesis, the asymptotic critical values of the test are 0.0343, 0.0563, and 0.0762 for the 1%, 5%, and 10% left tail tests.

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Consequently, we are in favor of the null hypothesis of asymptotic Brownian motion for the USD/GBP exchange rates, while we unambiguously reject the null hypothesis of asymptotic Brownian motion for the volatility index. According to our variance ratio test, the volatility index is clearly distinguishable from Brownian motion in the limit.

7. CONCLUSION

This paper defines scale functions and speed densities for jump diffusion models, in parallel with their definitions for diffusion models without jumps. Unlike diffusion models without jumps, scale functions and speed densities may not exist for jump diffusion models. However, when they do exist, they play exactly the same roles for jump diffusion models as they do for diffusion models without jumps: the scale functions transform jump diffusions into martingales and the speed densities represent the invariant measures of jump diffusions. Moreover, the scale functions and the speed densities fully characterize the asymptotics of the general additive functionals and the martingale transforms of jump diffusion models, exactly as they do in the case of diffusion models without jumps. Indeed, assuming that scale functions and speed densities exist, we develop the additive functional and the martingale transform asymptotics of jump diffusion models and present them explicitly in terms of their scale functions and speed densities. They are applicable for a wide variety of jump diffusion models including null recurrent as well as positive recurrent jump diffusion models, and for a very general class of additive functionals and martingale transforms. We also provide some sufficient conditions for the existence of scale functions and speed densities for jump diffusion models.

Appendix A. Useful Lemmas

Throughout, we write $f \in RV_k$ if $f : D \to \mathbb{R}$ is regularly varying with index k at the boundaries of D. The reader is referred to, for example, Bingham, Goldie, and Teugels (1989) for the notion and properties of regularly varying functions.

A.1. Lemmas

LEMMA A1. *Let Assumptions 2.1, 3.1, and 3.3 hold, and define* $B_t = (X^s \circ A^{-1})_t$ *with* $dA_t = \left[1/m_s(X_t^s)\right]dt$. *If we let* $B_t^T = T_r^{-1} B_{T_r^2 t}$ *and denote its local time as* L^T, *where* $T_r = T$ *or* $T_r = T^{1/(r+2)}$ *depending upon whether* m_s *is integrable or regularly varying with index* $r > -1$, *then there exists a standard Brownian motion* B° *with local time* L° *such that* $B^T \to_d B^\circ$ *in* $\mathcal{C}[0,1]$ *and*

$$\sup_{t\in[0,K]} \sup_{x\in[-K,K]} \left|L^T(t,x) - L^\circ(t,x)\right| \to_p 0$$

for any $0 < K < \infty$ *as* $T \to \infty$.

LEMMA A2. *Let Assumptions 2.1, 3.1, and 4.1 hold. Moreover, let X be regular with index* $r > -1$*, and let* W^T *be defined as* $W_t^T = T^{-1/2}W_{Tt}$*. If we define* V^T *as*

$$V_t^T = \frac{1}{\sqrt{T}}\int_0^{Tt}\left[\frac{v(X_{u-},Z_u)}{\sqrt{v_2\lambda}(X_u)}dN_u\big(\lambda(X_{u-})\big) - \frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\right]$$

for $v : \mathcal{D}\times\mathbb{R}\to\mathbb{R}$ *with* $\sup_{x\in\mathcal{D}}\big(v_4/v_2^2\big)(x) < \infty$ *jointly with* W^T*, where* $v_k(x) = \int_{\mathbb{R}} v^k(x,z)\phi(z)dz$ *for* $k > 0$*, then* $V^T \to_d V^\circ$ *in* $\mathcal{C}[0,1]$ *jointly with* $W^T \to_d W^\circ$ *as* $T\to\infty$*, where* V° *is a standard Brownian motion independent of* W°*.*

A.2. Proofs of Lemmas

A.2.1. *Proof of Lemma A1.* **Convergence of B^T** It follows from Itô's lemma and (5) that

$$\begin{aligned} dX_t^s &= -(\nu_s\lambda_s)(X_t^s)dt + \sigma_s(X_t^s)dW_t \\ &\quad + \Big(s\big[s^{-1}(X_{t-}^s) + (\nu\circ s^{-1})(X_{t-}^s) + (\tau\circ s^{-1})(X_{t-}^s)Z_t\big] - X_{t-}^s\Big)dN_t\big(\lambda(X_{t-})\big), \end{aligned} \qquad \textbf{(A.1)}$$

where

$$\nu_s(x) = \int_{\mathbb{R}}\Big(s\big[s^{-1}(x) + (\nu\circ s^{-1})(x) + (\tau\circ s^{-1})(x)z\big] - x\Big)\phi(z)dz.$$

We note that

$$dA_t^{-1} = m_s(B_t)dt, \qquad \textbf{(A.2)}$$

and redefine W and N, up to the distributional equivalence, as

$$d(W\circ A^{-1})_t =_d A_t^{-1/2}dW_t \qquad \textbf{(A.3)}$$

$$d(N\circ A^{-1})_t\Big(\lambda(X\circ A^{-1})_t\Big) =_d dN\Big((m_s\lambda_s)(B_{t-})\Big). \qquad \textbf{(A.4)}$$

Under the change of variable $t\mapsto A_t^{-1}$, the stochastic differential equation in (A.1) reduces to

$$\begin{aligned} dB_t &= -(m_s\nu_s\lambda_s)(B_t)dt + (m_s^{1/2}\sigma_s)(B_t)dW_t \\ &\quad + \Big(s\big[s^{-1}(B_{t-}) + (\nu\circ s^{-1})(B_{t-}) + (\tau\circ s^{-1})(B_{t-})Z_t\big] - B_{t-}\Big)dN_t\big((m_s\lambda_s)(B_{t-})\big), \end{aligned} \qquad \textbf{(A.5)}$$

up to the distributional equivalence, due to (A.2), (A.3), and (A.4). Furthermore, we may write the stochastic differential equation in (A.5) in terms of B_t^T as

$$\begin{aligned} dB_t^T &= -T_r(m_s\nu_s\lambda_s)(T_rB_t^T)dt + (m_s^{1/2}\sigma_s)(T_rB_t^T)dW_t + \frac{1}{T_r}\Big(s\big[s^{-1}(T_rB_{t-}^T) + (\nu\circ s^{-1})(T_rB_{t-}^T) \\ &\quad + (\tau\circ s^{-1})(T_rB_{t-}^T)Z_t\big] - T_rB_{t-}^T\Big)dN_t\big(T_r^2(m_s\lambda_s)(T_rB_{t-}^T)\big), \end{aligned} \qquad \textbf{(A.6)}$$

again up to the distributional equivalence.

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Denote by $\mathcal{A}_T$ the infinitesimal generator of B^T. To deduce the stated result, it suffices to show that

$$\mathcal{A}_T f(x) \to \frac{1}{2} f^{\cdot\cdot}(x) \tag{A.7}$$

as $T \to \infty$ locally uniformly in $x \in K$ for all $f \in C^2(K)$, the set of twice continuously differentiable functions vanishing outside K, for an arbitrary compact subset K of $\mathbb{R}$. This implies that $\mathcal{A}_T$ converges to the infinitesimal generator of standard Brownian motion (see, e.g., Jacod and Shiryaev, 2003, Thm. IX.4.8 and Rem. IX.4.13). However, it follows from (A.6) that

$$\begin{aligned}\mathcal{A}_T f(x) &= -T_r(m_s\nu_s\lambda_s)(T_r x) f^{\cdot}(x) + \frac{1}{2}(m_s\sigma_s^2)(T_r x) f^{\cdot\cdot}(x) \\ &\quad + T_r^2(m_s\lambda_s)(T_r x)\int_{\mathbb{R}}\Big[f\Big(T_r^{-1}s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]\Big)-f(x)\Big]\phi(z)dz\end{aligned} \tag{A.8}$$

on $x \in K$ for any $f \in C^2(K)$. Moreover, we have

$$\begin{aligned}&f\Big(T_r^{-1}s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]\Big)-f(x) \\ &= f^{\cdot}(x)\Big(T_r^{-1}s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]-x\Big) \\ &\quad + \frac{1}{2} f_T^{\cdot\cdot}(x,z)\Big(T_r^{-1}s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]-x\Big)^2,\end{aligned} \tag{A.9}$$

where

$$f_T^{\cdot\cdot}(x,z) = f^{\cdot\cdot}\left[x-\delta_T\left(\frac{1}{T_r}s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]-x\right)\right] \tag{A.10}$$

for some $0 \le \delta_T \le 1$. Note that we have

$$\begin{aligned}&\int_{\mathbb{R}}\Big(s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]-T_r x\Big)\phi(z)dz = \nu_s(T_r x) \\ &\int_{\mathbb{R}}\Big(s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]-T_r x\Big)^2\phi(z)dz = \omega_s^2(T_r x).\end{aligned} \tag{A.11}$$

Therefore, we deduce from (A.8), (A.9), and (A.11) that

$$\mathcal{A}_T f(x) = \frac{1}{2} f^{\cdot\cdot}(x) + R_T(x), \tag{A.12}$$

where

$$R_T(x) = \frac{1}{2}(m_s\lambda_s)(T_r x)\int_{\mathbb{R}}\big[f_T^{\cdot\cdot}(x,z)-f^{\cdot\cdot}(x)\big]\Big(s\big[s^{-1}(T_r x)+(\nu\circ s^{-1})(T_r x)+(\tau\circ s^{-1})(T_r x)z\big]-T_r x\Big)^2\phi(z)dz$$

for any $f \in C^2(K)$, from which (A.7) follows if we show that

$$R_T(x) \to 0 \tag{A.13}$$

as $T \to \infty$ uniformly in $x \in K$ for all $f \in C^2(K)$.

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To show (A.13), we write $R_T(x) = (m_s\lambda_s)(T_r x)\big(R_{1T}(x) + R_{2T}(x)\big)$, where

$$R_{1T}(x) = \frac{1}{2}\int_{\mathbb{R}\backslash[-c,c]} \big[f_T^{\cdot\cdot}(x,z) - f^{\cdot\cdot}(x)\big]\Big(s\big[s^{-1}(T_r x) + (\nu\circ s^{-1})(T_r x) + (\tau\circ s^{-1})(T_r x)z\big] - T_r x\Big)^2 \phi(z)dz,$$

$$R_{2T}(x) = \frac{1}{2}\int_{[-c,c]} \big[f_T^{\cdot\cdot}(x,z) - f^{\cdot\cdot}(x)\big]\Big(s\big[s^{-1}(T_r x) + (\nu\circ s^{-1})(T_r x) + (\tau\circ s^{-1})(T_r x)z\big] - T_r x\Big)^2 \phi(z)dz$$

for $c > 0$.

We first consider $(m_s\lambda_s)(T_r x)R_{1T}(x)$. Due to the boundedness of $f^{\cdot\cdot}$ on K, there exists $M_1 > 0$ such that

$$\sup_{z\in\mathbb{R}\backslash[-c,c]} \sup_{x\in K} \big|f_T^{\cdot\cdot}(x,z) - f^{\cdot\cdot}(x)\big| < M_1 \quad \textbf{(A.14)}$$

for any given $c > 0$, any given $f \in C^2(K)$, and all large T. Moreover, we can find $M_2 > 0$ such that

$$(m_s\omega_s^2\lambda_s)(T_r x) \le M_2 \quad \textbf{(A.15)}$$

uniformly in $x \in K$ for all large T. Therefore, for any $\varepsilon > 0$, we can find a large enough $c > 0$ such that

$$\frac{1}{2}(m_s\lambda_s)(T_r x)\int_{\mathbb{R}\backslash[-c,c]} \Big(s\big[s^{-1}(T_r x) + (\nu\circ s^{-1})(T_r x) + (\tau\circ s^{-1})(T_r x)z\big] - T_r x\Big)^2 \phi(z)dz \le \varepsilon \quad \textbf{(A.16)}$$

uniformly in $x \in K$ for all large T. Consequently, it follows from (A.14) and (A.16) that, for any $\varepsilon > 0$,

$$\big|(m_s\lambda_s)(T_r x)R_{1T}(x)\big| \le M_1\varepsilon \quad \textbf{(A.17)}$$

uniformly in $x \in K$ for some $M_1 > 0$ and all large T.

For $(m_s\lambda_s)(T_r x)R_{2T}(x)$, recall that we have $s^{-1}(x) \in RV_\kappa$ for $\kappa > 0$, and $\nu(x) \in RV_{\kappa_1}$ and $\tau(x) \in RV_{\kappa_2}$ for $\kappa_1, \kappa_2 < 1$. Therefore, we may deduce that

$$\frac{s^{-1}(T_r x)}{s^{-1}(T_r)} + \frac{(\nu\circ s^{-1})(T_r x)}{s^{-1}(T_r)} + \frac{(\tau\circ s^{-1})(T_r x)}{s^{-1}(T_r)}z \to x^\kappa \quad \textbf{(A.18)}$$

as $T \to \infty$ locally uniformly in $x \in \mathbb{R}$ for any given z, due to the local boundedness and the asymptotic monotonicity of $\nu\circ s^{-1}$ and $\tau\circ s^{-1}$, and the uniform convergence property of regularly varying function with positive index (see, e.g., Bingham et al., 1989, Thm. 1.5.2). Moreover, due to (A.18) and the uniform convergence property of regularly varying functions, it follows that

$$\frac{s[s^{-1}(T_r x) + (\nu\circ s^{-1})(T_r x) + (\tau\circ s^{-1})(T_r x)z]}{T_r}$$
$$= \frac{(s\circ s^{-1})(T_r)}{T_r}\,\frac{s[s^{-1}(T_r x) + (\nu\circ s^{-1})(T_r x) + (\tau\circ s^{-1})(T_r x)z]}{(s\circ s^{-1})(T_r)} \to x$$

as $T \to \infty$ uniformly in $x \in K$ for any given z. This implies that for any $\varepsilon > 0$, we have

$$\sup_{x\in K}\left|\frac{s[s^{-1}(T_r x) + (\nu\circ s^{-1})(T_r x) + (\tau\circ s^{-1})(T_r x)z]}{T_r} - x\right| \le \varepsilon \quad \textbf{(A.19)}$$

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uniformly in $z \in [-c,c]$ for all large T. Therefore, from (A.10), (A.19), and the continuity of $f^{\cdot\cdot}$ on K, we obtain that for any $\varepsilon > 0$,

$$\sup_{z\in[-c,c]} \sup_{x\in K} \left| f_T^{\cdot\cdot}(x,z) - f^{\cdot\cdot}(x) \right| \le \varepsilon \tag{A.20}$$

for any given $c > 0$, any given $f \in C^2(K)$, and all large T. Consequently, we deduce from (A.11), (A.15), and (A.20) that for any $\varepsilon > 0$,

$$\left|(m_s\lambda_s)(T_r x)R_{2T}(x)\right| \le (m_s\omega_s^2\lambda_s)(T_r x)\,\varepsilon \le M_2\varepsilon \tag{A.21}$$

uniformly in $x \in K$ for some $M_2 > 0$ and all large T.

Now (A.13) follows from (A.17) and (A.21), which establishes (A.7) from (A.12). The proof is therefore complete.

Convergence of L^T Let $B^T \to_{a.s.} B^\circ$ in $\mathcal{C}[0,1]$ by changing the underlying probability space if necessary. Due to the Meyer–Itô theorem (see, e.g., Protter, 2005, Thm. IV.70), we have

$$\begin{aligned} L^T(t,x) = {} & |B_t^T - x| - |B_0^T - x| - \int_0^t \mathrm{sgn}(B_{u-}^T - x)dB_u^T \\ & - \sum_{0\le u\le t} \left[|B_u^T - x| - |B_{u-}^T - x| - \mathrm{sgn}(B_{u-}^T - x)\Delta B_u^T \right] \end{aligned} \tag{A.22}$$

for $t \ge 0$ and $x \in \mathbb{R}$, where $\mathrm{sgn}(x) = 1\{x > 0\} - 1\{x \le 0\}$ and $\Delta B_u^T = B_u^T - B_{u-}^T$. Clearly, we have

$$|B_t^T - x| \to_{a.s.} |B_t^\circ - x| \quad \text{and} \quad |B_0^T - x| \to_{a.s.} |B_0^\circ - x| \tag{A.23}$$

as $T \to \infty$ locally uniformly in $t \ge 0$ and $x \in \mathbb{R}$.

Write

$$\int_0^t \mathrm{sgn}(B_{u-}^T - x)dB_u^T = \int_0^t \mathrm{sgn}(B_u^\circ - x)dB_u^T + R_t^T(x), \tag{A.24}$$

where

$$R_t^T(x) = \int_0^t \mathrm{sgn}(B_{u-}^T - x)dB_u^T - \int_0^t \mathrm{sgn}(B_u^\circ - x)dB_u^T.$$

It follows from Theorem 2.2 of Kurtz and Protter (1991) that

$$\int_0^t \mathrm{sgn}(B_u^\circ - x)dB_u^T \to_p \int_0^t \mathrm{sgn}(B_u^\circ - x)dB_u^\circ \tag{A.25}$$

as $T \to \infty$ locally uniformly in $t \ge 0$ and $x \in \mathbb{R}$. In fact, we may readily show that B^T satisfies C2.2(i) of Kurtz and Protter (1991). We use their notations for Y, δ, J, M, and A, and let $Y_T = B^T$. If we choose $\delta = \infty$, then $J_\delta(x)(t) = 0$ for all $x \in \mathbb{R}$ and $t \ge 0$, and consequently, $Y_T^\delta = Y_T$. Moreover, we have $A_T^\delta = 0$, since B^T is a martingale. Also, since $\mathbb{E}[B^T]_t = t$ for all $t \ge 0$, we may deduce that

$$\mathbb{E}[M_T^\delta]_t = \mathbb{E}[Y_T]_t = \mathbb{E}[B^T]_t = t < \infty$$

for all $t \in [0,K]$ uniformly in $T > 0$.

