R. A. Blumenthal
Nagoya Math. J.
Vol. 90 (1983), 145-153

RIEMANNIAN FOLIATIONS WITH PARALLEL CURVATURE

ROBERT A. BLUMENTHAL

§ 1. Introduction

Let M be a smooth compact manifold and let $\mathscr{F}$ be a smooth codimension q Riemannian foliation of M. Let $T(M)$ be the tangent bundle of M and let $E \subset T(M)$ be the subbundle tangent to $\mathscr{F}$. We may regard the normal bundle $Q = T(M)/E$ of $\mathscr{F}$ as a subbundle of $T(M)$ satisfying $T(M) = E \oplus Q$. Let g be a smooth Riemannian metric on Q invariant under the natural parallelism along the leaves of $\mathscr{F}$. This is equivalent to the existence of a bundle-like metric [16] and to the existence of a transverse $O(q)$-structure [5]. Recall that a connection ∇ on Q is basic if the induced parallel translation along a path lying in a leaf of $\mathscr{F}$ agrees with the natural parallelism along the leaves and that such a connection is characterized by the condition that $\nabla_X Y = [X, Y]_Q$ for all vector fields X tangent to E and Y tangent to Q where $[X, Y]_Q$ denotes the Q-component of the Lie bracket of X and Y [3]. The torsion of ∇ is the tensor field of type (1, 2) on M defined by $T(X, Y) = \nabla_X Y_Q - \nabla_Y X_Q - [X, Y]_Q$ where X and Y are vector fields on M. There is a unique torsion-free metric-preserving basic connection ∇ on Q [9], [11] defined as follows. Let $x \in M$. Let $f: U \to V$ be a submersion whose level sets are the leaves of $\mathscr{F}|U$ where U is a neighborhood of x in M and V is an open set in R^q. There is a unique Riemannian metric $\bar{g}$ on V such that $f^*(\bar{g}) = g|U$. Let $\bar{\nabla}$ be the Riemannian connection on V. Then $\nabla|U = f^{-1}(\bar{\nabla})$. It is natural to study the relationship between the curvature of ∇ and the structure of the foliated manifold $(M, \mathscr{F})$.

In the present work we study the case of parallel curvature, that is $\nabla R = 0$ where $R(X, Y)Z$ denotes the curvature tensor of ∇.

Let $\mathscr{F}$ be a Riemannian foliation with parallel curvature of a compact manifold M.

THEOREM 1. *Let $\tilde{M}$ be the universal cover of M and let $\tilde{\mathscr{F}}$ be the lift*

Received March 15, 1982.

of $\mathscr{F}$ to $\tilde{M}$. Then $\tilde{M}$ fibers over a simply connected Riemannian symmetric space with the leaves of $\tilde{\mathscr{F}}$ as fibers.

Let $p \in M$. Let π_p be a two-dimensional subspace of Q_p and let $\{X, Y\}$ be an orthonormal basis of π_p. The (transverse) sectional curvature of π_p is defined by $K(\pi_p) = -g_p(R(X, Y)X, Y)$ and depends only on π_p. If $K(\pi_p) > 0$ (respectively, ≤ 0, ≥ 0) for all two-dimensional subspaces $\pi_p \subset Q_p$ and all $p \in M$, we say that $(M, \mathscr{F})$ has positive (respectively, non-positive, non-negative) sectional curvature.

COROLLARY 1. *If $(M, \mathscr{F})$ has non-positive sectional curvature, then $\tilde{M}$ is diffeomorphic to a product $\tilde{L} \times R^q$ where $\tilde{L}$ is the (common) universal cover of the leaves of $\mathscr{F}$ and the leaves of $\tilde{\mathscr{F}}$ are identified with the sets $\tilde{L} \times \{x\}$, $x \in R^q$.*

COROLLARY 2. *If $\pi_1(M)$ is finite, then $(M, \mathscr{F})$ has non-negative sectional curvature and all the leaves of $\mathscr{F}$ are compact with finite holonomy.*

