

QUASIMODULAR FORMS AND COHOMOLOGY

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Abstract

We construct linear maps from the spaces of quasimodular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ to some cohomology spaces of the group Γ and prove that these maps are equivariant with respect to appropriate Hecke operator actions. The results are obtained by using the fact that there is a correspondence between quasimodular forms and certain finite sequences of modular forms.

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1. Introduction

It is well known that there is a Hecke equivariant map from modular forms of a given weight for a discrete subgroup Γ of $SL(2, \mathbb{R})$ to cocycles in some cohomology group of Γ , which determines the Eichler–Shimura isomorphism (see, for example, [3]). The goal of this paper is to study a quasimodular analogue of such a map.

Quasimodular forms generalise modular forms and were introduced by Kaneko and Zagier in [4]. Various arithmetic and geometric aspects of quasimodular forms have been investigated actively in recent years (see, for example, [2, 6, 8, 10, 12]). Given integers m and λ with $m \geq 0$, a holomorphic function f on the Poincaré upper half-plane \mathcal{H} is a quasimodular form for Γ of weight λ and depth at most m if there are holomorphic functions f_0, f_1, \dots, f_m on \mathcal{H} such that

$$(cz + d)^{-\lambda} f\left(\frac{az + b}{cz + d}\right) = f_0(z) + \frac{c}{cz + d} f_1(z) + \cdots + \left(\frac{c}{cz + d}\right)^m f_m(z)$$

for all $z \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. It is known that there is a correspondence between quasimodular forms and certain sequences of modular forms (see also [5]). More precisely, a quasimodular form can be expressed as a linear combination of derivatives of the corresponding modular forms.

Hecke operators play an important role in the theory of modular forms or more general automorphic forms, and they can be extended to the case of quasimodular

forms. If $\mathfrak{S}^n(\mathbb{C}^2)$ denotes the complex vector space \mathbb{C}^{n+1} equipped with the Γ -module structure given by the n th symmetric tensor representation of Γ , then a modular form of weight $n + 2$ determines a cocycle in the cohomology group $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$, which can be used to establish the Eichler–Shimura isomorphism. Such a map allows us to introduce Hecke operators on the cohomology space of the form $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ in such a way that the given map is Hecke equivariant. Hecke operators on the cohomology of more general groups were also investigated by Rhie and Whaples in [11].

In this paper we construct linear maps from the spaces of quasimodular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ to some cohomology spaces of the group Γ with coefficients of the form $\mathfrak{S}^\ell(\mathbb{C}^2)$ for some nonnegative integer ℓ and prove that these maps are equivariant with respect to appropriate Hecke operator actions. The results are obtained by using the above-mentioned correspondence between quasimodular forms and certain finite sequences of modular forms.

2. Quasimodular forms

Let \mathcal{H} be the Poincaré upper half-plane on which the group $SL(2, \mathbb{R})$ acts as usual by linear fractional transformations. Thus we may write

$$\gamma z = \frac{az + b}{cz + d}$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$. For the same z and γ , we set

$$\mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{R}(\gamma, z) = \frac{c}{cz + d}. \tag{2.1}$$

The resulting maps $\mathfrak{J}, \mathfrak{R} : SL(2, \mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C}$ can be shown to satisfy

$$\mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma'z)\mathfrak{J}(\gamma', z), \tag{2.2}$$

$$\mathfrak{R}(\gamma\gamma', z) = \mathfrak{J}(\gamma', z)^{-2}\mathfrak{R}(\gamma, \gamma'z) + \mathfrak{R}(\gamma', z), \tag{2.3}$$

for all $\gamma, \gamma' \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$.

Let R be the ring of holomorphic functions $f : \mathcal{H} \rightarrow \mathbb{C}$ satisfying the growth condition

$$|f(z)| \ll \left(\frac{\mathfrak{J}z}{1 + |z|^2} \right)^{-\nu}$$

for some $\nu > 0$ (see, for example, [7, Section 17.1] for a more precise description of this condition). We fix a nonnegative integer m and denote by $R_m[X]$ the complex vector space of polynomials in X over R of degree at most m . If a polynomial $\Phi(z, X) \in R_m[X]$ is of the form

$$\Phi(z, X) = \sum_{r=0}^m \phi_r(z)X^r \tag{2.4}$$

and λ is an integer with $\lambda > 2m$, we introduce two additional polynomials

$$(\Xi_\lambda^m \Phi)(z), (\Lambda_\lambda^m \Phi)(z) \in R_m[X],$$

defined by

$$(\Xi_\lambda^m \Phi)(z, X) = \sum_{r=0}^m \phi_r^\Xi(z) X^r, \quad (\Lambda_\lambda^m \Phi)(z, X) = \sum_{r=0}^m \phi_r^\Lambda(z) X^r, \tag{2.5}$$