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Now we show that $R_t^T(x)$ in (A.24) is asymptotically negligible locally uniformly in $t \geq 0$ and $x \in \mathbb{R}$. Note that it is a martingale, and

$$\sup_{x\in[-K,K]}\sup_{t\in[0,K]}[R^T(x)]_t = \sup_{x\in[-K,K]}\sup_{t\in[0,K]}\int_0^t \left[\mathrm{sgn}(B_u^T - x) - \mathrm{sgn}(B_u^\circ - x)\right]^2 d[B^T]_u$$
$$\leq 4 \sup_{x\in[-K,K]}\int_0^K 1\left\{\mathrm{sgn}(B_u^T - x) \neq \mathrm{sgn}(B_u^\circ - x)\right\} d[B^T]_u.$$

However, since $B_t^T \to_{a.s.} B_t^\circ$ uniformly in $t \in [0,K]$, we can find a sequence $a_T \to 0$ such that

$$\left\{\mathrm{sgn}(B_u^T - x) \neq \mathrm{sgn}(B_u^\circ - x)\right\} \subset \left\{|B_u^\circ - x| < a_T\right\}$$

for all large T. Moreover, since $[B^T]_t \to_p t$ locally uniformly in $t \geq 0$, we can find a sequence $b_T \to 0$ such that

$$\int_0^K 1\left\{|B_u^\circ - x| < a_T\right\} d[B^T]_u \leq \int_0^K 1\left\{|B_u^\circ - x| < a_T\right\} du + b_T$$

for all large T. Therefore,

$$\sup_{x\in[-K,K]}\int_0^K 1\left\{\mathrm{sgn}(B_u^T - x) \neq \mathrm{sgn}(B_u^\circ - x)\right\} d[B^T]_u \leq \sup_{x\in[-K,K]}\int_0^K 1\left\{|B_u^\circ - x| < a_T\right\} du + b_T,$$

where $a_T \to 0$ and $b_T \to 0$ as $T \to \infty$, and upon noting that

$$\sup_{x\in[-K,K]}\int_0^K 1\left\{|B_u^\circ - x| < a_T\right\} du \leq M a_T \left(\sup_{x\in[-K,K]} L(K,x) + c_T\right)$$

for a sequence $c_T \to 0$ and some $M > 0$, we may deduce that

$$\sup_{x\in[-K,K]}\sup_{t\in[0,K]}[R^T(x)]_t \to_{a.s.} 0 \tag{A.26}$$

locally uniformly in $t \geq 0$ and $x \in \mathbb{R}$. Consequently, it follows from (A.24), (A.25), and (A.26) that

$$\int_0^t \mathrm{sgn}(B_{u-}^T - x) dB_u^T \to_{a.s.} \int_0^t \mathrm{sgn}(B_u^\circ - x) dB_u^\circ \tag{A.27}$$

uniformly in $t \in [0,K]$ and $x \in [-K,K]$ as $T \to \infty$.

Last, we show the asymptotic negligibility of

$$S_t^T(x) = -\sum_{0\leq u\leq t}\left[|B_u^T - x| - |B_{u-}^T - x| - \mathrm{sgn}(B_{u-}^T - x)\Delta B_u^T\right].$$

Let a_T and b_T be increasing sequences of T such that $a_T \sup_{t\in[0,K]}|B_t^T - B_t^\circ| \to_{a.s.} M_1$ and $b_T \sup_{t\in[0,K]}|\Delta B_t^T| \to_{a.s.} M_2$ for any $K > 0$ and some $M_1, M_2 > 0$ as $T \to \infty$. Then we have

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$$
\begin{aligned}
S_t^T(x) &\le 2\sum_{0\le u\le t}\left|\Delta B_u^T\right| 1\left\{|B_{u-}^T - x| < |\Delta B_u^T|\right\} \\
&\le 2\sup_{0\le u\le t}\left|\Delta B_u^T\right| \int_0^T 1\left\{|B_{v-}^\circ - x| \le 1/a_T\right\} dN_v\left(T_r^2(m_s\lambda_s)(T_r B_{v-}^T)\right),
\end{aligned}
\tag{A.28}
$$

where the second inequality is due to

$$
\left\{|B_{u-}^T - x| < |\Delta B_u^T|\right\} \subseteq \left\{|B_{u-}^\circ - x| \le 1/a_T\right\}
$$

for all u such that $\Delta B_u^T \ne 0$ and all large T. The number of jumps in $\{B_t^T\}$ on a unit interval $t\in[0,1]$ is increasing at an order smaller than or equal to b_T^2, since otherwise there arises a contradiction to the local uniform convergence of $[B^T]_t \to_p t$. Therefore, we have

$$
\int_0^T 1\left\{|B_{v-}^\circ - x| \le 1/a_T\right\} dN_v\left(T_r^2(m_s\lambda_s)(T_r B_{v-}^T)\right),
$$

that is, the number of jumps of $\{B_t^T\}$ on $t \in \left\{v : |B_{v-}^\circ - x| \le 1/a_T\right\}$, is of the order $O(b_T^2 a_T^{-2+\varepsilon})$ uniformly in $x\in[-K,K]$ as $T\to\infty$, due to the uniform convergence order of $B^T \to_{a.s.} B^\circ$ and the modulus of continuity of B°. We also have $\sup_{0\le u\le t}|\Delta B_u^T| = O(b_T^{-1})$ as $T\to\infty$, and $b_T/a_T < \infty$ for all large T. Therefore, it follows that

$$
\sup_{t\in[0,K]}\sup_{x\in[-K,K]} S_t^T(x) \to_{a.s.} 0 \tag{A.29}
$$

as $T\to\infty$.

Consequently, we deduce from (A.22), (A.23) (A.27), and (A.29) that

$$
L^T(t,x) \to_{a.s.} |B_t^\circ - x| - |B_0^\circ - x| + \int_0^t \operatorname{sgn}(B_u^\circ - x)\, dB_u^\circ = L^\circ(t,x)
$$

as $T\to\infty$ locally uniformly in $t\ge 0$ and $x\in\mathbb{R}$. This completes the proof.

A.2.2. *Proof of Lemma A2.* Let Tt be an integer for notational simplicity, and write

$$
\begin{aligned}
V_t^T &= \frac{1}{\sqrt{T}}\int_0^{Tt}\left[\frac{v(X_{u-},Z_u)}{\sqrt{\lambda v_2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\right] \\
&= \frac{1}{\sqrt{T}}\sum_{i=1}^{Tt}\int_{i-1}^{i}\left[\frac{v(X_{u-},Z_u)}{\sqrt{\lambda v_2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\right].
\end{aligned}
$$

It follows from Jeanblanc et al. (2009, Sect. 8.8.4 and Prop. 8.8.6.1) that

$$
\begin{aligned}
&\mathbb{E}\int_{i-1}^{i}\left[\frac{v(X_{u-},Z_u)}{\sqrt{\lambda v_2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\right] = 0, \\
&\mathbb{E}\left(\int_{i-1}^{i}\left[\frac{v(X_{u-},Z_u)}{\sqrt{\lambda v_2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\right]\right)^2 = 1
\end{aligned}
\tag{A.30}
$$

for all $1\le i\le Tt$. Furthermore, we may also deduce from Jeanblanc et al. (2009, Sect. 8.8.4 and Prop. 8.8.6.1) that

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$$\mathbb{E}\biggl(\int_{i-1}^{i}\biggl[\frac{v^2(X_{u-},Z_u)}{(\lambda v_2)(X_{u-})}dN_u\bigl(\lambda(X_{u-})\bigr)-du\biggr]\biggr)^2=\mathbb{E}\biggl(\int_{i-1}^{i}\frac{v^2(X_{u-},Z_u)}{(\lambda v_2)(X_{u-})}dN_u\bigl(\lambda(X_{u-})\bigr)\biggr)^2-1$$
$$=\mathbb{E}\int_{i-1}^{i}\frac{v_4(X_u)}{(\lambda v_2^2)(X_u)}du\le c\,\mathbb{E}\int_{i-1}^{i}\frac{1}{\lambda(X_u)}du$$

for all $1\le i\le Tt$ and some $c>0$. Moreover, note that $T^{-1+\varepsilon}\sup_{t\in[0,T]}\mathbb{E}\bigl(1/\lambda(X_t)\bigr)<\infty$ for some $\varepsilon>0$ and all large T, due to Assumption 4.1. Therefore, it follows from the Burkholder–Davis–Gundy inequality that

$$\frac{1}{T}\mathbb{E}\biggl(\int_{i-1}^{i}\biggl[\frac{v(X_{u-},Z_u)}{\sqrt{\lambda v_2}(X_{u-})}dN_u\bigl(\lambda(X_{u-})\bigr)-\frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\biggr]\biggr)^4$$
$$\le\frac{1}{T}\mathbb{E}\biggl(\int_{i-1}^{i}\frac{v^2(X_{u-},Z_u)}{(\lambda v_2)(X_{u-})}dN_u\bigl(\lambda(X_{u-})\bigr)\biggr)^2$$
$$=\frac{c}{T}\mathbb{E}\biggl(\int_{i-1}^{i}\frac{1}{\lambda(X_u)}du\biggr)+\frac{1}{T}\to 0 \qquad \textbf{(A.31)}$$

as $T\to\infty$ for all $1\le i\le Tt$ and some $c>0$. Consequently, we obtain from (A.30), (A.31), and the functional CLT in, for example, Theorem 4.1 of Hall and Heyde (1980) that

$$V^T\to_d V^\circ, \qquad \textbf{(A.32)}$$

where V° is standard Brownian motion.

To show the independence between W° and V°, let

$$M_t^T=\frac{1}{\sqrt{T}}\int_0^{Tt}P_u dW_u$$

for an arbitrary bounded predictable process P. We have $\mathbb{E}(M_t^T)^2<\infty$ and $\mathbb{E}(V_t^T)^2<\infty$ for all $T>0$ and $t\in[0,1]$, due to Jeanblanc et al. (2009, Section 8.8.4). Therefore, we deduce from the covariance extension of the Itô isometry that

$$\mathbb{E}\bigl(M_t^T V_t^T\bigr)=\mathbb{E}[M^T,V^T]_t=\mathbb{E}\frac{1}{T}\int_0^{Tt}P_u d\bigl[W,V\bigr]_u=0 \qquad \textbf{(A.33)}$$

for all $T>0$ and $t\in[0,1]$, where

$$V_t=\int_0^t\biggl[\frac{v(X_{u-},Z_u)}{\sqrt{\lambda v_2}(X_{u-})}dN_u(\lambda(X_{u-}))-\frac{\sqrt{\lambda}v_1}{\sqrt{v_2}}(X_u)du\biggr],$$

since $[W,V]_t=0$ for all $t\ge 0$. Consequently, the independence between W° and V° follows from (A.32), (A.33), and Exercises IV.2.22 and V.4.25 of Revuz and Yor (1999), which completes the proof.

Appendix B. Proofs of Main Theorems

For $f,g:D\to\mathbb{R}$, we redefine $f\sim g$ if $g(x)/f(x)\to c$ for some $c\ne 0$ as x approaches the boundaries of D. Moreover, we write $f\succ g$ if $g(x)/f(x)\to 0$ as x approaches the

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boundaries of D. If either $f \succ g$ or $f \sim g$, we write $f \succcurlyeq g$. We also let $T_r = T^{1/(r+2)}$ and $\zeta_k(x) = \int_{\mathbb{R}} \left[s\big(x - \nu(x) - \tau(x)z\big) - s(x)\right]^k \phi(z)dz$ for $k > 0$.[17]

B.1. Proof of Proposition 3.2

To avoid unnecessary complications in dealing with the case of $\mathcal{D} \neq \mathbb{R}$, we let all functions in this proof take zero values outside of $\mathcal{D}$, and we also let $0/0 = 0$.

Let s be a solution to the integro-differential equation

$$(s^{\cdot}\mu)(x) + \frac{1}{2}(s^{\cdot\cdot}\sigma^2)(x) = -\lambda(x)\int_{\mathbb{R}}\Big(s\big[x+\nu(x)+\tau(x)z\big] - s(x)\Big)\phi(z)dz. \tag{B.34}$$

Changing the order of integrals, we have

$$\begin{aligned}\int_0^{\infty}\int_x^{x+\nu(x)+\tau(x)z} s^{\cdot}(y)dy\phi(z)dz &= \int_x^{\infty}\int_{\frac{y-x-\nu(x)}{\tau(x)}}^{\infty} s^{\cdot}(y)\phi(z)dzdy \\ &= \int_x^{\infty} s^{\cdot}(y)\left[1 - \Phi\left(\frac{y-x-\nu(x)}{\tau(x)}\right)\right]dy \\ &= \tau(x)\int_0^{\infty} s^{\cdot}\big(x+\nu(x)+\tau(x)y\big)\big(1-\Phi(y)\big)dy. \end{aligned} \tag{B.35}$$

Similarly, we may deduce that

$$\int_{-\infty}^{0}\int_x^{x+\nu(x)+\tau(x)z} s^{\cdot}(y)dy\phi(z)dz = -\tau(x)\int_0^{\infty} s^{\cdot}\big(x+\nu(x)+\tau(x)y\big)\Phi(y)dy. \tag{B.36}$$

Therefore, we may rewrite the integro-differential equation in (B.34) as

$$(\mu s^{\cdot})(x) + \frac{1}{2}(\sigma^2 s^{\cdot\cdot})(x) = -(\tau\lambda)(x)\int_{\mathbb{R}} s^{\cdot}\big(x+\nu(x)+\tau(x)z\big)\varphi(z)dz \tag{B.37}$$

due to (B.35) and (B.36).

For $f: \mathbb{R} \to \mathbb{R}$, let $s^{\cdot}(x) = \exp[F(x)]$, where $F(x) = \int_{-\infty}^{x} f(z)dz$. The integro-differential equation in (B.37) reduces to

$$\mu(x)\exp[F(x)] + \frac{1}{2}(\sigma^2 f)(x)\exp[F(x)] = -(\tau\lambda)(x)\int_{\mathbb{R}}\exp[F(x+\nu(x)+\tau(x)z)]\varphi(z)dz \tag{B.38}$$

if we replace $s^{\cdot}(\cdot)$ and $s^{\cdot\cdot}(\cdot)$ with $\exp[F(\cdot)]$ and $f(\cdot)\exp[F(\cdot)]$, respectively. Dividing both sides of (B.38) by $\exp[F(x)]$, we have

$$\left(\mu + \frac{1}{2}\sigma^2 f\right)(x) = -(\tau\lambda)(x)\int_{\mathbb{R}}\exp\left(\int_x^{x+\nu(x)+\tau(x)z} f(y)dy\right)\varphi(z)dz. \tag{B.39}$$

Therefore, we may see that there exists a strictly positive solution $s^{\cdot}$ to (B.37), as long as there exists a solution f to (B.39).

[17] Note that we have $\zeta_2(x) = \omega^2(x)$.

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To show the existence of a solution to (B.39), define a functional operator A as

$$(Af)(x) = -\frac{2\mu}{\sigma^2}(x) - \frac{2\tau\lambda}{\sigma^2}(x)\int_{\mathbb{R}} \exp\left(\int_x^{x+\nu(x)+\tau(x)z} f(y)dy\right)\varphi(z)dz. \tag{B.40}$$

Clearly, a fixed point f of A, that is, f satisfying $f = Af$, is a solution to (B.39). We define $\mathcal{F}_c$ as a space of functions that are bounded by $c > 0$ in absolute value, that is, $\sup_{x\in\mathbb{R}} |f(x)| \le c$ for $c > 0$. Also let $\mathcal{F}_c$ be endowed with the sup metric ρ. For any $f \in \mathcal{F}_c$, we have

$$|F(x+z) - F(x)| \le c|z| \tag{B.41}$$

for all $x, z \in \mathbb{R}$, due to the mean value theorem. It follows that

$$\sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}} \exp\left(\int_x^{x+\nu(x)+\tau(x)z} f(y)dy\right)\varphi(z)dz\right| \le \int_{\mathbb{R}} \exp\left[c(\bar{\nu}+\bar{\tau}|z|)\right]|\varphi(z)|dz = P_{c,\bar{\nu},\bar{\tau}},$$

which yields, together with $|(\mu/\sigma^2)(x)| \le \bar{\mu}_\sigma$, $(\tau\lambda/\sigma^2)(x) \le \bar{\tau}\bar{\lambda}_\sigma$, and $\bar{\mu}_\sigma + \bar{\tau}\bar{\lambda}_\sigma P_{c,\bar{\nu},\bar{\tau}} \le c$, that $\sup_{x\in\mathbb{R}} |(Af)(x)| \le c$ for all $f \in \mathcal{F}_c$. Therefore, A is a well-defined operator on $\mathcal{F}_c$.