Remark. If $\mathscr{F}$ is a flat Riemannian foliation of the compact manifold M, then $(M, \mathscr{F})$ has zero curvature and zero sectional curvature and so, by Corollary 1, $\tilde{M} \cong \tilde{L} \times R^q$ and $\tilde{\mathscr{F}}$ is the product foliation. A theorem of G. Reeb [14] states that if $\mathscr{F}$ is a codimension one foliation of a compact manifold M defined by a closed nonsingular one-form, then $\tilde{M} \cong \tilde{L} \times R$ and $\tilde{\mathscr{F}}$ is the product foliation. It is easy to see that such a codimension one foliation admits a flat Riemannian structure and so we obtain Reeb's theorem from Corollary 1.

A differential r-form ω on M is called base-like if on each coordinate neighborhood U with coordinates $(x^1, \cdots, x^k, y^1, \cdots, y^q)$ respecting the foliation $\mathscr{F}$, the local expression of ω is of the form

$$\sum_{1 \leq i_1 < \cdots < i_r \leq q} a_{i_1, \cdots, i_r}(y^1, \cdots, y^q) dy^{i_1} \wedge \cdots \wedge dy^{i_r}$$

[16], [17]. Since the exterior derivative of a base-like form is again base-like, one can construct the base-like cohomology algebra $H^*(M, \mathscr{F}) = \sum_{r \geq 0} H^r(M, \mathscr{F})$ of the foliated manifold $(M, \mathscr{F})$. For each $r \geq 0$, let $\beta_r(M, \mathscr{F})$ be the dimension of $H^r(M, \mathscr{F})$.

THEOREM 2. *If $(M, \mathscr{F})$ has positive sectional curvature, then $\beta_1(M, \mathscr{F}) = 0$.*

Recall from [13] the definition of the growth of a leaf L of $\mathscr{F}$. Pick a Riemannian metric on M (bundle-like or not) and restrict to obtain a

Riemannian metric on L. Let $p \in L$ and define the growth function of L at p by $g_p(r) = \mathrm{vol}(B_p(r))$ where $B_p(r)$ denotes the open ball in L of radius r centered at p. The growth type of L is then defined to be the growth type of the function $g_p : R^+ \to R^+$ and is independent of the choice of metric on M and of $p \in L$ [13].

THEOREM 3. *The growth type of each leaf of $\mathscr{F}$ is dominated by the growth type of $\pi_1(M)$.*

§ 2. Proofs

Let $\pi : F(Q) \to M$ be the frame bundle of Q, let ω be the connection form on $F(Q)$ associated to ∇, and let $H \subset T(F(Q))$ be the corresponding horizontal distribution. Let $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ be an R^q-cocycle defining $\mathscr{F}$. Let $F(R^q)$ be the frame bundle of R^q. Then $\{(\pi^{-1}(U_\alpha), f_{\alpha *}, g_{\alpha\beta *})\}_{\alpha,\beta \in A}$ is an $F(R^q)$-cocycle on $F(Q)$ and defines a codimension $q(q+1)$ foliation $\mathscr{F}'$ of $F(Q)$. Let $E' \subset T(F(Q))$ be the integrable distribution whose integral manifolds are the leaves of $\mathscr{F}'$. Since ∇ is basic we have $E' \subset H$ [12]. Let θ be the R^q-valued one-form on $F(Q)$ defined by $\theta_u(Y) = u^{-1}(\pi_{*u}(Y)_Q)$ for $u \in F(Q)$, $Y \in T_u(F(Q))$ where $u : R^q \to Q_{\pi(u)}$ denotes the vector space isomorphism which sends the standard basis of R^q to the frame u of $Q_{\pi(u)}$. The torsion form of ∇ is the R^q-valued two-form Θ on $F(Q)$ defined by $\Theta_u(X, Y) = (d\theta)_u(X_H, Y_H)$ for $u \in F(Q)$ and X, $Y \in T_u(F(Q))$. Since ∇ is the Riemannian basic connection, we have $\Theta = 0$.