where

$$\begin{aligned} \phi_r^\Xi &= \frac{1}{r!} \sum_{j=0}^{m-r} \frac{1}{j!(\lambda - 2r - j - 1)!} \phi_{m-r-j}^{(j)}, \\ \phi_r^\Lambda &= (\lambda + 2r - 2m - 1) \sum_{j=0}^r \frac{(-1)^j}{j!} (m - r + j)! \\ &\quad \times (2r + \lambda - 2m - j - 2)! \phi_{m-r+j}^{(j)}, \end{aligned}$$

for each $r \in \{0, 1, \dots, m\}$. These formulas determine isomorphisms

$$\Lambda_\lambda^m, \Xi_\lambda^m : R_m[X] \xrightarrow{\sim} R_m[X] \tag{2.6}$$

with

$$(\Lambda_\lambda^m)^{-1} = \Xi_\lambda^m \tag{2.7}$$

(see [5]).

Given $\gamma \in SL(2, \mathbb{R})$, $\lambda \in \mathbb{Z}$, $f \in R$ and $\Phi(z, X) \in R_m[X]$ as in (2.4), we set

$$\begin{aligned} (f \mid_\lambda \gamma)(z) &= \mathfrak{I}(\gamma, z)^{-\lambda} f(\gamma z), \\ (\Phi \mid_\lambda^X \gamma)(z, X) &= \sum_{r=0}^m (\phi_r \mid_{\lambda+2r} \gamma)(z) X^r, \end{aligned} \tag{2.8}$$

$$(\Phi \parallel_\lambda \gamma)(z, X) = \mathfrak{I}(\gamma, z)^{-\lambda} \Phi(\gamma z, \mathfrak{I}(\gamma, z)^2 (X - \mathfrak{R}(\gamma, z))) \tag{2.9}$$

for all $z \in \mathcal{H}$. Using (2.2) and (2.3), it can be shown that the operations \mid_λ , \mid_λ^X and \parallel_λ determine right actions of $SL(2, \mathbb{R})$ on R for the first one and on $R_m[X]$ for the other two. Furthermore, the two actions on $R_m[X]$ are compatible in such a way that

$$\begin{aligned} ((\Xi_\lambda^m \Phi) \parallel_\lambda \gamma)(z, X) &= \Xi_\lambda^m (\Phi \mid_{\lambda-2m}^X \gamma)(z, X), \\ ((\Lambda_\lambda^m \Phi) \mid_{\lambda-2m}^X \gamma)(z, X) &= \Lambda_\lambda^m (\Phi \parallel_\lambda \gamma)(z, X) \end{aligned}$$

for all $\gamma \in SL(2, \mathbb{R})$, where Ξ_λ^m and Λ_λ^m are the isomorphisms in (2.6) (see also [5]).

We now fix a discrete subgroup Γ of $SL(2, \mathbb{R})$ and consider the restrictions of the $SL(2, \mathbb{R})$ actions described above to Γ .

DEFINITION 2.1. (i) An element $f \in R$ is a *modular form for Γ of weight λ* if it satisfies

$$f \mid_\lambda \gamma = f$$

for all $\gamma \in \Gamma$.

(ii) A modular polynomial for Γ of weight λ and degree at most m is an element $F(z, X) \in R_m[X]$ satisfying

$$F \mid_{\lambda}^X \gamma = F$$

for all $\gamma \in \Gamma$.

(iii) An element $\Phi(z, X) \in R_m[X]$ is a quasimodular polynomial for Γ of weight λ and degree at most m if it satisfies

$$\Phi \parallel_{\lambda} \gamma = \Phi$$

for all $\gamma \in \Gamma$.

We denote by $M_{\lambda}(\Gamma)$ the space of modular forms for Γ of weight λ and by $MP_{\lambda}^m(\Gamma)$ and $QP_{\lambda}^m(\Gamma)$ the spaces of modular polynomials and quasimodular polynomials, respectively, for Γ of weight λ and degree at most m . From (2.6) we see that the maps Ξ_{λ}^m and Λ_{λ}^m induce the isomorphisms

$$\Xi_{\lambda}^m : MP_{\lambda-2m}^m(\Gamma) \rightarrow QP_{\lambda}^m(\Gamma), \quad \Lambda_{\lambda}^m : QP_{\lambda}^m(\Gamma) \rightarrow MP_{\lambda-2m}^m(\Gamma) \tag{2.10}$$

for each $\lambda \in \mathbb{Z}$ with $\lambda > 2m$.