Due to the contraction mapping theorem, it suffices to show that there exists $0 \le \alpha < 1$ such that

$$\sup_{x\in\mathbb{R}}\left|(Af_1)(x) - (Af_2)(x)\right| = \rho(Af_1, Af_2) \le \alpha\rho(f_1, f_2) = \alpha \sup_{x\in\mathbb{R}}\left|f_1(x) - f_2(x)\right| \tag{B.42}$$

for any $f_1, f_2 \in \mathcal{F}_c$. It follows from the mean value theorem that

$$\begin{aligned}
&\int_{\mathbb{R}}\left[\exp\left(\int_x^{x+\nu(x)+\tau(x)z} f_1(y)dy\right) - \exp\left(\int_x^{x+\nu(x)+\tau(x)z} f_2(y)dy\right)\right]\varphi(z)dz \\
&= \int_{\mathbb{R}}\int_x^{x+\nu(x)+\tau(x)z}(f_1(y) - f_2(y))dy \exp\left(\int_x^{x+\nu(x)+\tau(x)z}\tilde{f}(v)dv\right)\varphi(z)dz
\end{aligned} \tag{B.43}$$

for some $\tilde{f} \in \mathcal{F}_c$ such that $\int_x^{x+\nu(x)+\tau(x)z}\tilde{f}(v)dv$ takes a value between $\int_x^{x+\nu(x)+\tau(x)z} f_1(v)dv$ and $\int_x^{x+\nu(x)+\tau(x)z} f_2(v)dv$. Furthermore, we have

$$\begin{aligned}
&\int_0^\infty \int_x^{x+\nu(x)+\tau(x)z}(f_1(y) - f_2(y))dy \exp\left(\int_x^{x+\nu(x)+\tau(x)z}\tilde{f}(v)dv\right)\varphi(z)dz \\
&= \int_x^\infty (f_1(y) - f_2(y))\int_{\frac{y-x-\nu(x)}{\tau(x)}}^\infty \exp\left(\int_x^{x+\nu(x)+\tau(x)z}\tilde{f}(v)dv\right)\varphi(z)dzdy \\
&= \int_x^\infty (f_1(y) - f_2(y))\int_{\frac{y-x-\nu(x)}{\tau(x)}}^\infty \frac{\exp\left(\tilde{F}(x+\nu(x)+\tau(x)z)\right)}{\exp(\tilde{F}(x))}\varphi(z)dzdy \\
&= \tau(x)\int_0^\infty \left(f_1(x+\nu(x)+\tau(x)y) - f_2(x+\nu(x)+\tau(x)y)\right)\int_y^\infty \frac{\exp\left(\tilde{F}(x+\nu(x)+\tau(x)z)\right)}{\exp(\tilde{F}(x))}\varphi(z)dzdy,
\end{aligned} \tag{B.44}$$

where $\tilde{F}(x) = \int_{-\infty}^x \tilde{f}(z)dz$. Therefore, it follows from (B.40), (B.43), and (B.44) that

$$(Af_1 - Af_2)(x) = \frac{2\tau^2\lambda}{\sigma^2}(x)\int_{\mathbb{R}}\left(f_1(x+\nu(x)+\tau(x)y) - f_2(x+\nu(x)+\tau(x)y)\right)\Psi_x(y)dy, \tag{B.45}$$

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where

$$\Psi_x(y) = 1\{y \geq 0\} \int_y^{\infty} \exp\left[\tilde{F}\big(x+\nu(x)+\tau(x)z\big) - \tilde{F}(x)\right]\varphi(z)dz \\ - 1\{y < 0\} \int_{-\infty}^{y} \exp\left[\tilde{F}\big(x+\nu(x)+\tau(x)z\big) - \tilde{F}(x)\right]\varphi(z)dz.$$

Since $\tilde{F}$ satisfies the Lipschitz condition in (B.41), we have $|\Psi_x(y)| \leq |\Psi(y)|$ for all $y \in \mathbb{R}$, where $\Psi(y) = P_{c,\bar{\nu},\bar{\tau}} 1\{y \geq 0\} - \int_{-\infty}^{y} \exp[c(\bar{\nu}+\bar{\tau}|z|)]|\varphi(z)|dz$. We also have that

$$\sup_{x\in\mathbb{R}} \left|f_1(x+\nu(x)+\tau(x)y) - f_2(x+\nu(x)+\tau(x)y)\right| = \sup_{x\in\mathbb{R}} \left|f_1(x) - f_2(x)\right| \tag{B.46}$$

for all $y \in \mathbb{R}$. Therefore, we deduce from (B.45) and (B.46) that

$$\sup_{x\in\mathbb{R}} |Af_1 - Af_2|(x) \leq 2\bar{\tau}^2\bar{\lambda}_\sigma \int_{\mathbb{R}} |\Psi(y)| \sup_{x\in\mathbb{R}} \left|f_1(x+\nu(x)+\tau(x)y) - f_2(x+\nu(x)+\tau(x)y)\right| dy \\ = 2\bar{\tau}^2\bar{\lambda}_\sigma \sup_{x\in\mathbb{R}} \left|f_1(x) - f_2(x)\right| \int_{\mathbb{R}} |\Psi(y)|dy = 2\bar{\tau}^2\bar{\lambda}_\sigma Q_{c,\bar{\nu},\bar{\tau}} \sup_{x\in\mathbb{R}} \left|f_1(x) - f_2(x)\right|,$$

which yields (B.42), since $2\bar{\tau}^2\bar{\lambda}_\sigma Q_{c,\bar{\nu},\bar{\tau}} < 1$.

B.2. Proof of Lemma 3.3

We consider the following cases separately:

(i) $\sigma^2 \succcurlyeq (\nu^2+\tau^2)\lambda$, and $\mu \succcurlyeq \nu\lambda$,
(ii) $\sigma^2 \sim (\nu^2+\tau^2)\lambda$, $\nu^2 \prec \tau^2$, and $\mu \prec \nu\lambda$,
(iii) $\sigma^2 \succ (\nu^2+\tau^2)\lambda$, and $\mu \prec \nu\lambda$,
(iv) $\sigma^2 \prec (\nu^2+\tau^2)\lambda$, $\nu^2 \succcurlyeq \tau^2$, and $\mu \succ \nu\lambda$,
(v) $\sigma^2 \prec (\nu^2+\tau^2)\lambda$, $\nu^2 \prec \tau^2$, and $\mu \succcurlyeq \nu\lambda$,
(vi) $\sigma^2 \prec (\nu^2+\tau^2)\lambda$, $\nu^2 \prec \tau^2$, and $\mu \prec \nu\lambda$,
(vii) $\sigma^2 \preccurlyeq (\nu^2+\tau^2)\lambda$, $\nu^2 \succcurlyeq \tau^2$, and $\mu \preccurlyeq \nu\lambda$.

Here, we only provide the proofs of Cases (i), (ii), and (vii), since the proofs of other cases are analogous.

Proof of Case (i). In this case, we consider the differential equation

$$(s^{\cdot}\mu)(x) + \frac{1}{2}(s^{\cdot\cdot}\sigma^2)(x) = -(s^{\cdot}\nu\lambda)(x) - \frac{1}{2}[s^{\cdot\cdot}(\nu^2+\tau^2)\lambda](x), \tag{B.47}$$

which is a simplified version of the integro-differential equation in (5) ignoring the remainder term $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$, where $\tilde{x} = x + \alpha_z[\nu(x)+\tau(x)z]$ for some $\alpha_z \in [0,1]$. We will show that the solution to the differential equation in (B.47), which is given by

$$s^{\cdot}(x) = \exp\left(-\int_w^x \frac{2(\mu+\nu\lambda)}{\sigma^2+(\nu^2+\tau^2)\lambda}(u)du\right),$$

indeed guarantees that the remainder term is asymptotically negligible, in the sense that it is of smaller order than all four terms appearing in the differential equation in (B.47) at the boundaries of $\mathcal{D}$.

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To compare the asymptotic orders of $\lambda(x)\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$ and $(s^{\cdot}\mu)(x)$ at the boundaries of $\mathcal{D}$, we consider their ratio

$$\frac{\lambda(x)}{(s^{\cdot}\mu)(x)}\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$$
$$=\frac{(\nu+\tau)\lambda}{\mu}(x)\frac{xs^{\cdot\cdot}(x)}{s^{\cdot}(x)}\frac{xs^{\cdots}(x)}{s^{\cdot\cdot}(x)}\int_{\mathbb{R}}\frac{s^{\cdots}(\tilde{x})}{s^{\cdots}(x)}\frac{[\nu(x)+\tau(x)z]^3}{x^2(\nu+\tau)(x)}\phi(z)dz. \quad \textbf{(B.48)}$$

Similarly as in (B.61) and (B.67), we can show that

$$\frac{xs^{\cdot\cdot}(x)}{s^{\cdot}(x)}\to\kappa, \quad \frac{xs^{\cdots}(x)}{s^{\cdot\cdot}(x)}\to\kappa-1, \quad \int_{\mathbb{R}}\frac{s^{\cdots}(\tilde{x})}{s^{\cdots}(x)}\frac{[\nu(x)+\tau(x)z]^3}{x^2(\nu+\tau)(x)}\phi(z)dz\to 0 \quad \textbf{(B.49)}$$

as x approaches the boundaries of $\mathcal{D}$. Moreover, we have $\mu(x)x\succcurlyeq(\tau^2\lambda)(x)$, since otherwise we cannot have $s^{\cdot}\in RV_r$ for $r>-1$, due to the Karamata representation of regularly varying functions. Therefore, it follows from $\mu(x)x\succcurlyeq(\tau^2\lambda)(x)$ and $x\succ\tau(x)$ that

$$\mu(x)\succcurlyeq(\tau\lambda)(x). \quad \textbf{(B.50)}$$

Therefore, we deduce from (B.48), (B.49), and (B.50) that

$$\lambda(x)\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz\prec s^{\cdot}\mu(x) \quad \textbf{(B.51)}$$

at the boundaries of $\mathcal{D}$.

To compare $\lambda(x)\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$ with $(s^{\cdot\cdot}\sigma^2)(x)$, we look at their ratio

$$\frac{(\nu^2+\tau^2)\lambda}{\sigma^2}(x)\int_{\mathbb{R}}\frac{s^{\cdots}(\tilde{x})}{s^{\cdot\cdot}(x)}\frac{[\nu(x)+\tau(x)z]^3}{(\nu^2+\tau^2)(x)}\phi(z)dz \quad \textbf{(B.52)}$$
$$=-\frac{(\nu^2+\tau^2)\lambda}{\sigma^2}(x)\frac{xs^{\cdots}(x)}{s^{\cdot\cdot}(x)}\int_{\mathbb{R}}\frac{s^{\cdots}(\tilde{x})}{s^{\cdots}(x)}\frac{[\nu(x)+\tau(x)z]^3}{x(\nu^2+\tau^2)(x)}\phi(z)dz.$$

It follows from (B.49) and (B.52) that

$$\lambda(x)\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz\prec(s^{\cdot\cdot}\sigma^2)(x) \quad \textbf{(B.53)}$$

at the boundaries of $\mathcal{D}$. The asymptotic negligibility of $\lambda(x)\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$ relative to the other terms $(s^{\cdot}\nu\lambda)(x)$ and $[s^{\cdot\cdot}(\nu^2+\tau^2)\lambda](x)$ is easily obtained from (B.51) and (B.53), given that $\mu\succcurlyeq\nu\lambda$ and $\sigma^2\succcurlyeq(\nu^2+\tau^2)\lambda$. Therefore, the proof for Case (i) is complete.

Proof of Case (ii). For this case, we consider the differential equation

$$\frac{1}{2}(s^{\cdot\cdot}\sigma^2)(x)=-(s^{\cdot}\nu\lambda)(x)-\frac{1}{2}[s^{\cdot\cdot}(\nu^2+\tau^2)\lambda](x), \quad \textbf{(B.54)}$$

whose omitted remainder terms are $\lambda(x)\int_{\mathbb{R}} s^{\cdots}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$ and $(s^{\cdot}\mu)(x)$. The solution to the differential equation in (B.54) is given by

$$s^{\cdot}(x)=\exp\left(-\int_w^x\frac{2\nu\lambda}{\sigma^2+(\nu^2+\tau^2)\lambda}(u)du\right). \quad \textbf{(B.55)}$$

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To compare the asymptotic orders of $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$ and $[s^{\cdot\cdot}(\nu^2+\tau^2)\lambda](x)$, we consider their ratio

$$\int_{\mathbb{R}} \frac{s^{\cdot\cdot\cdot}(\tilde{x})}{s^{\cdot\cdot}(x)} \frac{[\nu(x)+\tau(x)z]^3}{(\nu^2+\tau^2)(x)}\phi(z)dz = \frac{x s^{\cdot\cdot\cdot}(x)}{s^{\cdot\cdot}(x)} \int_{\mathbb{R}} \frac{s^{\cdot\cdot\cdot}(\tilde{x})}{s^{\cdot\cdot\cdot}(x)} \frac{[\nu(x)+\tau(x)z]^3}{x(\nu^2+\tau^2)(x)}\phi(z)dz. \tag{B.56}$$

This ratio is asymptotically negligible due to (B.49), from which we deduce that $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz \prec [s^{\cdot\cdot}(\nu^2+\tau^2)\lambda](x)$. We may similarly show that $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz \prec (s^{\cdot\cdot}\sigma^2)(x)$.

To show $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz \prec (s^{\cdot}\nu\lambda)(x)$, note that we cannot have $s^{\cdot} \in RV_r$ for $r > -1$ if $\nu(x)x \prec \tau^2(x)$. This is because, if $\nu(x)x \prec \tau^2(x)$, then $(\nu\lambda)(x)x/[\sigma^2+(\nu^2+\tau^2)\lambda](x) \to 0$ as x approaches the boundaries of $\mathcal{D}$, which implies that $s(x)=\int_w^x s^{\cdot}(z)dz$ is slowly varying. Therefore, we may conclude that $\nu(x)x \succcurlyeq \tau^2(x)$, from which it follows that $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz \prec (s^{\cdot}\nu\lambda)(x)$. Similarly, we should have

$$s^{\cdot\cdot}\sigma^2 \succcurlyeq s^{\cdot}\nu\lambda \quad \text{and} \quad s^{\cdot\cdot}(\nu^2+\tau^2)\lambda \succcurlyeq s^{\cdot}\nu\lambda, \tag{B.57}$$

since otherwise, we cannot have $s^{\cdot} \in RV_r$ for $r > -1$. Therefore, we may deduce that $s^{\cdot\cdot}\sigma^2 \succ s^{\cdot}\mu$ and $s^{\cdot\cdot}(\nu^2+\tau^2)\lambda \succ s^{\cdot}\mu$ from (B.57). Furthermore, $s^{\cdot}\nu\lambda \succ s^{\cdot}\mu$ directly follows from $\nu\lambda \succ \mu$, from which the proof for Case (ii) is complete.

Proof of Case (vii). In this case, we consider the differential equation

$$(s^{\cdot}\mu)(x)+\frac{1}{2}(s^{\cdot\cdot}\sigma^2)(x) = -(s^{\cdot}\nu\lambda)(x)-\frac{1}{2}[s^{\cdot\cdot}(\nu^2+\tau^2)\lambda](x),$$

whose omitted remainder term is $\lambda(x)\int_{\mathbb{R}} s^{\cdot\cdot\cdot}(\tilde{x})[\nu(x)+\tau(x)z]^3\phi(z)dz$. The solution to this differential equation is given by

$$s^{\cdot}(x) = \exp\left(-\int_w^x \frac{2(\mu+\nu\lambda)}{\sigma^2+(\nu^2+\tau^2)\lambda}(u)du\right).$$

For $s(x)=\int_w^x s^{\cdot}(z)dz$ to be regularly varying, we must have $[\nu\lambda/(\nu^2\lambda)](x)x \to c$ for some $|c|<\infty$ as x approaches the boundaries of $\mathcal{D}$, due to the Karamata representation of regularly varying functions. However, this is impossible due to Assumption 2.1(c), which implies $\nu(x)x \succ \nu^2(x)$. Therefore, Case (vii) does not arise under our assumption.

B.3. Proof of Proposition 3.4

By the mean value theorem, we have

$$\begin{aligned}\frac{\omega^2}{s^{\cdot 2}(\nu^2+\tau^2)}(x) &= 1+\int_{\mathbb{R}} \frac{s^{\cdot\cdot}(\tilde{x})}{s^{\cdot}(x)} \frac{[\nu(x)+\tau(x)z]^3}{(\nu^2+\tau^2)(x)}\phi(z)dz + \frac{1}{4}\int_{\mathbb{R}} \frac{s^{\cdot\cdot 2}(\tilde{x})}{s^{\cdot 2}(x)} \frac{[\nu(x)+\tau(x)z]^4}{(\nu^2+\tau^2)(x)}\phi(z)dz \\ &= 1+R_1(x)+R_2(x)\end{aligned} \tag{B.58}$$

with $\tilde{x} = x+\alpha_z[\nu(x)+\tau(x)z]$ for some $0 \le \alpha_z \le 1$.

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We will show that $R_1(x)$ is asymptotically negligible as x approaches the right boundary of $\mathcal{D}$. Define

$$f(x,z) = \frac{[\nu(x)+\tau(x)z]^3}{x(\nu^2+\tau^2)(x)},$$

and rewrite $R_1(x)$ as

$$R_1(x) = \frac{xs^{\cdot\cdot}(x)}{s^{\cdot}(x)} \int_{\mathbb{R}} \frac{s^{\cdot\cdot}(\tilde{x})}{s^{\cdot\cdot}(x)} f(x,z)\phi(z)dz. \tag{B.59}$$

However, since

$$s^{\cdot\cdot}(x) = -\frac{2s^{\cdot}\mu}{\sigma^2}(x) - \frac{2\lambda}{\sigma^2}(x)\int_{\mathbb{R}} \Big(s\big[x+\nu(x)+\tau(x)z\big]-s(x)\Big)\phi(z)dz \tag{B.60}$$

and $s^{\cdot}$, μ, σ^2, λ, ν, and τ are all asymptotically monotonic, $s^{\cdot\cdot}$ is asymptotically monotone. Therefore, due to the monotone density theorem (see, e.g., Lamperti, 1958, Thm. 2), we have

$$\frac{xs^{\cdot\cdot}(x)}{s^{\cdot}(x)} \to \kappa_+ \tag{B.61}$$

as x approaches the right boundary of $\mathcal{D}$, if $s^{\cdot} \in RV_{\kappa_+}$ at the right boundary of $\mathcal{D}$.

Let $\nu \in RV_{\kappa_\nu}$ and $\tau \in RV_{\kappa_\tau}$ for some $\kappa_\nu, \kappa_\tau < 1$, and $\phi(\iota^3) < \infty$. In this case, we may write

$$\left|\int_{\mathbb{R}} f(x,z)\phi(z)dz\right| \le g(x)\int_{\mathbb{R}} |z|^3\phi(z)dz$$

for some function g such that $g(x) \to 0$ as x approaches the right boundary of $\mathcal{D}$, and therefore, it follows that

$$\int_{\mathbb{R}} f(x,z)\phi(z)dz \to 0 \tag{B.62}$$

as x approaches the right boundary of $\mathcal{D}$.