For each $h \in R^q$ let $B(h)$ be the unique horizontal vector field on $F(Q)$ satisfying $\pi_{*u}(B(h)_u) = u(h)$ for all $u \in F(Q)$. Let $E^i = B(e_i)$ for $i = 1, \cdots, q$ where $\{e_1, \cdots, e_q\}$ is the standard basis of R^q and let $Q' \subset T(F(Q))$ be the q-plane bundle spanned by $E^1, \cdots, E^q$. Then $H = E' \oplus Q'$ and so $T(F(Q)) = E' \oplus Q' \oplus V$ where V is the bundle of vertical vectors. Hence we may regard $Q' \oplus V$ as the normal bundle of $\mathscr{F}'$. Let E_h^k be the $q \times q$ matrix with a 1 in the h^{th} column and k^{th} row and 0 elsewhere and let $\sigma(E_h^k)$ be the corresponding fundamental vector field on $F(Q)$. Then $\{E^i, \sigma(E_h^k) : i, h, k = 1, \cdots, q\}$ is a trivialization of the normal bundle of $\mathscr{F}'$. Recall that a vector field Y on $F(Q)$ which is normal to $\mathscr{F}'$ is said to be parallel along the leaves to $\mathscr{F}'$ if $(f_{\alpha *})_*(Y|\pi^{-1}(U_\alpha))$ is a well-defined vector field on $f_{\alpha *}(\pi^{-1}(U_\alpha)) \subset F(R^q)$ for each $\alpha \in A$. This is equivalent to Y being invariant under the natural parallelism along the leaves and is characterized by the condition that $[X, Y]$ is tangent to $\mathscr{F}'$ whenever X is a vector field tangent to $\mathscr{F}'$ [5]. Since the fundamental vector fields on $F(R^q)$ are preserved by the maps $g_{\alpha\beta *}$, it follows that the vector fields $\sigma(E_h^k)$ are parallel along the leaves to $\mathscr{F}'$. Since ∇ is the Riemannian basic connection, it is transversely projectable and hence $[X, E^i]$ is tangent to $\mathscr{F}'$ for $i = 1, \cdots, q$ whenever X is a vector field tangent to $\mathscr{F}'$ [10], and so $E^1, \cdots, E^q$ are parallel along the leaves to $\mathscr{F}'$. Hence $\{E^i, \sigma(E_h^k): i, h, k = 1, \cdots, q\}$ is an e-structure for $\mathscr{F}'$ [5].

Let $u \in F(Q)$ and let $X \in T_u(F(Q))$. Then there is a unique expression $X = X_{E'} + X_{Q'} + X_V$. Thus $\theta_u(X) = \theta_u(X_{Q'}) = \theta_u(B(h)_u) = h$ for some $h \in R^q$ and so $X_{Q'} = B(\theta_u(X))_u$. Also $\omega_u(X) = \omega_u(X_V) = \omega_u(\sigma(A)_u) = A$ for some $A \in \mathrm{gl}(q, R)$ and so $X_V = \sigma(\omega_u(X))_u$. Letting $X = [E^i, E^j]_u$, we obtain

$$[E^i, E^j]_{Q'_u} = B(\theta_u([E^i, E^j]_u))_u \quad \text{and} \quad [E^i, E^j]_{V_u} = \sigma(\omega_u([E^i, E^j]_u))_u \,.$$

Since

$$\begin{aligned} -\theta_u([E^i, E^j]_u) &= E_u^i\theta(E^j) - E_u^j\theta(E^i) - \theta_u([E^i, E^j]_u) \\ &= (d\theta)_u(E_u^i, E_u^j) = \Theta_u(E_u^i, E_u^j) = 0 \,, \end{aligned}$$

we have $[E^i, E^j]_{Q'} = 0$. Let Ω be the curvature form of ∇ and write $\Omega = \sum_{h,k=1}^q \Omega_k^h E_h^k$ where the Ω_k^h are two-forms on $F(Q)$. Since

$$\begin{aligned} -\omega_u([E^i, E^j]_u) &= E_u^i\omega(E^j) - E_u^j\omega(E^i) - \omega_u([E^i, E^j]_u) \\ &= (d\omega)_u(E_u^i, E_u^j) = \Omega_u(E_u^i, E_u^j) \,, \end{aligned}$$