DEFINITION 2.2. Given an integer λ , an element $\phi \in R$ is a quasimodular form for Γ of weight λ and depth at most m if there are functions $\phi_0, \phi_1, \dots, \phi_m \in R$ satisfying

$$(\phi \mid_{\lambda} \gamma)(z) = \sum_{r=0}^m \phi_r(z) \mathfrak{R}(\gamma, z)^r \tag{2.11}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\mathfrak{R}(\gamma, z)$ is as in (2.1). We denote by $QM_{\lambda}^m(\Gamma)$ the space of quasimodular forms for Γ of weight λ and depth at most m .

Let $\phi \in R$ be a quasimodular form belonging to $QM_{\lambda}^m(\Gamma)$ satisfying (2.11). Then it is known that for $0 \leq k \leq m$ the function ϕ_k is a quasimodular form belonging to $QM_{\lambda-2k}^{m-k}(\Gamma)$ (see, for example, [7]). In particular, since quasimodular forms of depth 0 are modular forms, ϕ_m is a modular form belonging to $M_{\lambda-2m}(\Gamma)$. We define the polynomial $(Q_{\lambda}^m \phi)(z, X) \in R_m[X]$ associated with ϕ by

$$(Q_{\lambda}^m \phi)(z, X) = \sum_{r=0}^m \phi_r(z) X^r. \tag{2.12}$$

Since ϕ determines the functions ϕ_r uniquely, $Q_{\lambda}^m \phi$ is well defined. Furthermore, it can be shown that formula (2.12) determines the isomorphism

$$Q_{\lambda}^m : QM_{\lambda}^m(\Gamma) \rightarrow QP_{\lambda}^m(\Gamma) \tag{2.13}$$

for each $\lambda \in \mathbb{Z}$ whose inverse is given by

$$(Q_{\lambda}^m)^{-1}(\Phi(z, X)) = \Phi(z, 0)$$

for $\Phi(z, X) \in QP_{\lambda}^m(\Gamma)$ (see also [1]). For $0 \leq k \leq m$ we define the projection map

$$\Pi_k^m : R_m[X] \rightarrow R \tag{2.14}$$

by setting

$$\Pi_k^m \Phi = \phi_k \tag{2.15}$$

for $\Phi(z, X)$ as in (2.4). Then

$$\Pi_k^m(MP_\lambda^m(\Gamma)) \subset M_{\lambda+2k}(\Gamma), \quad \Pi_k^m(QP_\lambda^m(\Gamma)) \subset QM_{\lambda-2k}^{m-k}(\Gamma); \tag{2.16}$$

hence by restriction we obtain the maps

$$\Pi_k^m : MP_\lambda^m(\Gamma) \rightarrow M_{\lambda+2k}(\Gamma), \quad \Pi_k^m : QP_\lambda^m(\Gamma) \rightarrow QM_{\lambda-2k}^{m-k}(\Gamma).$$

3. Group cohomology

Given a positive integer n , let $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ be the standard basis for the complex vector space \mathbb{C}^{n+1} , whose elements are regarded as column vectors, and set

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = \sum_{k=0}^n z_1^{n-k} z_2^k \mathbf{e}_{k+1} \in \mathbb{C}^{n+1}$$

for $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$. Then the n th symmetric tensor power

$$\rho_n : GL(2, \mathbb{C}) \rightarrow GL(n + 1, \mathbb{C})$$

of the standard representation of $GL(2, \mathbb{C})$ on \mathbb{C}^2 is given by

$$\rho_n(\gamma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = \left(\gamma \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^n$$

for all $\gamma \in GL(2, \mathbb{C})$. We now define the vector-valued function $\mathbf{v}_n : \mathcal{H} \rightarrow \mathbb{C}^{n+1}$ on \mathcal{H} by

$$\mathbf{v}_n(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}^n = \sum_{k=0}^n z^{n-k} \mathbf{e}_{k+1} = \sum_{k=0}^n z^k \mathbf{e}_{n-k+1} \tag{3.1}$$

for all $z \in \mathcal{H}$. Then, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$,

$$\rho_n(\gamma) \mathbf{v}_n(z) = \begin{pmatrix} az + b \\ cz + d \end{pmatrix}^n = (cz + d)^n \mathbf{v}_n(\gamma z) = \mathfrak{J}(\gamma, z)^n \mathbf{v}_n(\gamma z), \tag{3.2}$$

where $\mathfrak{J}(\gamma, z)$ is as in (2.2).