For $s^{\cdot\cdot} \in RV_{\kappa-1}$ with $\kappa-1<0$, it follows immediately from (B.59), (B.61), and (B.62) that $R_1(x) \to 0$, since $s^{\cdot\cdot}$ is bounded. Therefore, in what follows, we will only consider the case $s^{\cdot\cdot} \in RV_{\kappa-1}$ with $\kappa-1 \ge 0$.

By a change of variables, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{s^{\cdot\cdot}(\tilde{x})}{s^{\cdot\cdot}(x)} f(x,z)\phi(z)dz \\
&= \frac{x}{\alpha_z\tau(x)}\int_{\mathbb{R}} \frac{s^{\cdot\cdot}(x+xu)}{s^{\cdot\cdot}(x)} f\left(x, \frac{xu-\alpha_z\nu(x)}{\alpha_z\tau(x)}\right)\phi\left(\frac{xu-\alpha_z\nu(x)}{\alpha_z\tau(x)}\right)du \\
&= \frac{x}{\alpha_z\tau(x)}\int_{-1}^{\infty} \frac{s^{\cdot\cdot}\big(x(1+u)\big)}{s^{\cdot\cdot}(x)} f\left(x, \frac{xu-\alpha_z\nu(x)}{\alpha_z\tau(x)}\right)\phi\left(\frac{xu-\alpha_z\nu(x)}{\alpha_z\tau(x)}\right)du \\
&\quad + \frac{xs^{\cdot\cdot}(-x)}{\alpha_z\tau(x)s^{\cdot\cdot}(x)}\int_{-\infty}^{-1} \frac{s^{\cdot\cdot}\big(-x|1+u|\big)}{s^{\cdot\cdot}(-x)} f\left(x, \frac{xu-\alpha_z\nu(x)}{\alpha_z\tau(x)}\right)\phi\left(\frac{xu-\alpha_z\nu(x)}{\alpha_z\tau(x)}\right)du.
\end{aligned} \tag{B.63}$$

Let $s^{\cdot} \in RV_{\kappa_+}$ and $s^{\cdot} \in RV_{\kappa_-}$ at the right and the left boundaries of $\mathcal{D}$, respectively. Then, due to the Karamata theorem, we have $s^{\cdot\cdot} \in RV_{\kappa_+-1}$ and $s^{\cdot\cdot} \in RV_{\kappa_--1}$ at the right and the

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left boundaries of $\mathcal{D}$, respectively, from which we deduce that

$$\left|\frac{s^{\cdot\cdot}(xu)}{s^{\cdot\cdot}(x)}\right| \leq c + u^{\kappa_+ + \varepsilon - 1}, \qquad \left|\frac{s^{\cdot\cdot}(-xu)}{s^{\cdot\cdot}(-x)}\right| \leq c + u^{\kappa_- + \varepsilon - 1}$$

for some $c > 0$, any $u, \varepsilon > 0$ and all large x. Therefore, the sum of the last two terms in (B.63) is bounded by

$$\begin{aligned} &\frac{x}{\alpha_z \tau(x)} \int_{-1}^{\infty} \left(c + (1+u)^{\kappa_+ + \varepsilon - 1}\right) f\left(x, \frac{xu - \alpha_z \nu(x)}{\alpha_z \tau(x)}\right) \phi\left(\frac{xu - \alpha_z \nu(x)}{\alpha_z \tau(x)}\right) du \\ &+ \frac{x s^{\cdot\cdot}(-x)}{\alpha_z \tau(x) s^{\cdot\cdot}(x)} \int_{-\infty}^{-1} \left(c + |1+u|^{\kappa_- + \varepsilon - 1}\right) f\left(x, \frac{xu - \alpha_z \nu(x)}{\alpha_z \tau(x)}\right) \phi\left(\frac{xu - \alpha_z \nu(x)}{\alpha_z \tau(x)}\right) du \end{aligned} \tag{B.64}$$

for some $c > 0$, any $\varepsilon > 0$ and all large x.

Using a change of variables again, we may write (B.64) as

$$\begin{aligned} &\int_{-[x/\alpha_z + \nu(x)]/\tau(x)}^{\infty} \left(c + \left(1 + \frac{\alpha_z[\nu(x) + \tau(x) z]}{x}\right)^{\kappa_+ + \varepsilon - 1}\right) f(x,z)\phi(z) dz \\ &+ \frac{s^{\cdot\cdot}(-x)}{s^{\cdot\cdot}(x)} \int_{-\infty}^{-[x/\alpha_z + \nu(x)]/\tau(x)} \left(c + \left|1 + \frac{\alpha_z[\nu(x) + \tau(x) z]}{x}\right|^{\kappa_- + \varepsilon - 1}\right) f(x,z)\phi(z) dz. \end{aligned} \tag{B.65}$$

It follows from (B.63), (B.64), and (B.65) that

$$\begin{aligned} \left|\int_{\mathbb{R}} \frac{s^{\cdot\cdot}(\tilde{x})}{s^{\cdot\cdot}(x)} f(x,z)\phi(z) dz\right| \leq\; & c \int_{-[x+\nu(x)]/\tau(x)}^{\infty} \left(1 + \frac{|z|^{k_+}}{x^{1-\kappa_\tau + k_+}}\right) \frac{1}{x^k} \phi(z) dz \\ & + c \frac{s^{\cdot\cdot}(-x)}{s^{\cdot\cdot}(x)} \int_{-\infty}^{-[x+\nu(x)]/\tau(x)} \left(1 + \frac{|z|^{k_-}}{x^{1-\kappa_\tau + k_-}}\right) \frac{1}{x^k} \phi(z) dz \end{aligned} \tag{B.66}$$

for some $c > 0$, any $k_+ > \kappa_+ - 1$, any $k_- > \kappa_- - 1$, any $0 < k < 1 - \kappa_\nu \vee \kappa_\tau$, and all large x, since $\nu \in RV_{\kappa_\nu}$ and $\tau \in RV_{\kappa_\tau}$ for some $\kappa_\nu, \kappa_\tau < 1$. Moreover, we have $\int_{\mathbb{R}} |z|^{\kappa_+ \vee \kappa_- - 1 + \varepsilon} \phi(z) dz < \infty$ for any small $\varepsilon > 0$. Therefore, the right-hand side of (B.66) is bounded by

$$\begin{aligned} &\frac{1}{x^{1-\kappa_\nu \vee \kappa_\tau - \varepsilon}} \left(a + b \frac{s^{\cdot\cdot}(-x)}{s^{\cdot\cdot}(x)} \int_{-\infty}^{-[x+\nu(x)]/\tau(x)} \phi(z) dz\right) \\ &= \frac{1}{x^{1-\kappa_\nu \vee \kappa_\tau - \varepsilon}} \left(a + b \frac{s^{\cdot\cdot}(-x)}{s^{\cdot\cdot}(x)} \Phi\left(-\frac{x + \nu(x)}{\tau(x)}\right)\right) \\ &\leq \frac{1}{x^{1-\kappa_\nu \vee \kappa_\tau - \varepsilon}} \left(a + b \frac{s^{\cdot\cdot}(-x)}{s^{\cdot\cdot}(x)} \Phi(-x^{1-\kappa_\tau + \varepsilon})\right) \to 0 \end{aligned}$$

for some $a, b > 0$ and any $\varepsilon > 0$ as x approaches the right boundary of $\mathcal{D}$, where $\Phi(x) = \int_{-\infty}^{x} \phi(z) dz$, from which it follows that

$$\int_{\mathbb{R}} \frac{s^{\cdot\cdot}(\tilde{x})}{s^{\cdot\cdot}(x)} f(x,z)\phi(z) dz \to 0 \tag{B.67}$$

as x approaches the right boundary of $\mathcal{D}$.

As was to be shown, we obtain from (B.59), (B.61), and (B.67) that $R_1(x) \to 0$ as x approaches the right boundary of $\mathcal{D}$. The asymptotic negligibility at the left boundary can

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be shown similarly. The proof of the asymptotic negligibility of $R_2(x)$ is entirely analogous, and therefore, it is omitted.

B.4. Proof of Theorem 4.1

The proof is omitted, since it is essentially the same as the proof of Theorem 4.4(a).

B.5. Proof of Proposition 4.2

Write $X^T = B^T \circ A^T$, where

$$A_t^T = T_r^{-2} \int_0^{Tt} 1/m(X_u)du$$

and B^T is defined in Lemma A1. As in the proof of Lemma A1, we may assume without loss of generality that $B^T \to_{a.s.} B^\circ$ as $T \to \infty$. To derive the stated result, it suffices to show that

$$\sup_{t\in[0,K]} \left| \frac{1}{T_r^r} \int_0^t m\big(T_r B_u^T\big)du - \int_{\mathbb{R}} \bar{m}(x)L^\circ(t,x)dx \right| \to_{a.s.} 0 \tag{B.68}$$

as $T \to \infty$ for any $K < \infty$. Once (B.68) is established, we have $A^T \to_{a.s.} \bar{A}$, where

$$\bar{A}_t = \inf \left\{ v \middle| \int_{\mathbb{R}} \bar{m}(x)L^\circ(v,x)dx > t \right\},$$

as shown in Lemma A.1 of Kim and Park (2017).

To derive (B.68), we write

$$\frac{1}{T_r^r} \int_0^t m\big(T_r B_u^T\big)du - \int_{\mathbb{R}} \bar{m}(x)L^\circ(t,x)dx$$
$$= \frac{1}{T_r^r} \int_0^t \big[m\big(T_r B_u^T\big) - m\big(T_r B_u^\circ\big)\big]du + \left[\int_0^t \frac{m\big(T_r B_u^\circ\big)}{T_r^r} du - \int_{\mathbb{R}} \bar{m}(x)L^\circ(t,x)dx \right] = P_t^T + Q_t^T.$$

As shown in Lemma A.4 of Kim and Park (2017), we have $\sup_{t\in[0,K]} |Q_t^T| \to_{a.s.} 0$ as $T \to \infty$ for arbitrary finite $K > 0$.

To obtain the asymptotics of P_t^T, define for any given $\epsilon > 0$

$$m_T^\epsilon(x) = \frac{m(T_r x)}{T_r^r} 1\{|x| > \epsilon\} + \tilde{m}_T(x) 1\{|x| \le \epsilon\},$$

where we let $\tilde{m}_T$ be a nonnegative and differentiable function such that

$$\tilde{m}_T(\pm\epsilon) = \frac{m(\pm\epsilon T_r)}{T_r^r} \qquad \text{and} \qquad \tilde{m}_T^{\cdot}(\pm\epsilon) = \frac{T_r m^{\cdot}(\pm\epsilon T_r)}{T_r^r}$$

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for all T, and $\sup_{|x|<\epsilon}|\tilde{m}_T(x)| < M_\epsilon$, $\sup_{|x|<\epsilon}|\tilde{m}^{\cdot}_T(x)| < M_\epsilon$ for some $M_\epsilon > 0$ and all large T.[18] Then we have from the occupation times formula that

$$|P_t^T| = \frac{1}{T_r^r}\left|\int_{\mathbb{R}} m(T_r x)\big[L^T(t,x) - L^\circ(t,x)\big]dx\right| \le P_t^{1T} + P_t^{2T} + P_t^{3T},$$

where

$$P_t^{1T} = \sup_{|x|\le\epsilon}\left|L^T(t,x) - L^\circ(t,x)\right|\frac{1}{T_r^r}\int_{|x|\le\epsilon} m(T_r x)dx,$$

$$P_t^{2T} = \left|\int_0^t \big[m_T^\epsilon(B_u^T) - m_T^\epsilon(B_u^\circ)\big]du\right|,$$

$$P_t^{3T} = \sup_{|x|\le\epsilon}\left|L^T(t,x) - L^\circ(t,x)\right|\int_{|x|\le\epsilon}\tilde{m}_T(x)dx.$$

For P_t^{2T}, we obtain from the mean value theorem that

$$P_t^{2T} \le t\sup_{u\in[0,t]}\left|B_u^T - B_u^\circ\right|\left(\frac{T_r\sup_{|x|\in[\epsilon,M]}\left|m^{\cdot}(T_r x)\right|}{T_r^r} + \sup_{|x|\le\epsilon}\left|\tilde{m}^{\cdot}_T(x)\right|\right) \tag{B.69}$$

for some $M > 0$ such that $\sup_{u\in[0,t]}|B_u^\circ| < M$. Moreover, we have

$$\frac{T_r\sup_{|x|\in[\epsilon,M]}\left|m^{\cdot}(T_r x)\right|}{T_r^r} < \infty \tag{B.70}$$

for all large T, due to the monotone density theorem (see, e.g., Soulier, 2009, Thm. 1.20). Therefore, we obtain from (B.69), (B.70), Lemma A1, and the uniform boundedness of $\tilde{m}^{\cdot}_T$, that $\sup_{t\in[0,K]}P_t^{2T} \to_{a.s.} 0$ as $T\to\infty$.

For $P_{1t,T}$ and $P_{3t,T}$, we have from Lemma A1 that

$$\sup_{t\in[0,K]}\sup_{|x|\le\epsilon}\left|L^T(t,x) - L^\circ(t,x)\right| \to_{a.s.} 0$$

for any $K > 0$ and all large T. Also, we have

$$\frac{1}{T_r^r}\int_{|x|\le\epsilon} m(T_r x)dx \to \int_{|x|\le\epsilon}\bar{m}(x)dx < \infty$$

as $T\to\infty$, due to the Karamata theorem and the Potter theorem (see, e.g., Soulier, 2009, Prop. 1.18). Therefore, we deduce that $\sup_{t\in[0,K]}P_t^{1T} \to_{a.s.} 0$ and $\sup_{t\in[0,K]}P_t^{3T} \to_{a.s.} 0$ as $T\to\infty$. Consequently, we have (B.68), and the proof is complete.

B.6. Proof of Proposition 4.3

In the first part of the proof, we show the joint convergence to X°, W°, and J°. In the second part, we further decompose J° into N° and Z°. For clarity, here we put the superscript and subscript "s" in X, m, $\bar{m}$, σ^2, ω^2, and λ, and write them as X^s, m_s, $\bar{m}_s$, σ_s^2, ω_s^2, and λ_s, respectively.

[18] The existence of such m_T^ϵ is guaranteed by the regular variation of m and the monotone density theorem.

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Joint Convergence to X°, W°, and J° Rewrite the SDE in (A.1) as

$$dX_t^s = \sigma_s(X_t^s)dW_t + \left(\omega_s\lambda_s^{1/2}\right)(X_{t-}^s)\sqrt{T}dJ_{t/T}^T, \tag{B.71}$$

where

$$J_t^T = \frac{1}{\sqrt{T}}\int_0^{Tt}\left[\frac{s[X_{u-}+\nu(X_{u-})+\tau(X_{u-})Z_u]-s(X_{u-})}{(\omega\lambda^{1/2})(X_{u-})}dN_u\left(\lambda(X_{u-})\right)-\frac{\zeta_1\lambda^{1/2}}{\omega}(X_u)du\right]. \tag{B.72}$$

Multiplying $m_s^{1/2}(X_{t-}^s)$ on both sides of (B.71), we have

$$\frac{1}{\sqrt{T}}\int_0^{Tt} m_s^{1/2}(X_{u-}^s)dX_u^s = \int_0^t \left(m_s^{1/2}\sigma_s\right)(T_rX_u^T)dW_u^T + \int_0^t \left(m_s^{1/2}\omega_s\lambda_s^{1/2}\right)(T_rX_{u-}^T)dJ_u^T. \tag{B.73}$$

It follows from Lemma A2 that $J^T \to_d J^\circ$ as $T\to\infty$ for some Brownian motion J° independent of W°. As in the proof of Proposition 3.4, we may show that $(\zeta_4/\zeta_2^2)(x)\le M$ for some $M>0$ and all $x\in\mathcal{D}$, which is required in Lemma A2. Moreover, we have

$$\begin{aligned}\left(m_s\sigma_s^2\right)(x) &\sim p_c 1\{x\ge 0\}+q_c 1\{x<0\},\\ \left(m_s\omega_s^2\lambda_s\right)(x) &\sim (1-p_c)1\{x\ge 0\}+(1-q_c)1\{x<0\}\end{aligned}$$

as x approaches the boundaries of $\mathbb{R}$, and $m_s\sigma_s^2$ and $m_s\omega_s^2\lambda_s$ are locally bounded. Therefore, it follows that

$$\begin{aligned}\int_0^t \left(m_s^{1/2}\sigma_s\right)(T_rX_u^T)dW_u^T &\to_d \int_0^t \left(\sqrt{p_c}1\{X_u^\circ\ge 0\}+\sqrt{q_c}1\{X_u^\circ<0\}\right)dW_u^\circ,\\ \int_0^t \left(m_s^{1/2}\omega_s\lambda_s^{1/2}\right)(T_rX_{u-}^T)dJ_u^T &\to_d \int_0^t \left(\sqrt{1-p_c}1\{X_u^\circ\ge 0\}+\sqrt{1-q_c}1\{X_u^\circ<0\}\right)dJ_u^\circ\end{aligned} \tag{B.74}$$

as $T\to\infty$ for $t\ge 0$. See the proof of Theorem 4.4(b) for more details.