we have that

$$[E^i, E^j]_V = -\sum_{h,k=1}^q \Omega_k^h(E^i, E^j)\sigma(E_h^k) \,.$$

Let $u_0 \in F(Q)$ and let $P(u_0) = \{u \in F(Q) : u$ can be joined to u_0 by a horizontal curve$\}$, the holonomy bundle through u_0. Let $u \in P(u_0)$ and let $p = \pi(u) \in M$. Let $c : (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $c(0) = p$, and let $c^* : (-\varepsilon, \varepsilon) \to F(Q)$ be the unique horizontal lift of c satisfying $c^*(0) = u$. Fix $1 \le i, j, l \le q$ and let $X(t) = c^*(t)_i$, $Y(t) = c^*(t)_j$, and $Z(t) = c^*(t)_l$. Then X, Y and Z are normal vector fields along c which are parallel along c and hence, since R is parallel, $\nabla_{\dot{c}(0)}R(X, Y)Z = 0$. Defining $f: (-\varepsilon, \varepsilon) \to R^q$ by $f(t) = c^*(t)^{-1}(R(X, Y)Z)_{c(t)}$, we have that $\nabla_{c(0)}R(X, Y)Z = u(f'(0))$ and so $f'(0) = 0$. But

$$(R(X, Y)Z)_{c(t)} = c^*(t)(\Omega_{c^*(t)}(E_{c^*(t)}^i, E_{c^*(t)}^j)c^*(t)^{-1}(c^*(t)_l))$$

and hence $f(t) = \Omega_{c^*(t)}(E_{c^*(t)}^i, E_{c^*(t)}^j)\cdot e_l = l^{\text{th}}$ column of $\Omega_{c^*(t)}(E_{c^*(t)}^i, E_{c^*(t)}^j)$. Thus $\dot{c}^*(0)(l^{\text{th}}$ column of $\Omega(E^i, E^j)) = f'(0) = 0$ and hence

$$W(l^{\text{th}} \text{ column of } \Omega(E^i, E^j)) = 0$$

for all $W \in H_u$. Thus $\Omega^h_k(E^i, E^j)$ is constant on $P(u_0)$ for each $1 \leq h, k \leq q$.

Let $\Phi(u_0)$ be the holonomy group with reference point u_0; that is, $\Phi(u_0) = \{A \in \mathrm{gl}(q, R) : u_0$ and $u_0 A$ can be joined by a horizontal curve$\}$. Then $P(u_0)$ is a reduced bundle with structure group $\Phi(u_0)$ such that the natural parallelism along the leaves of $\mathscr{F}$ carries elements of $P(u_0)$ to elements of $P(u_0)$ and so $P(u_0)$ is a transverse $\Phi(u_0)$-structure for $\mathscr{F}$ [5] and ω is reducible to a basic connection in $P(u_0)$. Let $V' \subset T(P(u_0))$ be the subbundle consisting of vectors tangent to the fibers of $P(u_0)$. Then $T(P(u_0)) = E' \oplus Q' \oplus V'$ and $\mathscr{F}'$ is a foliation of $P(u_0)$ whose tangent bundle is E'. Let $A_1, \cdots, A_r$ be a basis of the Lie algebra of $\Phi(u_0)$. Then $\{E^1, \cdots, E^q, \sigma(A_1), \cdots, \sigma(A_r)\}$ is an e-structure for $\mathscr{F}'$ on $P(u_0)$. On $P(u_0)$ we have

$$[E^i, E^j]_{Q'} = 0$$

$$[E^i, E^j]_{V'} = \sum_{k=1}^{r} f^k_{ij}\sigma(A_k)$$

$$[\sigma(A_i), \sigma(A_j)] = \sum_{k=1}^{r} c^k_{ij}\sigma(A_k)$$

$$[\sigma(A_i), E^j] = B(A_i \cdot e_j) = \sum_{k=1}^{q} b^k_{ij}E^k$$

where f^k_{ij}, c^k_{ij}, and b^k_{ij} are constants.