We denote by $\mathfrak{S}^n(\mathbb{C}^2)$ the complex vector space \mathbb{C}^{n+1} equipped with the structure of a $GL(2, \mathbb{C})$ module given by

$$(\gamma, v) \mapsto (\det \gamma)^{-n/2} \rho_n(\gamma)v$$

for $\gamma \in GL(2, \mathbb{C})$ and $v \in \mathbb{C}^{n+1}$. If Γ is a discrete subgroup of $SL(2, \mathbb{R}) \subset GL(2, \mathbb{C})$ as in Section 2, its first cohomology group with coefficients in $\mathfrak{S}^n(\mathbb{C}^2)$ can be described

as follows. The set $Z^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ of 1-cocycles consists of all maps $u : \Gamma \rightarrow \mathbb{C}^{n+1}$ satisfying

$$u(\gamma\gamma') = u(\gamma) + \rho_n(\gamma)u(\gamma') \tag{3.3}$$

for all $\gamma, \gamma' \in \Gamma$. Given an element $v_0 \in \mathbb{C}^{n+1}$, the set $B^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ of coboundaries consists of the maps $v : \Gamma \rightarrow \mathbb{C}^{n+1}$ such that

$$v(\gamma) = (\rho_n(\gamma) - 1)v_0$$

for all $\gamma \in \Gamma$, where 1 is the identity map on \mathbb{C}^{n+1} . Then the first cohomology group of Γ with coefficients in $\mathfrak{S}^n(\mathbb{C}^2)$ is given by

$$H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2)) = \frac{Z^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))}{B^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))}. \tag{3.4}$$

We now fix a nonnegative integer m and a point $z_0 \in \mathcal{H}$ and consider a quasimodular form $\phi \in QM_{2\nu}^m(\Gamma)$ satisfying (2.11) with $\lambda = 2\nu$ for some integer $\nu > m$. If k and r are integers with $0 \leq r \leq k \leq m$, we set

$$\widehat{\omega}_{m,\nu}^{k,r}(\phi)(\gamma) = \sum_{\ell=0}^{2(k+\nu-m-1)} \int_{z_0}^{\gamma z_0} \phi_{m-k+r}^{(r)}(z) z^\ell \mathbf{e}_{2(k+\nu-m)-\ell-1} dz \in \mathbb{C}^{2(k+\nu-m)-1} \tag{3.5}$$

for all $\gamma \in \Gamma$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_{2(k+\nu-m)-1}\}$ is the standard basis for $\mathbb{C}^{2(k+\nu-m)-1}$. Note that the integral is independent of the choice of the path $z_0 \rightarrow \gamma z_0$ because the functions ϕ_k are holomorphic. We now define the map

$$\mathcal{L}_{m,\nu}^k(\phi) : \Gamma \rightarrow \mathbb{C}^{2(k+\nu-m)-1} \tag{3.6}$$

associated with ϕ by

$$\mathcal{L}_{m,\nu}^k(\phi)(\gamma) = \sum_{r=0}^k \frac{(-1)^r}{r!} (2(k+\nu-m-1)-r)!(m-k+r)! \widehat{\omega}_{m,\nu}^{k,r}(\gamma) \tag{3.7}$$

for all $\gamma \in \Gamma$.

THEOREM 3.1. *The map $\mathcal{L}_{m,\nu}^k(\phi)$ in (3.6) is a cocycle belonging to $Z^1(\Gamma, \mathfrak{S}^{2(k+\nu-m)}(\mathbb{C}^2))$, which induces a complex linear map*

$$\mathcal{L}_{m,\nu}^k : QM_{2\nu}^m(\Gamma) \rightarrow H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m)}(\mathbb{C}^2)) \tag{3.8}$$

sending a quasimodular form $\phi(z, X) \in QM_{2\nu}^m(\Gamma)$ to the cohomology class of $\mathcal{L}_{m,\nu}^k(\phi)$ in $H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m)}(\mathbb{C}^2))$.

PROOF. Let $\phi \in QM_{2\nu}^m(\Gamma)$, so that $\mathcal{Q}_{2\nu}^m(\phi) \in QP_{2\nu}^m(\Gamma)$. Then by (2.10) the polynomial $(\Lambda_{2\nu}^m \circ \mathcal{Q}_{2\nu}^m)(\phi)$ belongs to $MP_{2(\nu-m)}^m(\Gamma)$; hence, from (2.16),

$$(\Pi_k^m \circ \Lambda_{2\nu}^m \circ \mathcal{Q}_{2\nu}^m)(\phi) \in M_{2(k+\nu-m)}(\Gamma)$$

for $0 \leq k \leq m$. If ϕ satisfies (2.11) with $\lambda = 2\nu$, using (2.5), (2.12) and (2.15),

$$\begin{aligned}
 (\Pi_k^m \circ \Lambda_{2\nu}^m \circ \mathcal{Q}_{2\nu}^m)(\phi)(z) &= (2(k + \nu - m) - 1) \\
 &\times \sum_{r=0}^k \frac{(-1)^r}{r!} (2(k + \nu - m - 1) - r)! (m - k + r)! \phi_{m-k+r}^{(r)}.
 \end{aligned}
 \tag{3.9}$$