Now we show that

$$\int_0^t \bar{m}_s^{1/2}(X_{u-}^T)dX_u^T \to_d \int_0^t \bar{m}_s^{1/2}(X_u^\circ)dX_u^\circ \tag{B.75}$$

as $T\to\infty$. Let

$$\int_0^t \bar{m}_s^{1/2}(X_{u-}^T)dX_u^T = \int_0^t \bar{m}_s^{1/2}(X_u^\circ)dX_u^T + U_t^T, \tag{B.76}$$

where

$$U_t^T = \int_0^t \left[\bar{m}_s^{1/2}(X_u^T)-\bar{m}_s^{1/2}(X_u^\circ)\right]dX_u^T.$$

We have

$$\int_0^t \bar{m}_s^{1/2}(X_u^\circ)dX_u^T \to_d \int_0^t \bar{m}_s^{1/2}(X_u^\circ)dX_u^\circ \tag{B.77}$$

as $T\to\infty$, due to Jacod and Shiryaev (2003, Cor. IX.5.18).

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As before, we may assume without loss of generality that $X^T \to_{a.s.} X^\circ$. Note that U^T is a martingale whose quadratic variation is given by

$$[U^T]_t = \int_0^t \left[\bar{m}_s^{1/2}(X_{u-}^T) - \bar{m}_s^{1/2}(X_u^\circ)\right]^2 d[X^T]_u. \tag{B.78}$$

Moreover, it follows from (A.1) that

$$\begin{aligned} d[X^T]_t &= \frac{T}{T_r^2}\sigma_s^2(T_r X_t^T)dt \\ &\quad + \frac{1}{T_r^2}\Big(s\big[s^{-1}(T_r X_{t-}^T) + (\nu\circ s^{-1})(T_r X_{t-}^T) + (\tau\circ s^{-1})(T_r X_{t-}^T)Z_t\big] - T_r X_{t-}^T\Big)^2 dN_t\big(T\lambda_s(T_r X_{t-}^T)\big) \\ &= \frac{T}{T_r^2}\frac{1}{m_s(T_r X_t^T)}dt + \frac{\sqrt{T}}{T_r^2}(\zeta_{4s}\lambda_s)^{1/2}(T_r X_{t-}^T)dV_t^T, \end{aligned} \tag{B.79}$$

where $\zeta_{4s} = \zeta_4 \circ s^{-1}$ and

$$\begin{aligned} V_t^T &= \int_0^t \left(\frac{\big(s\big[s^{-1}(T_r X_{u-}^T) + (\nu\circ s^{-1})(T_r X_{u-}^T) + (\tau\circ s^{-1})(T_r X_{u-}^T)Z_u\big] - T_r X_{u-}^T\big)^2}{\sqrt{T}(\zeta_{4s}\lambda_s)^{1/2}(T_r X_{u-}^T)} dN_u\big(T\lambda_s(T_r X_{u-}^T)\big) \right. \\ &\qquad \left. - \frac{\sqrt{T}(\omega_s^2\lambda_s)(T_r X_u^T)}{(\zeta_{4s}\lambda_s)^{1/2}(T_r X_u^T)}du\right) \\ &= \frac{1}{\sqrt{T}}\int_0^{Tt}\left(\frac{\big(s\big[X_{u-} + \nu(X_{u-}) + \tau(X_{u-})Z_u\big] - s(X_{u-})\big)^2}{(\zeta_4\lambda)^{1/2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \frac{(\omega^2\lambda^{1/2})(X_u)}{\zeta_4^{1/2}(X_u)}du\right). \end{aligned}$$

Therefore, we have

$$[U^T]_t = P_t^T + Q_t^T, \tag{B.80}$$

where

$$\begin{aligned} P_t^T &= \frac{T}{T_r^2}\int_0^t \big[\bar{m}_s^{1/2}(X_u^T) - \bar{m}_s^{1/2}(X_u^\circ)\big]^2 \frac{1}{m_s(T_r X_u^T)}du \\ Q_t^T &= \frac{\sqrt{T}}{T_r^2}\int_0^t \big[\bar{m}_s^{1/2}(X_{u-}^T) - \bar{m}_s^{1/2}(X_u^\circ)\big]^2 (\zeta_{4s}\lambda_s)^{1/2}(T_r X_{u-}^T)dV_u^T, \end{aligned}$$

from (B.79).

For P^T, we have

$$\begin{aligned} P_t^T &= \int_0^t \big[\bar{m}_s^{1/2}(X_u^T) - \bar{m}_s^{1/2}(X_u^\circ)\big]^2 \frac{T_r^r}{m_s(T_r X_u^T)}du \\ &\to_d \int_0^t \big[\bar{m}_s^{1/2}(X_u^\circ) - \bar{m}_s^{1/2}(X_u^\circ)\big]^2 \frac{1}{\bar{m}_s(X_u^\circ)}du = 0 \end{aligned}$$

as $T \to \infty$ for all $t \geq 0$. See the proof of Theorem 4.4(b) for more details. For Q^T, we may readily establish that $Q_t^T \to_p 0$ as $T \to \infty$ for all $t \geq 0$. In fact, we may show that $(\zeta_8/\zeta_4^2)(x) \leq M$ for some $M > 0$ and all $x \in \mathcal{D}$, similarly as in the proof of Proposition 3.4, and therefore, it follows from Lemma A2 that $V^T \to_d V^\circ$ as $T \to \infty$ for some standard Brownian motion V°. Also, we note that ζ_{4s} is locally bounded, due to the local integrability of s and the local boundedness of s^{-1}, ν and τ, and deduce from Assumptions 2.1(c), 3.1, and 4.1(b) that $\zeta_{4s}\lambda_s \sim [s^{\cdot 4}(\nu^4 + \tau^4)\lambda] \circ s^{-1} \in RV_k$ for some $k < -r + 2$ as in the proof of

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Proposition 3.4. Therefore, $\zeta_{4s}\lambda_s$ satisfies the conditions in Definition 4.1, and as a result, it follows that

$$\kappa\big((\zeta_{4s}\lambda_s)^{1/2},T_r\big)^{-1}\int_0^t\big[\bar m_s^{1/2}(X_{u-}^T)-\bar m_s^{1/2}(X_u^\circ)\big]^2(\zeta_{4s}\lambda_s)^{1/2}(T_rX_{u-}^T)dR_u^T$$
$$\to_d \int_0^t\big[\bar m_s^{1/2}(X_u^\circ)-\bar m_s^{1/2}(X_u^\circ)\big]^2h\big((\zeta_{4s}\lambda_s)^{1/2},X_u^\circ\big)dR_u^\circ=0 \quad \textbf{(B.81)}$$

as $T\to\infty$ for $t\geq 0$. See the proof of Theorem 4.4(b) for more details. Furthermore, we have

$$\frac{\sqrt{T}}{T_r^2}\kappa\big((\zeta_{4s}\lambda_s)^{1/2},T_r\big)\to 0 \quad \textbf{(B.82)}$$

as $T\to\infty$, since $\zeta_{4s}\lambda_s\in RV_k$ for some $k<-r+2$. Therefore, we deduce from (B.81) and (B.82) that $Q_t^T\to_p 0$ as $T\to\infty$ for all $t\geq 0$.

The asymptotic negligibility of P^T and Q^T implies that $U_t^T\to_p 0$ as $T\to\infty$, from which, together with (B.76) and (B.77), we establish (B.75). Consequently, we obtain (16) from (B.73), (B.74), and (B.75).

Decomposition of J° into $N^\circ$and Z° In this part, we further decompose J° into N° and Z°. Write

$$s[x+\nu(x)+\tau(x)z]-s(x)=(s^{\cdot}\nu)(x)+(s^{\cdot}\tau)(x)z+r(x,z), \quad \textbf{(B.83)}$$

where $r(x,z)=s[x+\nu(x)+\tau(x)z]-s(x)-(s^{\cdot}\nu)(x)-(s^{\cdot}\tau)(x)z$. Then it follows from (B.72) and (B.83) that

$$J_t^T=P_t^T+Q_t^T+R_t^T, \quad \textbf{(B.84)}$$

where

$$P_t^T=\frac{1}{\sqrt{T}}\int_0^{Tt}\frac{s^{\cdot}\nu}{\omega}(X_{u-})\left[\frac{1}{\lambda^{1/2}(X_{u-})}dN_u\big(\lambda(X_{u-})\big)-\lambda^{1/2}(X_u)du\right]=\int_0^{Tt}\frac{s^{\cdot}\nu}{\omega}(X_{u-})dN_{u/T}^T,$$
$$Q_t^T=\frac{1}{\sqrt{T}}\int_0^{Tt}\frac{s^{\cdot}\tau}{\omega}(X_{u-})\frac{Z_u}{\lambda^{1/2}(X_{u-})}dN_u\big(\lambda(X_{u-})\big)=\int_0^{Tt}\frac{s^{\cdot}\tau}{\omega}(X_{u-})dM_{u/T}^T,$$
$$R_t^T=\frac{1}{\sqrt{T}}\int_0^{Tt}\frac{r_2^{1/2}}{\omega}(X_{u-})\left[\frac{r(X_{u-},Z_u)}{(r_2\lambda)^{1/2}(X_{u-})}dN_u\big(\lambda(X_{u-})\big)-\frac{r_1\lambda^{1/2}}{r_2^{1/2}}(X_u)du\right]=\int_0^{Tt}\frac{r_2^{1/2}}{\omega}(X_{u-})dV_{u/T}^T,$$

where in turn $r_1(x)=\int_{\mathbb{R}}r(x,z)\phi(z)dz=\zeta_1(x)-(s^{\cdot}\nu)(x)$, $r_2(x)=\int_{\mathbb{R}}r^2(x,z)\phi(z)dz$, and

$$V_t^T=\frac{1}{\sqrt{T}}\int_0^{Tt}\left[\frac{r(X_{u-},Z_u)}{(r_2\lambda)^{1/2}(X_{u-})}dN_u\big(\lambda(X_{u-})\big)-\frac{r_1\lambda^{1/2}}{r_2^{1/2}}(X_u)du\right].$$

To derive the asymptotics of P^T, Q^T, and R^T, we first show that $(s^{\cdot}\tau/\omega)\circ s^{-1}$, $(s^{\cdot}\nu/\omega)\circ s^{-1}$ and $(r_2^{1/2}/\omega)\circ s^{-1}$ satisfy the conditions in Definition 4.1. For $(s^{\cdot}\tau/\omega)\circ s^{-1}$, we have from Proposition 3.4 that

$$\frac{s^{\cdot 2}\tau^2}{\omega^2}(x)\sim\frac{\tau^2}{\nu^2+\tau^2}(x)\sim(1-p_z)1\{x\geq 0\}+(1-q_z)1\{x<0\} \quad \textbf{(B.85)}$$

as x approaches the boundaries of $\mathcal{D}$. For $(s^{\cdot}\nu/\omega)\circ s^{-1}$, we may similarly show that $(s^{\cdot}\nu/\omega)^2(x)\sim p_z1\{x\geq 0\}+q_z1\{x<0\}$ as x approaches the boundaries of $\mathcal{D}$. Moreover,

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$(s^{\cdot}\tau)/\omega$ and $(s^{\cdot}\nu)/\omega$ are locally bounded from $\omega(x) > 0$ for all $x \in \mathcal{D}$ and the differentiability of $s^{\cdot}$, ν and τ. Therefore, $(s^{\cdot}\tau/\omega)\circ s^{-1}$ and $(s^{\cdot}\nu/\omega)\circ s^{-1}$ satisfy the conditions in Definition 4.1.

For $(r_2^{1/2}/\omega)\circ s^{-1}$, we write

$$\frac{r_2(x)}{\big(s^{\cdot\cdot 2}(\nu^4+\tau^4)\big)(x)} = \frac{1}{4}\frac{\nu^4+6\nu^2\tau^2+4z_3\nu\tau^3+z_4\tau^4}{\nu^4+\tau^4}(x)+r(x), \tag{B.86}$$

where

$$r(x) = \frac{1}{6}\int_{\mathbb{R}}\left(\frac{1}{6}\frac{s^{\cdots 2}(\tilde{x})}{s^{\cdot\cdot 2}(x)}\frac{[\nu(z)+\tau(x)z]^6}{(\nu^4+\tau^4)(x)}+\frac{s^{\cdots}(\tilde{x})}{s^{\cdot\cdot}(x)}\frac{[\nu(x)+\tau(x)z]^5}{(\nu^4+\tau^4)(x)}\right)\phi(z)dz,$$

$z_k = \int_{\mathbb{R}} z^k\phi(z)dz$ for $k = 3,4$, and $\tilde{x} = x+\alpha_z[\nu(x)+\tau(x)z]$ for some $0 \le \alpha_z \le 1$. It follows from (5) that

$$\begin{aligned} s^{\cdots}(x) = &-\frac{2(s^{\cdot\cdot}\mu+s^{\cdot}\mu^{\cdot})}{\sigma^2}(x)-\frac{2s^{\cdot\cdot}\sigma^{\cdot}}{\sigma}(x)-\frac{2\lambda^{\cdot}}{\sigma^2}(x)\int_{\mathbb{R}}\Big(s\big[x+\nu(x)+\tau(x)z\big]-s(x)\Big)\phi(z)dz \\ &-\frac{2\lambda}{\sigma^2}(x)\int_{\mathbb{R}}\Big(s^{\cdot}\big[x+\nu(x)+\tau(x)z\big]-s^{\cdot}(x)\Big)\phi(z)dz \\ &-\frac{2\nu^{\cdot}\lambda}{\sigma^2}(x)\int_{\mathbb{R}}s^{\cdot}\big[x+\nu(x)+\tau(x)z\big]\phi(z)dz-\frac{2\tau^{\cdot}\lambda}{\sigma^2}(x)\int_{\mathbb{R}}s^{\cdot}\big[x+\nu(x)+\tau(x)z\big]z\phi(z)dz. \end{aligned}$$

Due to the differentiability and asymptotic monotonicity of $s^{\cdot}$, $s^{\cdot\cdot}$, μ, σ, λ, ν, and τ, we may deduce that $s^{\cdots}$ is locally bounded and asymptotically monotone, which implies that $r(x)$ is asymptotically negligible as x approaches the boundaries of $\mathcal{D}$ (see the proof of Proposition 3.4 for the details). From (B.86) and the asymptotic negligibility of $r(x)$, we deduce that $r_2 \sim cs^{\cdot\cdot 2}(\nu^4+\tau^4)$ for some $c \neq 0$ as x approaches the boundaries of $\mathcal{D}$. Therefore, we have $r_2^{1/2}/\omega \preccurlyeq f$ for some $f : \mathcal{D} \to \mathbb{R}$ such that $f \in RV_k$ with $k = \max\{a-1, b-1\}$, where a and b are constants such that $|\nu| \in RV_a$ and $|\tau| \in RV_b$, due to the monotone density theorem. Moreover, we deduce that $r_2^{1/2}/\omega$ is locally bounded from $\omega(x) > 0$ for all $x \in \mathcal{D}$ and the differentiability of r_2. Consequently, $(r_2^{1/2}/\omega)\circ s^{-1}$ satisfies the conditions in Definition 4.1.

Now, we derive the asymptotics of P^T, Q^T, and R^T. For R^T, we note that $(r_4/r_2^2)(x) \le M$ for some $M > 0$ and all $x \in \mathcal{D}$, where $r_4(x) = \int_{\mathbb{R}} r^4(x,z)\phi(z)dz$, which can be shown as in the proof of Proposition 3.4. Therefore, $V^T \to_d V^\circ$ as $T \to \infty$ for some standard Brownian motion V°, due to Lemma A2. However, since $V^T \to_d V^\circ$ as $T \to \infty$, and we have already shown that $(r_2^{1/2}/\omega)\circ s^{-1}$ satisfies the conditions in Definition 4.1, we may deduce that

$$R_t^T = \kappa(\xi, T_r)\frac{1}{\kappa(\xi, T_r)}\int_0^t \frac{r_2^{1/2}}{\omega}(X_{Tu-})dV_u^T = o(1)\left(\int_0^t h(\xi, X_u^\circ)V_u^\circ + o_p(1)\right) = o_p(1) \tag{B.87}$$

as $T \to \infty$ for $t \ge 0$, where $\xi = (r_2^{1/2}/\omega)\circ s^{-1}$. See the proof of Theorem 4.4(b) for details.

For P^T and Q^T, we first deduce from Lemma A2 that $N^T \to_d N^\circ$ and $Z^T \to_d Z^\circ$ as $T \to \infty$. Moreover, we have $(s^{\cdot}\tau/\omega)\circ s^{-1}(x) \sim \sqrt{1-p_z}1\{x \ge 0\}+\sqrt{1-q_z}1\{x < 0\}$ and $(s^{\cdot}\nu/\omega)\circ s^{-1}(x) \sim \sqrt{p_z}1\{x \ge 0\}+\sqrt{q_z}1\{x < 0\}$ at the boundaries of $\mathbb{R}$, and we have already shown that $(s^{\cdot}\tau/\omega)\circ s^{-1}$ and $(s^{\cdot}\nu/\omega)\circ s^{-1}$ satisfy the conditions in Definition 4.1.

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Therefore, we have

$$
\begin{aligned}
P_t^T &\to_d \int_0^t \Big(\sqrt{p_z}1\{X_u^\circ \geq 0\} + \sqrt{p_z}1\{X_u^\circ < 0\}\Big) dN_u^\circ, \\
Q_t^T &\to_d \int_0^t \Big(\sqrt{1-p_z}1\{X_u^\circ \geq 0\} + \sqrt{1-q_z}1\{X_u^\circ < 0\}\Big) dZ_u^\circ,
\end{aligned}
\tag{B.88}
$$

as $T \to \infty$ for $t \geq 0$. See the proof of Theorem 4.4(b) for the details. Now it follows from (B.84), (B.87), and (B.88) that

$$
\begin{aligned}
J_t^\circ = &\int_0^t \Big(\sqrt{p_z}1\{X_u^\circ \geq 0\} + \sqrt{p_z}1\{X_u^\circ < 0\}\Big) dN_u^\circ \\
&+ \int_0^t \Big(\sqrt{1-p_z}1\{X_u^\circ \geq 0\} + \sqrt{1-q_z}1\{X_u^\circ < 0\}\Big) dZ_u^\circ
\end{aligned}
\tag{B.89}
$$

for all $t \geq 0$. Consequently, the statement of the proposition follows from (B.73), (B.74), (B.75), and (B.89), which completes the proof.