Let G be the unique simply connected Lie group with Lie algebra $\mathfrak{g}$ spanned by elements $Z_1, \cdots, Z_q, B_1, \cdots, B_r$ satisfying

$$[Z_i, Z_j] = \sum_{k=1}^{r} f^k_{ij}B_k$$

$$[B_i, B_j] = \sum_{k=1}^{r} c^k_{ij}B_k$$

$$[B_i, Z_j] = \sum_{k=1}^{q} b^k_{ij}Z_k\,.$$

Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ spanned by $B_1, \cdots, B_r$ and let $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ spanned by $Z_1, \cdots, Z_q$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Let $X \in \mathfrak{g}$ and write X uniquely as $X = Y + Z$ where $Y \in \mathfrak{h}$, $Z \in \mathfrak{m}$. Let $\tau(X) = Y - Z$. Then τ is an automorphism of $\mathfrak{g}$ and τ^2 is the identity. Since G is simply connected there is an automorphism $F : G \to G$ such that $F_* = \tau$. Let H be the identity component of the subgroup of G fixed by F. Then H is a closed Lie subgroup of G and the triple (G, H, F) is a symmetric space.

Let $u \in P(u_0)$. Since $E^1, \cdots, E^q$, $\sigma(A_1), \cdots, \sigma(A_r)$ are parallel along the leaves to $\mathscr{F}'$, there is a neighborhood W of u in $P(u_0)$ and a smooth submersion $\bar{f}: W \to G$ such that

$$\begin{aligned} &\operatorname{kernel}(\bar{f}_{*y}) = E'_y \\ &\bar{f}_{*y}(E^i_y) = Z_{i\bar{f}(y)},\ i = 1, \cdots, q \\ &\bar{f}_{*y}(\sigma(A_j)_y) = B_{j\bar{f}(y)},\ j = 1, \cdots, r \end{aligned}$$

for all $y \in W$. Let $U = \pi(W)$, a neighborhood of $\pi(u)$ in M. Then $\bar{f}$ induces a smooth submersion $f: U \to G/H$ such that $\operatorname{kernel}(f_{*p}) = E_p$ for all $p \in U$ and the diagram

$$\begin{array}{ccc} W & \xrightarrow{\ \bar{f}\ } & G \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ U & \xrightarrow[\ f\ ]{} & G/H \end{array}$$

commutes. Let $\bar{\theta}$ be the canonical left-invariant $\mathfrak{g}$-valued one-form on G and let $\bar{\theta}_{\mathfrak{h}}$ be the $\mathfrak{h}$-component of $\bar{\theta}$. Then $\bar{\theta}_{\mathfrak{h}}$ defines a G-invariant connection in the principal H-bundle $G \to G/H$ which induces the canonical linear connection on the symmetric space G/H [8], and $\bar{f}^*\bar{\theta}_{\mathfrak{h}} = \omega$. Thus if $F(G/H)$ is the frame bundle of G/H and ϖ is the connection form on $F(G/H)$ corresponding to the canonical linear connection on G/H, we have that $(f_*)^*\varpi = \omega$ on $F(Q)/_U$. Thus we can find a G/H-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ defining $\mathscr{F}$ such that $(f_{\alpha *})^*\varpi = \omega$ on $F(Q)/_{U_\alpha}$. If $U_\alpha \cap U_\beta \neq \phi$ then, since $(f_{\alpha *})^*\varpi = \omega = (f_{\beta *})^*\varpi$ on $F(Q)/_{U_\alpha \cap U_\beta}$, it follows that $(g_{\alpha\beta *})^*\varpi = \varpi$ on $F(G/H)/_{f_\beta(U_\alpha \cap U_\beta)}$. Without loss of generality we may assume that $U_\alpha \cap U_\beta$ is connected whenever it is non-empty. Hence, since ϖ is a complete analytic linear connection on the simply connected analytic manifold G/H, $g_{\alpha\beta}$ extends to an affine isomorphism of G/H [8].