B.7. Proof of Theorem 4.4

Proof of Part (a). In the sequel, we provide the proofs for the three asymptotics presented in Part (a). We let $T_r = T^{1/(r+2)}$ in what follows.

Proof of First Asymptotics. We deduce from a change of variables and the occupation times formula that

$$
\begin{aligned}
\frac{1}{T_r}\int_0^T f(X_t)dt &= \frac{T}{T_r}\int_0^1 f\big(T_r(B^T \circ A^T)_t\big)dt \\
&= T_r \int_0^{A_1^T} (mf)(T_r B_t^T)dt \\
&= \int_{\mathbb{R}} (mf)(x) L^T\left(A_1^T, \frac{x}{T_r}\right)dx,
\end{aligned}
\tag{B.90}
$$

where

$$
A_t^T = \inf\left\{u \left| \frac{T_r}{T}\int_{\mathbb{R}} L^T\left(u, \frac{x}{T_r}\right) m(x)dx > t\right.\right\} = \inf\left\{u \left| \int_{\mathbb{R}} L^T(u,x)\frac{m(T_r x)}{T_r^r}dx > t\right.\right\},
$$

and L^T is the local time of B^T. As before, we let $B^T \to_{a.s.} B^\circ$. We already obtained in the proof of Proposition 4.2 that $A_1^T \to_{a.s.} \bar{A}_1$ as $T \to \infty$. Moreover, due to Lemma A1 and the continuity of $L^\circ(t,x)$ in both $t \geq 0$ and $x \in \mathbb{R}$, we deduce that $\sup_{t\in[0,a]}\sup_{x\in[-a,a]} L^T(t,x) < \infty$ for any $a > 0$ uniformly in all large T. Therefore, we obtain from (B.90) and the dominated convergence theorem that

$$
\frac{1}{T_r}\int_0^T f(X_t)dt \to_d L^\circ(\bar{A}_1, 0)m_s(f_s)
\tag{B.91}
$$

as $T \to \infty$. Note that we have $L^\circ(\bar{A}_1, 0) =_d KE^{1/(r+2)}$, due to Remark 3.5 of Kim and Park (2017). Therefore, we obtain the first asymptotics of Part (a) from (B.91).

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Proof of Second Asymptotics. We deduce from a change of variables, (A.2) and (A.3) that

$$\begin{aligned}\frac{1}{\sqrt{T_r}}\int_0^T g_1(X_u)dW_u &= \frac{1}{\sqrt{T_r}}\int_0^1 g_1\big(T_r(B^T\circ A^T)_u\big)dW_{Tu}\\ &= \sqrt{\frac{T}{T_r}}\int_0^{A_1^T} g_1(T_rB_u^T)d(W^T\circ A^{-1T})_u\\ &=_d \sqrt{T_r}\int_0^{A_1^T}(m^{1/2}g_1)(T_rB_u^T)dW_u \end{aligned} \tag{B.92}$$

for all $T>0$, where B^T and W in the last line are defined in (A.6).

Then, we define

$$M_t^{1T} = \sqrt{T_r}\int_0^t (m^{1/2}g_1)(T_rB_u^T)dW_u \tag{B.93}$$

and $B_t^{1T} = m(g_1g_1')^{-1/2}M^{1T}\circ\rho_t$, where $\rho_t = \inf\{u|L^\circ(u,0)>t\}$. We deduce from a change of variables, the occupation times formula and Lemma A1 that

$$\begin{aligned}[B^{1T}]_t &= m(g_1g_1')^{-1/2}T_r\int_0^{\rho_t}(mg_1g_1')(T_rB_u^T)du\, m(g_1g_1')^{-1/2}\\ &= m(g_1g_1')^{-1/2}\int_{\mathbb{R}}(mg_1g_1')(x)L^T\left(\rho_t,\frac{x}{T_r}\right)dx\, m(g_1g_1')^{-1/2}\\ &\to_p L^\circ(\rho_t,0)m(g_1g_1')^{-1/2}m(g_1g_1')m(g_1g_1')^{-1/2} = tI_{g_1}\end{aligned} \tag{B.94}$$

locally uniformly in $t\geq 0$ as $T\to\infty$, where I_{g_1} is the identity matrix which has the same dimension as g_1. Therefore, we deduce from (B.94) that

$$B^{1T}\to_d B^1 \tag{B.95}$$

as $T\to\infty$, where B^1 is a standard vector Brownian motion which has the same dimension as g_1.

Next, to show the independence between B^1 and B°, we first deduce that

$$\|\langle M^{1T},B^T\rangle_t\| = \sqrt{T_r}\left\|\int_0^t (m\sigma g_1)(T_rB_u^T)du\right\| \leq c\sqrt{T_r}\int_0^t\left(|T_rB_u^T|+1\right)^{-1/2-\varepsilon}du \tag{B.96}$$

uniformly in $t\geq 0$ as $T\to\infty$ for some $c,\varepsilon>0$, where the equality is obtained from (A.6), (B.93), Section III.5 of Protter (2005) and the covariance extension of the Itô isometry, and the inequality from $m = 1/(\sigma^2+\omega^2\lambda)$ and the integrability of mg_1g_1'. Then, we may deduce that

$$T_r^{1/2+\varepsilon}\int_0^t\left(|T_rB_u^T|+1\right)^{-1/2-\varepsilon}du \to_d \int_0^t |B_u^\circ|^{-1/2-\varepsilon}du \tag{B.97}$$

as $T\to\infty$ locally uniformly in $t\geq 0$, from the uniform convergence of B^T and the asymptotic homogeneity of a function $(|x|+1)^{-1/2-\varepsilon}$ (see the proofs of Theorem 3.5 of Jeong and Park (2013) and Theorem 4.4(b) for more details). Therefore, we obtain from (B.96) and (B.97) that

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$$\left\| \langle M^{1T}, B^T \rangle_t \right\| \to_p 0 \tag{B.98}$$

uniformly in $t \geq 0$ as $T \to \infty$.

Due to (B.98) and Theorem I.4.2 of Jacod and Shiryaev (2003), M^{1T} and B^T are strongly orthogonal in the limit as $T \to \infty$. This implies that B^{1T} and B^T are also strongly orthogonal in the limit, since arbitrary time changes of strongly orthogonal martingales are also strongly orthogonal, due to Exercise IV.2.22 of Revuz and Yor (1999). Therefore, B^1 and B° are independent of each other, due to the strong orthogonality between B° and B^1, and Exercise V.4.25 of Revuz and Yor (1999).

Since the law of $L^\circ(\bar{A}_1, 0)$ is solely determined by B°, and B° is independent of B^1, we deduce that $L^\circ(\bar{A}_1, 0)$ is independent of B^1. Consequently, due to (B.92), (B.93), the independence between $L^\circ(\bar{A}_1, 0)$ and B^1, and $L^\circ(\bar{A}_1, 0) =_d KE^{1/(r+2)}$, we obtain that

$$\begin{aligned}\frac{1}{\sqrt{T_r}} \int_0^T g_1(X_u) dW_u &=_d M^{1T} \circ A_1^T = m(g_1 g_1')^{1/2} B^{1T} \circ L^\circ(A_1^T, 0) \\ &\to_d m(g_1 g_1')^{1/2} B^1 \circ L^\circ(\bar{A}_1, 0) =_d m(g_1 g_1')^{1/2} B^1 \circ (KE^{1/(r+2)})\end{aligned}$$

as $T \to \infty$, which complete the proof of the second asymptotics of Part (a).

Proof of Third Asymptotics. We deduce from a change of variables, (A.2) and (A.4) that

$$\frac{1}{\sqrt{T_r}} \int_0^T (g_2 \lambda^{1/2})(X_{u-}) \left[\frac{\upsilon(Z_u)}{\lambda^{1/2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \phi(\upsilon) \lambda^{1/2}(X_u) du \right] \tag{B.99}$$

$$=_d \frac{1}{\sqrt{T_r}} \int_0^{A_1^T} \big((m\lambda)^{1/2} g_2\big)(T_r B_{u-}^T) \left[\frac{\upsilon(Z_u)}{(m\lambda)^{1/2}(T_r B_{u-}^T)} dN_u\big(T_r^2 (m\lambda)(T_r B_{u-}^T)\big) - \frac{\phi(\upsilon)}{T^{-1} T_r^2} (m\lambda)^{1/2}(T_r B_u^T) du \right],$$

where B^T, Z and $N\big(T_r^2 (m\lambda)(T_r B^T)\big)$ are defined in (A.6). We let

$$M_t^{2T} = \frac{1}{\sqrt{T_r}} \int_0^t \big((m\lambda)^{1/2} g_2\big)(T_r B_{u-}^T) \times \left[\frac{\upsilon(Z_u)}{(m\lambda)^{1/2}(T_r B_{u-}^T)} dN_u\big(T_r^2 (m\lambda)(T_r B_{u-}^T)\big) - \frac{\phi(\upsilon)}{T^{-1} T_r^2} (m\lambda)^{1/2}(T_r B_u^T) du \right], \tag{B.100}$$

and $B_t^{2T} = m(g_2 g_2' \lambda)^{-1/2} \phi(\upsilon^2)^{-1/2} M^{2T} \circ \rho_t$, where $\rho_t = \inf\big\{u | L^\circ(u, 0) > t\big\}$.

To show $B^{2T} \to_d B^2$ as $T \to \infty$, where B^2 is a standard vector Brownian motion which has the same dimension as g_2, we write

$$[B^{2T}]_t = P_t^T + Q_t^T, \tag{B.101}$$

where

$$\begin{aligned} P_t^T &= m(g_2 g_2' \lambda)^{-1/2} T_r \int_0^{\rho_t} (m g_2 g_2' \lambda)(T_r B_u^T) du\, m(g_2 g_2' \lambda)^{-1/2}, \\ Q_t^T &= m(g_2 g_2' \lambda)^{-1/2} \frac{1}{T_r \phi(\upsilon^2)} \bigg(\int_0^{\rho_t} (g_2 g_2')(T_r B_{u-}^T) \upsilon^2(Z_u) dN_u\big(T_r^2 (m\lambda)(T_r B_{u-}^T)\big) \\ &\qquad - T_r \phi(\upsilon^2) \int_0^{\rho_t} (m g_2 g_2' \lambda)(T_r B_u^T) du \bigg) m(g_2 g_2' \lambda)^{-1/2}. \end{aligned}$$

To show the asymptotic negligibility of Q_t^T, let

$$R_t^T = \frac{1}{T_r}\left(\int_0^{A_t^T}(g_2g_2')(T_rB_{u-}^T)\upsilon^2(Z_u)dN_u\big(T_r^2(m\lambda)(T_rB_{u-}^T)\big) - T_r\phi(\upsilon^2)\int_0^{A_t^T}(mg_2g_2'\lambda)(T_rB_u^T)du\right).$$

Then R_t^T is a martingale whose quadratic variation is bounded in the sense that

$$\big\|[R^T]_t\big\| \le \frac{1}{T_r^2}\int_0^{A_t^T}\bar{g}_2^2(T_rB_{u-}^T)\upsilon^2(Z_u)dN_u\big(T_r^2(m\lambda)(T_rB_{u-}^T)\big) \tag{B.102}$$

for some locally bounded $\bar{g}_2 : \mathbb{R} \to \mathbb{R}$ such that $\|g_2(x)\| \le \bar{g}_2(x)$ for all $x \in \mathbb{R}$ and $m(\bar{g}_2^2\lambda) < \infty$. We deduce that

$$\begin{aligned}\frac{1}{T_r^2}\mathbb{E}\int_0^{A_t^T}\bar{g}_2^2(T_rB_{u-}^T)\upsilon^4(Z_u)dN_u\big(T_r^2(m\lambda)(T_rB_{u-}^T)\big) &= \frac{\phi(\upsilon^4)}{T_r^2}\mathbb{E}\int_0^t(\bar{g}_2^2\lambda)(T_rX_u^T)du \\ &\le \frac{\phi(\upsilon^4)}{T_r^2}\mathbb{E}\int_0^t c\big(1+\big|T_rX_u^T\big|\big)du \to 0\end{aligned} \tag{B.103}$$

locally uniformly in $t \ge 0$ for some $c > 0$ as $T \to \infty$, where we obtain the equality from a change of variables, (A.2), (A.4), and Section 8.8.4 of Jeanblanc et al. (2009), the inequality from the integrability of $m\bar{g}_2^2\lambda$, and the last convergence from Assumption 4.1(b). Due to (B.103) and the Markov inequality, we deduce that $[R^T]_t \to_p 0$ as $T \to \infty$ locally uniformly in $t \ge 0$. Then, we deduce from this asymptotic negligibility and the local uniform convergence $A^{-1T} \to_{a.s.} \bar{A}^{-1}$, which is already shown in the proof of Proposition 4.2, that $[R^T] \circ A^{-1T} \circ \rho_t \to_p 0$ as $T \to \infty$ locally uniformly in $t \ge 0$, which implies that $Q_t^T \to_p 0$ as $T \to \infty$ locally uniformly in $t \ge 0$.

We may also deduce similarly as in (B.94) that $P_t^T \to_p tI_{g_2}$ as $T \to \infty$ locally uniformly in $t \ge 0$, where I_{g_2} is the identity matrix which has the same dimension as g_2. Therefore, we obtain that

$$[B^{2T}]_t = P_t^T + Q_t^T = P_t^T + o_p(1) \to_p tI_{g_2} \tag{B.104}$$

as $T \to \infty$ locally uniformly in $t \ge 0$. Consequently, we deduce from (B.104) that $B^{2T} \to_d B^2$ as $T \to \infty$, where B^2 is a standard vector Brownian motion which has the same dimension as g_2.

For the independence between B^2 and B°, we deduce that

$$\begin{aligned}\big\|\langle M^{2T}, B^T\rangle_t\big\| &= \frac{1}{T_r^{3/2}}\left\|\left\langle\int_0^{\cdot}\Big(s\big[s^{-1}(T_rB_{u-}^T)+(\nu\circ s^{-1})(T_rB_{u-}^T)+(\tau\circ s^{-1})(T_rB_{u-}^T)Z_u\big]-T_rB_{u-}^T\Big)\times\right.\right. \\ &\qquad \left.\left. g_2(T_rB_{u-}^T)\upsilon(Z_u)dN_u\big(T_r^2(m\lambda)(T_rB_{u-}^T)\big)\right\rangle_t\right\| \\ &\le \sqrt{T_r\phi(\upsilon^2)\int_0^t(m\omega^2\lambda)(T_rB_{u-}^T)du\int_0^t(m\bar{g}_2^2\lambda)(T_rB_{u-}^T)du} \to_p 0\end{aligned} \tag{B.105}$$

as $T \to \infty$ locally uniformly in $t \ge 0$, where we obtain the equality from (A.6), (B.100), Section III.5 of Protter (2005), and the covariance extension of the Itô isometry, the inequality from the Hölder inequality and Section 8.8.4 and Proposition 8.8.6.1 of Jeanblanc et al. (2009), and the last convergence from $m = 1/(\sigma^2 + \omega^2\lambda)$ and the integrability of

$m\bar{g}_2^2\lambda$. Therefore, we obtain the independence between B^2 and B° similarly as we obtained the independence between B^1 and B°. Consequently, we deduce that

$$M^{2T} \circ A_1^T = m(g_2 g_2' \lambda)^{1/2} \phi(\upsilon^2)^{1/2} B^{2T} \circ L^\circ(A_1^T, 0)$$
$$\to_d m(g_2 g_2' \lambda)^{1/2} \phi(\upsilon^2)^{1/2} B^2 \circ L^\circ(\bar{A}_1, 0) =_d m(g_2 g_2' \lambda)^{1/2} \phi(\upsilon^2)^{1/2} B^2 \circ (KE^{1/(r+2)})$$

as $T \to \infty$.

For the independence between B^1 and B^2, we deduce from the covariance extension of the Itô isometry and $[W, V]_t = 0$ for $t \geq 0$, that

$$\mathbb{E}\left(M_t^{1T} \int_0^t P_u dM_u^{2T\prime} \right) = \sqrt{T_r}\, \mathbb{E}\left(\int_0^t \left(m g_1 g_2' \lambda^{1/2}\right)(T_r B_u^T) P_u d\big[W, V\big]_u \right) = 0 \qquad \textbf{(B.106)}$$

for all $t \geq 0$, all large T and any bounded predictable process P, where

$$V_t = \frac{1}{T_r} \int_0^t \left(\frac{\upsilon(Z_u)}{(m\lambda^{1/2})(T_r B_{u-}^T)} dN_u\big(T_r^2 (m\lambda)(T_r B_{u-}^T)\big) - \phi(\upsilon) \frac{T}{T_r^2} \lambda^{1/2} (T_r B_u^T) du \right).$$

Therefore, from (B.106) and Exercise IV.2.22 of Revuz and Yor (1999), we obtain the independence between B^1 and B^2, which completes the proof.

Proof of Part (b). Below we provide the proofs for the three asymptotics presented in Part (b).

Proofs of First and Second Asymptotics. Note that the asymptotics in (B.91) follows from (B.90) because we have

$$\int_{\mathbb{R}} (mf)(x) L^T \left(A_1^T, \frac{x}{T_r} \right) dx = L^T \left(A_1^T, 0 \right) \int_{\mathbb{R}} (mf)(x) dx + o_p(1) \to_d L^\circ \left(\bar{A}_1, 0 \right) \int_{\mathbb{R}} (mf)(x) dx$$

for m-integrable f. Clearly, these asymptotics do not apply to m-nonintegrable f. For m-nonintegrable f, we require m-asymptotic homogeneity so that

$$f(\lambda x) = \kappa(f, \lambda) h(f, x) \big(1 + o(1)\big)$$

for large λ and uniformly for all x in any compact interval. Then it follows that

$$\frac{1}{T} \kappa(f, T_r)^{-1} \int_0^T f(X_t) dt = \kappa(f, T_r)^{-1} \int_0^1 f(X_{Tt}) dt$$
$$= \kappa(f, T_r)^{-1} \int_0^1 f(T_r X_t^T) dt$$
$$= \int_0^1 h(f, X_t^T) dt + o_p(1) \to_d \int_0^1 h(f, X_t^\circ) dt,$$

where X° is the distributional limit of $X^T = (X_t^T)$ with $X_t^T = T_r^{-1} X_{Tt}$ for $t \in [0, 1]$. Subsequently, we develop this and other related asymptotics more rigorously.

The proofs of the first and second asymptotics are exactly the same as the proof of Theorem 3.5(b) in Jeong and Park (2013), except for the weak convergences in (B.71) and the equation right before (B.82) in their paper. Therefore, we only need to establish these two weak convergences.

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For the first weak convergence, which is given by

$$\int_0^{\bar{A}_1} \bar{m}(B_t^T)h(f,B_t^T)dt \to_d \int_0^{\bar{A}_1} \bar{m}(B_t^\circ)h(f,B_t^\circ)dt \tag{B.107}$$

as $T \to \infty$ in our notation, we use the Vitali convergence theorem. We let $B^T \to_{a.s.} B^\circ$ as before. To apply the Vitali convergence theorem, it is required to establish the pointwise convergence and uniform integrability. The pointwise convergence is easily obtained from $B^T \to_{a.s.} B^\circ$. On the other hand, the uniform integrability follows from

$$\int_0^{\bar{A}_1} \bar{m}(B_t^\circ)h(f,B_t^\circ)dt = \int_{\mathbb{R}} \bar{m}(x)h(f,x)L^T(\bar{A}_1,x)dx \leq \int_{\mathbb{R}} \bar{m}(x)h(f,x)L^\circ(\bar{A}_2+1,x)dx < \infty$$

for all large T, where we use the local integrability of $\bar{m}(\cdot)h(f,\cdot)$ and Lemma A1. Therefore, (B.107) follows from the Vitali convergence theorem.

The second weak convergence is given by

$$\int_0^{\bar{A}_1} h(g_1,B_t^T)d(W^T \circ \bar{A}^{-1})_t \to_d \int_0^{\bar{A}_1} h(g_1,B_t^\circ)d(W^\circ \circ \bar{A}^{-1})_t \tag{B.108}$$

as $T \to \infty$ in our notation, and we write

$$\int_0^{\bar{A}_1} h(g_1,B_t^T)d(W^T \circ \bar{A}^{-1})_t = P_T + Q_T, \tag{B.109}$$

where

$$P_T = \int_0^{\bar{A}_1} \left[h(g_1,B_t^T) - h(g_1,B_t^\circ)\right]d(W^T \circ \bar{A}^{-1})_t,$$
$$Q_T = \int_0^{\bar{A}_1} h(g_1,B_t^\circ)d(W^T \circ \bar{A}^{-1})_t,$$

which will be analyzed separately below.

For Q_T, note that

$$\mathbb{E}\left[W^T \circ \bar{A}^{-1}\right]_t = \mathbb{E}\bar{A}_t^{-1} = \mathbb{E}\int_{\mathbb{R}} \bar{m}(x)L^\circ(t,x)dx < \infty$$

for any given $t \geq 0$ and all T, since in particular W^T is a Brownian motion for all T. Therefore, the condition C2.2(i) of Kurtz and Protter (1991) holds, and it follows from their Theorem 2.2 that

$$Q_T \to_d \int_0^{\bar{A}_1} h(g_1,B_t^\circ)d(W^\circ \circ \bar{A}^{-1})_t \tag{B.110}$$

as $T \to \infty$. For P_T, we may readily deduce that it is a martingale whose quadratic variation

$$\int_0^{\bar{A}_1} \bar{m}(B_t)\left[h(g_1,B_t^T) - h(g_1,B_t^\circ)\right]\left[h(g_1,B_t^T) - h(g_1,B_t^\circ)\right]' dt \tag{B.111}$$

is asymptotically negligible (see the proof of Theorem 3.5 of Jeong and Park (2013) for more details), which implies that $P_T \to_p 0$ as $T \to \infty$. Consequently, (B.108) follows from (B.109) and (B.110).

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Proof of Third Asymptotics. We deduce from a change of variables that

$$\frac{1}{\sqrt{T}}\kappa(\sqrt{\lambda}g_2,T_r)^{-1}\int_0^T g_2(X_{t-})d\big[\upsilon(Z_t)N_t\big(\lambda(X_{t-})\big)-\phi(\upsilon)\lambda(X_t)dt\big]$$
$$=\frac{1}{\sqrt{T}}\kappa(\sqrt{\lambda}g_2,T_r)^{-1}\int_0^1 g_2\big(T_r(B^T\circ A^T)_{t-}\big)d\big[\upsilon(Z_{Tt})N_{Tt}\big(\lambda(X_{Tt-})\big)-T\phi(\upsilon)\lambda(X_{Tt})dt\big]$$
$$=\kappa(\sqrt{\lambda}g_2,T_r)^{-1}\sqrt{\phi(\upsilon^2)}\int_0^{A_1^T}(\sqrt{\lambda}g_2)\big(T_rB_{t-}^T\big)d(V^T\circ A^{-1T})_t, \tag{B.112}$$

where

$$V_t^T=\frac{1}{\sqrt{T\phi(\upsilon^2)}}\int_0^{Tt}\left[\frac{\upsilon(Z_u)}{\lambda^{1/2}(X_u)}dN_u\big(\lambda(X_{u-})\big)-\phi(\upsilon)\lambda^{1/2}(X_u)du\right].$$

Furthermore, we write

$$\kappa(\sqrt{\lambda}g_2,T_r)^{-1}\int_0^{A_1^T}(\sqrt{\lambda}g_2)\big(T_rB_{t-}^T\big)d(V^T\circ A^{-1T})_t=P_T+Q_T, \tag{B.113}$$

where

$$P_T=\int_0^{A_1^T}h\big(\sqrt{\lambda}g_2,B_{t-}^T\big)d(V^T\circ A^{-1T})_t$$
$$Q_T=\int_0^{A_1^T}\Big(\kappa(\sqrt{\lambda}g_2,T_r)^{-1}(\sqrt{\lambda}g_2)(T_rB_{t-}^T)-h(\sqrt{\lambda}g_2,B_{t-}^T)\Big)d(V^T\circ A^{-1T})_t,$$

which will be analyzed in the sequel.

For P_T, we write

$$P_T=P_{1T}+P_{2T}, \tag{B.114}$$

where

$$P_{1T}=\int_0^1\big(h(\sqrt{\lambda}g_2,X_{t-}^T)-h(\sqrt{\lambda}g_2,X_t^\circ)\big)dV_t^T,$$
$$P_{2T}=\int_0^1 h(\sqrt{\lambda}g_2,X_t^\circ)dV_t^T.$$

Note that P_{1T} is a martingale whose quadratic variation is

$$\int_0^1\big(h(\sqrt{\lambda}g_2,X_{t-}^T)-h(\sqrt{\lambda}g_2,X_t^\circ)\big)\big(h(\sqrt{\lambda}g_2,X_{t-}^T)-h(\sqrt{\lambda}g_2,X_t^\circ)\big)'d[V^T]_t,$$

and we may readily establish (see the proof of Theorem 3.5 of Jeong and Park (2013) for more details) that $P_{1T}=o_p(1)$ as $T\to\infty$.

The weak convergence of P_{2T} follows Theorem 2.2 of Kurtz and Protter (1991). To show that the required condition C2.2(i) holds, we use their notations Y, J, δ, A, and M, and let $(Y_T)_t=V_t^T$. We set $\delta=\infty$, so that $J_\delta(x)(t)=0$ for all x and $t\geq 0$. Then we have $Y_T^\delta=Y_T$ and $A_T^\delta=0$. However, we may deduce from Jeanblanc et al. (2009, Sect. 8.8.4 and Prop. 8.8.6.1) that

$$\mathbb{E}[M_T^\delta]_t = \mathbb{E}[Y_T]_t = \frac{1}{T\phi(\upsilon^2)}\mathbb{E}\int_0^{Tt}\frac{\upsilon^2(Z_u)}{\lambda(X_{u-})}dN_u\big(\lambda(X_{u-})\big) = t < \infty$$

for all $t \in [0,1]$, uniformly in $T > 0$. Therefore, it follows from Lemma A2 that

$$P_{2T} \to_d \int_0^1 h(\sqrt{\lambda}g_2, X_t^\circ)dV_t^\circ \tag{B.115}$$

as $T \to \infty$.

Next, we show that $Q_T = o_p(1)$ as $T \to \infty$. Note that Q_T is a martingale whose quadratic variation is

$$\int_0^{A_1^T}\Big(\kappa(\sqrt{\lambda}g_2, T_r)^{-1}(\sqrt{\lambda}g_2)(T_rB_{t-}^T) - h(\sqrt{\lambda}g_2, B_{t-}^T)\Big)\times$$
$$\Big(\kappa(\sqrt{\lambda}g_2, T_r)^{-1}(\sqrt{\lambda}g_2)(T_rB_{t-}^T) - h(\sqrt{\lambda}g_2, B_{t-}^T)\Big)'\frac{dN_t\big(T_r^2(m\lambda)(T_rB_{t-}^T)\big)}{T_r^2(m\lambda)(T_rB_{t-}^T)},$$

and we may deduce that it is asymptotically negligible, as in the proof of Theorem 3.5 of Jeong and Park (2013), if we establish the stochastic boundedness of

$$R_T = \int_0^{A_1^T}\frac{f(B_{t-}^T)}{T_r^2(m\lambda)\big(T_rB_{t-}^T\big)}dN_t\big(T_r^2(m\lambda)(T_rB_{t-}^T)\big) \tag{B.116}$$

for all large T, where $f : \mathbb{R} \to [0,\infty]$ is a function, which is locally bounded on $\mathbb{R}\backslash\{0\}$ and also locally integrable. In what follows, we let f be unbounded at the origin.

To show the stochastic boundedness of R_T, we write

$$R_T = \int_0^{S_T}\sum_{i\in\Gamma} f(B_{t_i-}^T)1\big\{t \in [k_{i-1}, k_i)\big\}dt, \tag{B.117}$$

where t_i for $i = 1,\ldots,N_T(\lambda(X_T))$ is the ith jump time of B^T, $S_T = \sum_{i\in\Gamma}1/\big(T_r^2(m\lambda)(T_rB_{t_i-}^T)\big)$, $\Gamma = \{1,\ldots,N_T(\lambda(X_T))\}$, and

$$k_i = \sum_{1\le j\le i}\frac{1}{T_r^2(m\lambda)\big(T_rB_{t_j-}^T\big)}.$$

Note that the partition

$$\big\{0, k_1, k_2, \ldots, k_{N_T(\lambda(X_T))}\big\} \tag{B.118}$$

is used in (B.117), instead of $\big\{0, t_1, t_2, \ldots, t_{N_T(\lambda(X_T))}\big\}$, that is, the partition defined by the actual jump times of B^T. We let $B^T \to_{a.s.} B^\circ$ as before.

First, we show that the partition defined in (B.118) and the partition defined by the actual jump times of B^T converge uniformly to each other. To be more specific, we let $t_0 = k_0 = 0$, and define $\tilde{t}_u = t_{uN_T(\lambda(X_T))}$ and $\tilde{k}_u = k_{uN_T(\lambda(X_T))}$ for $u \in [0,1]$, so that $\tilde{t}_u$ and $\tilde{k}_u$ are the $uN_T(\lambda(X_T))$th $N_T(\lambda(X_T))$-quantiles of $\{t_i\}$ and $\{k_i\}$, respectively. To simplify our proof, here we let $uN_T(\lambda(X_T))$ be an integer. Then we may deduce that

$$\sup_{u\in[0,1]}\big|\tilde{t}_u - \tilde{k}_u\big| \to_{a.s.} 0 \tag{B.119}$$

as $T \to \infty$. This will be shown below.

To establish (B.119), note that $N\big(T_r^2(m\lambda)(T_rB^T)\big) = N\circ\big(T\lambda(T_r)H^T\big)$, where

$$H_t^T = \frac{1}{T_r^r\lambda(T_r)}\int_0^t (m\lambda)(T_rB_u^T)du, \tag{B.120}$$

since $T_r^2 = T/T_r^r$. Also, let t_i^* be the ith jump time of $N\big(T\lambda(T_r)\big)$, and let $k_i^* = i/\big(T\lambda(T_r)\big)$ for $i = 1,\dots,N_T(\lambda(X_T))$. Then we have

$$\begin{aligned}\sup_{u\in[0,1]}\left|\tilde{t}_u - \tilde{k}_u\right| &= \sup_{u\in[0,1]}\left|\big(\tilde{t}\circ H^{-1T}\circ H^T\big)_u - \big(\tilde{k}\circ H^{-1T}\circ H^T\big)_u\right| \\ &= \sup_{u\in[0,1]}\left|\big(\tilde{t}^*\circ H^T\big)_u - \big(\tilde{k}^*\circ H^T\big)_u\right|,\end{aligned} \tag{B.121}$$

where $\tilde{t}_u^* = t_{uD_T}^*$, $\tilde{k}_u^* = k_{uD_T}^*$ for $D_T = N_T[\lambda(X_T)]/H_1^T$, and $H_t^{-1T} = \inf\big\{u\big|H_u^T > t\big\}$.

Due to (B.121), it suffices to show that

$$\sup_{u\in[0,c]}|\tilde{t}_u^* - \tilde{k}_u^*| \to_{a.s.} 0 \tag{B.122}$$

as $T\to\infty$ for any $0 < c < \infty$, and

$$\sup_{t\in[0,c]}\left|H_t^T - \bar{H}_t\right| \to_{a.s.} 0 \tag{B.123}$$

as $T\to\infty$ for any $0 < c < \infty$ and some continuous nondecreasing process $\bar{H}$, to establish (B.119). We may readily obtain (B.123) similarly as in the proof of Proposition 4.2. For (B.122), we let uD_T be an integer for simplicity. Since $\{t_i^*\}$ is a set of jump times of $N\big(T\lambda(T_r)\big)$, which is a constant intensity Poisson process, we have

$$\tilde{t}_u^* = \frac{1}{T\lambda(T_r)}\sum_{i=1}^{uD_T} e_i = \frac{D_T}{T\lambda(T_r)}\frac{1}{D_T}\sum_{i=1}^{uD_T} e_i, \qquad \tilde{k}_u^* = \frac{D_T}{T\lambda(T_r)}u, \tag{B.124}$$

where $\{e_i\}$ is a sequence of i.i.d. exponential random variables with rate parameter 1. Moreover, we have $D_T\to\infty$ and

$$\frac{D_T}{T\lambda(T_r)} = \frac{N\big(T\lambda(T_r)\big)\circ H^T\circ A_1^T}{T\lambda(T_r)H_1^T} \to_{a.s.} \frac{\bar{H}\circ\bar{A}_1}{\bar{H}_1} \tag{B.125}$$

as $T\to\infty$. Therefore, (B.122) follows from (B.124), (B.125), and the functional central limit theorem for i.i.d. random variables. Consequently, we obtain (B.119) from (B.122) and (B.123).

To show the stochastic boundedness of R_T, we decompose R_T into two parts: one consisting only of locally bounded parts of f and the other involving locally unbounded parts of f. To be more specific, we let a_T be an increasing sequence such that $a_T\sup_{u\in[0,1]}|\tilde{t}_u - \tilde{k}_u| \to_{a.s.} C_1$ for some $C_1 > 0$ as $T\to\infty$. Also, we let b_T be another increasing sequence such that $b_T\sup_{u\in[0,1]}\left|B_u^T - B_u^\circ\right| \to_{a.s.} C_2$ for some $C_2 > 0$ as $T\to\infty$, and define $c_T = \min\big\{a_T^{1/2-\varepsilon}, b_T\big\}$ for some small $\varepsilon > 0$. Then we write

$$R_T = R_{1T} + R_{2T}, \tag{B.126}$$

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where

$$R_{1T} = \int_0^{S_T} \sum_{i \in \Gamma \setminus \Omega} f(B_{t_i-}^T) 1\left\{t \in [k_{i-1}, k_i)\right\} dt,$$

$$R_{2T} = \sum_{i \in \Omega} \frac{f(B_{t_i-}^T)}{T_r^2(m\lambda)\left(T_r B_{t_i-}^T\right)}.$$

Here, we let $\Omega = \Omega_0 \cup \cdots \cup \Omega_K$, where

$$\Omega_j = \left\{t_i : t_i \in \left[\delta_j - \frac{1}{c_T}, \delta_j + \frac{1}{c_T}\right]\right\} \text{ for } j = 0, \ldots, K,$$

$$\delta_0 = 0,$$

$$\delta_j = \inf\left\{t : t > \xi_{j-1},\ B_t^T = 0 \ \text{ or } \ \operatorname{sgn}\left(B_{t-}^T\right) \neq \operatorname{sgn}\left(B_t^T\right)\right\} \text{ for } j = 1, \ldots, K,$$

$$\xi_j = \inf\left\{t : t \geq \delta_j,\ B_t^T \neq 0,\ ^\exists\varepsilon > 0 \text{ s.t. } \inf_{u \in [v, t+\varepsilon]} \left|B_u^T\right| > 0 \ \text{ for all } t < v < t+\varepsilon\right\} \text{ for } j = 0, \ldots, K,$$

and K is the total number of δ_j's such that $0 < \delta_j < \bar{A}_1$.