Hence $\mathscr{F}$ is transversely homogeneous. The foliated manifold $(\tilde{M}, \tilde{\mathscr{F}})$ admits a complete bundle-like metric, and so we have that $\tilde{\mathscr{F}}$ is regular [1]. Hence the space of leaves $\tilde{M}/\tilde{\mathscr{F}}$ is a complete, Riemannian, Hausdorff manifold and the natural projection $f: \tilde{M} \to \tilde{M}/\tilde{\mathscr{F}}$ is a fibration [16]. Let N denote the Riemannian manifold $\tilde{M}/\tilde{\mathscr{F}}$. Since the metric on N is induced by the bundle-like metric on $\tilde{M}$, it follows that the curvature tensor field of N is parallel. Thus N is a complete, simply connected, Riemannian locally symmetric space and hence is Riemannian symmetric [8] and so Theorem 1 is proved.

https://doi.org/10.1017/S0027763000020390 Published online by Cambridge University Press

If $(M, \mathscr{F})$ has non-positive sectional curvature, then N has non-positive sectional curvature. Since N is complete and simply connected we have that N is diffeomorphic to R^q [8] and hence the fibration $f : \tilde{M} \to N$ is a product, proving Corollary 1. See [2] for similar results in the flat non-Riemannian case.

If $\pi_1(M)$ is finite, then $\tilde{M}$ is compact and so the leaves of $\tilde{\mathscr{F}}$ are compact. Thus all the leaves of $\mathscr{F}$ are compact. Hence $\mathscr{F}$ is a closed metric foliation and so the holonomy group of each leaf is finite [15]. Since N is compact it has non-negative sectional curvature [18] and so $(M, \mathscr{F})$ has non-negative sectional curvature, proving Corollary 2.

We now prove Theorem 2. Each covering transformation σ of $\tilde{M}$ induces an isometry $\Psi(\sigma)$ of N. We thus obtain a homomorphism $\Psi : \pi_1(M) \to I(N)$ where $I(N)$ denotes the isometry group of N such that $\Psi(\sigma) \circ f = f \circ \sigma$ for all $\sigma \in \pi_1(M)$. Let Σ be the image of Ψ in $I(N)$ and let K be the closure of Σ. For each $r \geq 0$ let $A_K^r(N)$ be the space of K-invariant r-forms on N and let $A^r(M, \mathscr{F})$ be the space of base-like r-forms on M. Let $\eta \in A_K^r(N)$. Then $f^*\eta$ is an r-form on $\tilde{M}$ which is base-like with respect to $\tilde{\mathscr{F}}$. Since η is Σ-invariant it follows that $f^*\eta$ is $\pi_1(M)$-invariant and hence there exists a unique $\omega \in A^r(M, \mathscr{F})$ such that $f^*\eta = p^*\omega$ where $p : \tilde{M} \to M$ is the covering projection. Conversely, suppose $\omega \in A^r(M, \mathscr{F})$. Then $p^*\omega$ is base-like and hence there exists a unique r-form η on N such that $p^*\omega = f^*\eta$. Since $p^*\omega$ is $\pi_1(M)$-invariant it follows that η is Σ-invariant and hence K-invariant. Thus $\eta \in A_K^r(N)$. We have constructed an isomorphism of cochain complexes

$$\begin{array}{ccccc} A^0(M, \mathscr{F}) & \xrightarrow{d} & A^1(M, \mathscr{F}) & \xrightarrow{d} & \cdots \\ \updownarrow & & \updownarrow & & \\ A_K^0(N) & \xrightarrow{d} & A_K^1(N) & \xrightarrow{d} & \cdots \end{array}$$

and hence we obtain an isomorphism in cohomology $H^*(M, \mathscr{F}) \to H_K^*(N)$. Since $(M, \mathscr{F})$ has positive sectional curvature it follows that N has positive sectional curvature. Thus N is compact [18] and so K is compact. Hence the inclusion of the algebra of K-invariant forms on N into the algebra of differential forms on N induces an injection $H_K^*(N) \to H^*(N)$ [4], [6]. Thus $H^*(M, \mathscr{F})$ is isomorphic to a subalgebra of $H^*(N)$. Since N is simply connected we have that $H^1(N) = 0$ and hence $H^1(M, \mathscr{F}) = 0$.

Let $\Sigma\backslash N$ be the orbit space of Σ. To prove Theorem 3 we construct a map $h : M/\mathscr{F} \to \Sigma\backslash N$ and apply a result in [1]. Let L be a leaf of $\mathscr{F}$.

Choose a leaf $\tilde{L}$ of $\tilde{\mathscr{F}}$ such that $p(\tilde{L}) = L$. Let $x = f(\tilde{L}) \in N$. Then the orbit of x under Σ depends only on L and we denote it by $h(L)$. The growth type of a leaf L of $\mathscr{F}$ is dominated by the growth type of $h(L)$ [1] which in turn is dominated by the growth type of Σ. Hence, since Σ is a homomorphic image of $\pi_1(M)$, we have that the growth type of L is dominated by the growth type of $\pi_1(M)$.

§ 3. Examples

Example 1. Let G be a compact connected Lie group of dimension q and let $\mathfrak{g}$ be the Lie algebra of G. Let M be a compact manifold and suppose ω is a smooth $\mathfrak{g}$-valued one-form of rank q on M satisfying $d\omega + \frac{1}{2}[\omega, \omega] = 0$. Then ω defines a smooth codimension q foliation $\mathscr{F}$ on M which is a Lie foliation modeled on G [7]. Let $\langle\, ,\rangle$ be a bi-invariant Riemannian metric on G. Then $\langle\, ,\rangle$ induces on $(M, \mathscr{F})$ a Riemannian structure with parallel curvature and non-negative sectional curvature.

Example 2. Let H be a Lie subgroup of the n-dimensional torus T^n. Then the foliation $\mathscr{F}$ of T^n by the cosets of H admits a Riemannian structure with vanishing curvature and $\beta_1(T^n, \mathscr{F}) \neq 0$.

Example 3. Let M be the unit tangent bundle of the two-holed torus T_2. Let $\mathscr{F}$ be the foliation of M by the circle fibers. Then $\mathscr{F}$ admits a Riemannian structure with parallel curvature and negative sectional curvature and $\beta_1(M, \mathscr{F}) \neq 0$.

Example 4. Let $\Phi : \pi_1(T_2) \to SO(3)$ be a homomorphism whose image is dense in $SO(3)$. This defines a left action of $\pi_1(T_2)$ on S^2. Let H be the universal cover of T_2. Then H is a principal $\pi_1(T_2)$-bundle over T_2. Let $M = H \times_{\pi_1(T_2)} S^2$ be the associated bundle with fiber S^2. The foliation of $H \times S^2$ whose leaves are the sets $H \times \{x\}$, $x \in S^2$ passes to a foliation $\mathscr{F}$ of M all of whose leaves are dense. Since $\pi_1(T_2)$ acts on S^2 by isometries, $(M, \mathscr{F})$ admits a Riemannian structure with parallel curvature and positive sectional curvature. Also $\beta_1(M, \mathscr{F}) = 0$.

Example 5. Define a left action of the integers Z on S^2 by letting the generator act as

$$\begin{pmatrix} \cos 2\pi\alpha & \sin 2\pi\alpha & 0 \\ -\sin 2\pi\alpha & \cos 2\pi\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $0 < \alpha < 1$ is irrational. Let $M = R \times_Z S^2$. Then M is an S^2-bundle over S^1 with a codimension 2 foliation $\mathscr{F}$. The foliated manifold $(M, \mathscr{F})$ admits a Riemannian structure with parallel curvature and positive sectional curvature and $\beta_1(M, \mathscr{F}) = 0$. There are exactly two compact leaves. If L is a non-compact leaf then $\bar{L}$ is diffeomorphic to T^2. The foliation of $\bar{L}$ by the leaves of $\mathscr{F}$ is an irrational slope foliation and hence is a flat Riemannian foliation and $\beta_1(\bar{L}, \mathscr{F}) \neq 0$.

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Department of Mathematics
Saint Louis University
St. Louis, Missouri 63103