For R_{1T}, we let

$$S_T = \sum_{i \in \Gamma} \frac{1}{T_r^2(m\lambda)(T_r B_{t_i-}^T)} = \int_0^{A_1^T} \frac{1}{T_r^2(m\lambda)(T_r B_{t-}^T)} dN_t\left(T_r^2(m\lambda)(T_r B_{t-}^T)\right) = G_T(A_1^T). \tag{B.127}$$

To show the stochastic boundedness of $S_T = G_T(A_1^T)$, we deduce that

$$\mathbb{E} \sup_{u \in [0, c]} G_T(u) \leq \mathbb{E} \int_0^c \frac{1}{T_r^2(m\lambda)(T_r B_{t-}^T)} dN_t\left(T_r^2(m\lambda)(T_r B_{t-}^T)\right) = c < \infty \tag{B.128}$$

for all large T and any $c > 0$, see Section 8.8.4 of Jeanblanc et al. (2009). However, it follows from (B.128) and the Markov inequality that $G_T(u) = O_p(1)$ as $T \to \infty$ locally uniformly in $u > 0$, which, together with $A_1^T \to_{a.s.} \bar{A}_1$, implies that $G_T(A_1^T) = S_T = O_p(1)$ as $T \to \infty$.

Furthermore, we may deduce that

$$\sum_{i \in \Gamma \setminus \Omega} f(B_{t_i-}^T) 1\left\{t \in [k_{i-1}, k_i)\right\} \leq c_0 \left(|t|^{-1+\varepsilon} + \sum_{j=1}^K |t - c_j|^{-1+\varepsilon} + 1\right) \tag{B.129}$$

almost surely for all $0 \leq t \leq S_T$, all large T, and some $\varepsilon, c_0, \ldots, c_K > 0$, due to the definitions of a_T and b_T, the modulus of continuity of B°, the shrinking speed of each Ω_j for $j = 0, \ldots, K$, and the local integrability of f. Therefore, we obtain from (B.129) and $S_T = O_p(1)$ that

$$R_{1T} \leq c_0 \int_0^{S_T} \left(|t|^{-1+\varepsilon} + \sum_{j=1}^K \left|t - c_j\right|^{-1+\varepsilon} + 1\right) dt = O_p(1) \tag{B.130}$$

as $T \to \infty$.

For R_{2T}, we have

$$R_{2T} \leq \sum_{j=0}^{K} \left(\max_{i \in \Omega_j} f(B^T_{t_i-}) \sum_{i \in \Omega_j} \frac{1}{T_r^2 (m\lambda)\big(T_r B^T_{t_i-}\big)} \right). \tag{B.131}$$

Define

$$D_{1j} = \max_{i \in \Omega_j} f(B^T_{t_i-}) \qquad \text{and} \qquad D_{2j} = \sum_{i \in \Omega_j} \frac{1}{T_r^2 (m\lambda)\big(T_r B^T_{t_i-}\big)}$$

for $j = 0, \ldots, K$. Note that D_{2j} is the length of the interval given by Ω_j for $j = 0, \ldots, K$, and therefore, $D_{2j} = O_p(1/c_T)$ as $T \to \infty$ by the definition of Ω_j, for $j = 0, \ldots, K$. To analyze D_{1j}, we let τ_j be the jump time of B^T closest to δ_j for $j = 0, \ldots, K$. As $T \to \infty$, $f(B^T_{\tau_j-})$ becomes the largest value among $\{f(B^T_{t_i-})\}_{i \in \Omega_j}$ for $j = 0, \ldots, K$. By the definitions of $1/a_T$ and $1/b_T$ and the modulus of continuity of B°, we may therefore deduce that there exists a subsequence of B^T such that $B^T_{\tau_j-}$ converges to zero at the speed of $1/c_T$ for $j = 0, \ldots, K$. For such a subsequence of B^T, it follows from the local integrability of f that $f(B^T_{\tau_j-})$ diverges to infinity at a rate slower than c_T, which implies that $D_{1j} = O_p(c_T)$ as $T \to \infty$ for $j = 0, \ldots, K$. Consequently, we have

$$R_{2T} = O_p(c_T) O_p(1/c_T) = O_p(1) \tag{B.132}$$

from (B.131) as $T \to \infty$.

Now it follows from (B.126), (B.130), and (B.132) that $R_T = O_p(1)$ as $T \to \infty$. Therefore, we may deduce that Q_T is a martingale whose quadratic variation is of order $o_p(1)$ as $T \to \infty$, which implies that $Q_T \to_p 0$ as $T \to \infty$. Finally, we may deduce from (B.112), (B.113), (B.114), (B.115), $P_{1T} \to_p 0$, and $Q_T \to_p 0$ that

$$\frac{1}{\sqrt{T}} \kappa(\sqrt{\lambda} g_2, T_r)^{-1} \int_0^T g_2(X_{t-}) d\big[\upsilon(Z_t) N_t(\lambda(X_{t-})) - \phi(\upsilon)\lambda(X_t)dt\big] \to_d \sqrt{\phi(\upsilon^2)} \int_0^1 h(\sqrt{\lambda} g_2, X_t^\circ) dV_t^\circ$$

as $T \to \infty$.

Covariances of Limit Brownian Motions. Write

$$Z^T = \frac{\phi(\iota\upsilon)}{\sqrt{\phi(\upsilon^2)}} V^T + \sqrt{\phi(\upsilon_c^2)}\, U^T, \tag{B.133}$$

where $\upsilon_c(z) = z - \phi(\iota\upsilon)\upsilon(z)/\phi(\upsilon^2)$ and

$$\begin{aligned} V_t^T &= \frac{1}{\sqrt{T\phi(\upsilon^2)}} \int_0^{Tt} \left[\frac{\upsilon(Z_u)}{\lambda^{1/2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \phi(\upsilon)\lambda^{1/2}(X_u) du \right], \\ U_t^T &= \frac{1}{\sqrt{T\phi(\upsilon_c^2)}} \int_0^{Tt} \left[\frac{\upsilon_c(Z_u)}{\lambda^{1/2}(X_{u-})} dN_u\big(\lambda(X_{u-})\big) - \phi(\upsilon_c)\lambda^{1/2}(X_u) du \right]. \end{aligned} \tag{B.134}$$

We have $V^T \to_d V^\circ$ and $U^T \to_d U^\circ$ as $T \to \infty$ from Lemma A2, where V° and U° are standard Brownian motions. Moreover,

$$\mathbb{E}\left(V_t^T \int_0^t P_u dU_u^T\right) = \mathbb{E}\int_0^t P_u\left(\phi(\iota\upsilon) - \frac{\phi(\iota\upsilon)\phi(\upsilon^2)}{\phi(\upsilon^2)}\right)du = 0$$

for any bounded predictable process P and $t \geq 0$, due to Section 8.8.4 and Proposition 8.8.6.1 of Jeanblanc et al. (2009). Therefore, V° and U° are independent of each other, due to Revuz and Yor (1999, Exers. IV.2.22 and V.4.25). Furthermore, we have $\phi(\upsilon_c^2) = 1 - \phi(\iota\upsilon)^2/\phi(\upsilon^2)$. Consequently, it follows from (B.133) and the independence between V° and U° that $\mathbb{E}Z_t^\circ V_t^\circ = t\phi(\iota\upsilon)/\sqrt{\phi(\upsilon^2)}$.

To complete the proof, we write

$$N^T = \frac{\phi(\upsilon)}{\sqrt{\phi(\upsilon^2)}}V^T + \sqrt{\phi(\upsilon_c^2)}U^T,$$

where $\upsilon_c(z) = 1 - \phi(\upsilon)\upsilon(z)/\phi(\upsilon^2)$, and V^T and U^T are defined as in (B.134), from which we may easily show that $\mathbb{E}N_t^\circ V_t^\circ = t\phi(\upsilon)/\sqrt{\phi(\upsilon^2)}$ similarly as above.

Appendix C. Continuous Time Approximation

Here, we present the proofs of the continuous time approximations used in Section 6, and provide the precise conditions required for their validity.

Assumption A1. (a) μ, σ^2, ν, τ^2, and λ are piecewise infinitely differentiable and regularly varying at the boundaries, and they and their derivatives are asymptotically monotone at the boundaries and bounded by locally bounded regularly varying functions with index $p \geq 1$, (b) $\sup_{t\in[0,T]}|X_t| = O_p(T^q)$ as $T \to \infty$, (c) $\sup_{t\in[0,T]}\left(\mathbb{E}|X_t|^k\right)^{1/k} = O(T^q)$ as $T \to \infty$ for some $k \geq 4p$, (d) $\sqrt{\Delta}T^{3pq+1} \to 0$ as $T \to \infty$ and $\Delta \to 0$.

LEMMA A3. *Under Assumption A1, we have*

$$\Delta\sum_{i=1}^n X_{i\Delta}^2 = \int_0^T X_t^2 dt + O_p(\Delta T^{2pq+1})$$

as $T \to \infty$ *and* $\Delta \to 0$.

Proof. From Itô's lemma, we have

$$\begin{aligned}\Delta\sum_{i=1}^n X_{i\Delta}^2 &= \int_0^T X_t^2 dt - \sum_{i=1}^n \int_{(i-1)\Delta}^{i\Delta}(X_t^2 - X_{(i-1)\Delta}^2)dt \\ &= \int_0^T X_t^2 dt - A_T - B_T - C_T,\end{aligned} \tag{C.135}$$

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where

$$A_T = \sum_{i=1}^{n} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{t} f_A(X_s) ds dt,$$

$$B_T = \sum_{i=1}^{n} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{t} f_B(X_s) dW_s dt,$$

$$C_T = \sum_{i=1}^{n} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{t} \Big(f\big[X_{s-} + \nu(X_{s-}) + \tau(X_{s-}) Z_s\big] - f(X_{s-}) \Big) dN_s\big(\lambda(X_{s-})\big) dt,$$

and $f(x) = x^2$, $f_A(x) = (\mu f^{\cdot} + \sigma^2 f^{\cdot\cdot}/2)(x)$ and $f_B(x) = (\sigma f^{\cdot})(x)$.

For A_T, we have

$$A_T \leq \frac{n\Delta^2}{2} \sup_{0 \leq t \leq T} |f_A(X_t)| = O_p(\Delta T^{2pq+1})$$

as $T \to \infty$ and $\Delta \to 0$, and for B_T, we have

$$\mathbb{E} B_T^2 \leq c n \Delta^3 \sup_{0 \leq t \leq T} \mathbb{E} f_B^2(X_t) = O(\Delta^2 T^{4pq+1})$$

as $T \to \infty$ and $\Delta \to 0$ for some $c > 0$, from which we deduce that $B_T = O_p(\Delta T^{2pq+1/2})$. Readers are referred to the proof of Lemma A1 in Jeong and Park (2013) for the details. For C_T, due to Jeanblanc et al. (2009, Prop. 8.8.6.1), we obtain that

$$\mathbb{E}|C_T| \leq \mathbb{E} \sum_{i=1}^{n} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{t} g(X_s) \lambda(X_s) ds dt = O(\Delta T^{2pq+1})$$

as $T \to \infty$ and $\Delta \to 0$, where

$$g(x) = \int_{\mathbb{R}} \big| f\big(x + \nu(x) + \tau(x) z\big) - f(x) \big| \phi(z) dz,$$

from which we deduce that $C_T = O_p(\Delta T^{2pq+1})$. Therefore, we obtain

$$\Delta \sum_{i=1}^{n} X_{i\Delta}^2 = \int_0^T X_t^2 dt + O_p(\Delta T^{2pq+1})$$

as $T \to \infty$ and $\Delta \to 0$, which completes the proof. □

LEMMA A4. *Under Assumption A1, we have*

$$\sum_{i=1}^{n} (X_{i\Delta} - X_{(i-1)\Delta})^2 = \int_0^T \sigma^2(X_t) dt + \int_0^T \big(\nu(X_{t-}) + \tau(X_{t-}) Z_t\big)^2 dN_t\big(\lambda(X_{t-})\big) + O_p(\sqrt{\Delta} T^{3pq+1})$$

as $T \to \infty$ *and* $\Delta \to 0$.

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Proof. We first write that

$$\begin{aligned}
&\sum_{i=1}^{n}(X_{i\Delta}-X_{(i-1)\Delta})^2 \\
&=\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\mu(X_t)dt+\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t+\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)dN_t\big(\lambda(X_{t-})\big)\right)^2 \\
&=A_T+B_T+C_T, \qquad \textbf{(C.136)}
\end{aligned}$$

where

$$\begin{aligned}
A_T&=\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t\right)^2, \\
B_T&=\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)dN_t\big(\lambda(X_{t-})\big)\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
C_T=&\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\mu(X_t)dt\right)^2+2\sum_{i=2}^{n}\int_{(i-1)\Delta}^{i\Delta}\mu(X_t)dt\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t \\
&+2\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta}\mu(X_t)dt\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)dN_t\big(\lambda(X_{t-})\big) \\
&+2\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)dN_t\big(\lambda(X_{t-})\big).
\end{aligned}$$

To obtain the leading term of A_T, we write that

$$\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t\right)^2=\int_0^T\sigma^2(X_t)dt+R_{1T},$$

where

$$R_{1T}=\sum_{i=1}^{n}\left[\left(\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t\right)^2-\int_{(i-1)\Delta}^{i\Delta}\sigma^2(X_t)dt\right].$$

Due to Itô's lemma, we have

$$R_{1T}=2\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)U_{i,t}dW_t,$$

where $U_{i,t} = \int_{(i-1)\Delta}^{t} \sigma(X_s)dW_s$. Then we obtain from the Hölder inequality and the Burkholder–Davis–Gundy inequality that

$$\begin{aligned}
\mathbb{E}R_{1T}^2 &= 4\mathbb{E}\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta} \sigma^2(X_t)U_{i,t}^2 dt \\
&\leq \sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta} \left(\mathbb{E}_{(i-1)\Delta}\sigma^4(X_t)\mathbb{E}_{(i-1)\Delta}U_{i,t}^4\right)^{1/2} dt \\
&\leq c\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta} \left(\mathbb{E}_{(i-1)\Delta}\sigma^4(X_t)\mathbb{E}_{(i-1)\Delta}\left[\int_{(i-1)\Delta}^{t}\sigma^2(X_s)ds\right]^2\right)^{1/2} dt
\end{aligned} \tag{C.137}$$

for some $c > 0$. We further deduce from (C.137) that

$$\begin{aligned}
\mathbb{E}R_{1T}^2 &\leq c\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta} \left(\mathbb{E}_{(i-1)\Delta}\sigma^4(X_t)\mathbb{E}_{(i-1)\Delta}\int_{(i-1)\Delta}^{t}\int_{(i-1)\Delta}^{t}\sigma^2(X_r)\sigma^2(X_s)drds\right)^{1/2} dt \\
&= O(\Delta T^{2pq+1})
\end{aligned} \tag{C.138}$$

as $T \to \infty$ and $\Delta \to 0$, from which $R_{1T} = O_p(\sqrt{\Delta}T^{pq+1/2})$ follows. Consequently, we obtain that

$$\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\sigma(X_t)dW_t\right)^2 = \int_0^T \sigma^2(X_t)dt + O_p(\sqrt{\Delta}T^{pq+1/2}) \tag{C.139}$$

as $T \to \infty$ and $\Delta \to 0$.

For the leading term of B_T, we write that

$$\sum_{i=1}^{n}\left(\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)dN_t\big(\lambda(X_{t-})\big)\right)^2 = \int_0^T\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)^2 dN_t\big(\lambda(X_{t-})\big) + R_{2T},$$

where

$$\begin{aligned}
R_{2T} = \sum_{i=1}^{n}\Bigg[&\left(\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)dN_t\big(\lambda(X_{t-})\big)\right)^2 \\
&-\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)^2 dN_t\big(\lambda(X_{t-})\big)\Bigg].
\end{aligned}$$

We have

$$R_{2T} = 2\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta}\big(\nu(X_{t-})+\tau(X_{t-})Z_t\big)V_{i,t-}dN_t\big(\lambda(X_{t-})\big),$$

where $V_{i,t} = \int_{(i-1)\Delta}^{t}\big(\nu(X_{s-})+\tau(X_{s-})Z_s\big)dN_s\big(\lambda(X_{s-})\big)$. Furthermore, using analogous techniques as in (C.137) and (C.138), we obtain from the Hölder inequality and the Burkholder–Davis–Gundy inequality that

$$\mathbb{E}R_{2T}^2 = 4\mathbb{E}\sum_{i=1}^{n}\int_{(i-1)\Delta}^{i\Delta}\big((\nu^2+\tau^2)\lambda\big)(X_t)V_{i,t}^2 dt = O(\Delta T^{4pq+1})$$

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as $T \to \infty$ and $\Delta \to 0$, from which we deduce that $R_{2T} = O_p(\sqrt{\Delta} T^{2pq+1/2})$. Consequently, we obtain that

$$\sum_{i=1}^{n} \left(\int_{(i-1)\Delta}^{i\Delta} \big(\nu(X_{t-}) + \tau(X_{t-}) Z_t\big) dN_t\big(\lambda(X_{t-})\big) \right)^2 = \int_0^T \big(\nu(X_{t-}) + \tau(X_{t-}) Z_t\big)^2 dN_t\big(\lambda(X_{t-})\big) + O_p(\sqrt{\Delta} T^{2pq+1/2}) \tag{C.140}$$

as $T \to \infty$ and $\Delta \to 0$.

Finally, we may readily show that $C_T = O_p(\sqrt{\Delta} T^{3pq+1})$ as $T \to \infty$ and $\Delta \to 0$, simply by repeating what we did to establish (C.139) and (C.140). Therefore, it follows from (C.136), (C.139), and (C.140) that

$$\sum_{i=1}^{n} (X_{i\Delta} - X_{(i-1)\Delta})^2 = \int_0^T \sigma^2(X_t) dt + \int_0^T \big(\nu(X_{t-}) + \tau(X_{t-}) Z_t\big)^2 dN_t\big(\lambda(X_{t-})\big) + O_p(\sqrt{\Delta} T^{3pq+1})$$

as $T \to \infty$ and $\Delta \to 0$, which completes the proof. □

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