On the other hand, using (3.1), (3.5) and (3.7),

$$\begin{aligned}
 \mathcal{L}_{m,\nu}^k(\phi)(\gamma) &= \sum_{r=0}^k \sum_{\ell=0}^{2(k+\nu-m-1)} \frac{(-1)^r}{r!} (2(k + \nu - m - 1) - r)! (m - k + r)! \\
 &\times \int_{z_0}^{\gamma z_0} \phi_{m-k+r}^{(r)}(z) z^\ell \mathbf{e}_{2(k+\nu-m)-\ell-1} dz \\
 &= \sum_{r=0}^k \frac{(-1)^r}{r!} (2(k + \nu - m - 1) - r)! (m - k + r)! \\
 &\times \int_{z_0}^{\gamma z_0} \phi_{m-k+r}^{(r)}(z) \mathbf{v}_{2(k+\nu-m-1)}(z) dz.
 \end{aligned}$$

From this and (3.9),

$$\mathcal{L}_{m,\nu}^k(\phi)(\gamma) = \int_{z_0}^{\gamma z_0} \frac{(\Pi_k^m \circ \Lambda_{2\nu}^m \circ \mathcal{Q}_{2\nu}^m)(\phi)(z)}{2(k + \nu - m) - 1} \mathbf{v}_{2(k+\nu-m-1)}(z) dz.
 \tag{3.10}$$

To verify the cocycle condition (3.3) for $\mathcal{L}_{m,\nu}^k(\phi)$, we set

$$f_k = \frac{(\Pi_k^m \circ \Lambda_{2\nu}^m \circ \mathcal{Q}_{2\nu}^m)(\phi)}{2(k + \nu - m) - 1} \in M_{2(k+\nu-m)}(\Gamma).
 \tag{3.11}$$

Then, for $\gamma, \gamma' \in \Gamma$,

$$\begin{aligned}
 \mathcal{L}_{m,\nu}^k(\phi)(\gamma\gamma') &= \int_{z_0}^{\gamma\gamma' z_0} f_k(z) \mathbf{v}_{2(k+\nu-m-1)}(z) dz \\
 &= \int_{z_0}^{\gamma z_0} f_k(z) \mathbf{v}_{2(k+\nu-m-1)}(z) dz + \int_{\gamma z_0}^{\gamma\gamma' z_0} f_k(z) \mathbf{v}_{2(k+\nu-m-1)}(z) dz \\
 &= \mathcal{L}_{m,\nu}^k(\phi)(\gamma) + \int_{z_0}^{\gamma' z_0} f_k(\gamma z) \mathbf{v}_{2(k+\nu-m-1)}(\gamma z) d(\gamma z).
 \end{aligned}$$

However, using (3.2),

$$\mathbf{v}_{2(k+\nu-m-1)}(\gamma z) = \mathfrak{I}(\gamma, z)^{-2(k+\nu-m-1)} \rho_{2(k+\nu-m-1)}(\gamma) \mathbf{v}_{2(k+\nu-m-1)}(z).
 \tag{3.12}$$

Using this and the relations

$$f_k(\gamma z) = \mathfrak{I}(\gamma, z)^{2(k+\nu-m)} f_k(z), \quad d(\gamma z) = \mathfrak{I}(\gamma, z)^{-2} dz,$$

we see that

$$f_k(\gamma z) \mathbf{v}_{2(k+\nu-m-1)}(\gamma z) d(\gamma z) = \rho_{2(k+\nu-m-1)}(\gamma) f_k(z) \mathbf{v}_{2(k+\nu-m-1)}(z) d(z); \tag{3.13}$$

hence, from (3.10),

$$\begin{aligned} \int_{z_0}^{\gamma' z_0} f_k(\gamma z) \mathbf{v}_{2(k+\nu-m-1)}(\gamma z) d(\gamma z) &= \rho_{2(k+\nu-m-1)}(\gamma) \int_{z_0}^{\gamma' z_0} f_k(z) \mathbf{v}_{2(k+\nu-m-1)}(z) d(z) \\ &= \rho_{2(k+\nu-m-1)}(\gamma) \mathcal{L}_{m,\nu}^k(\phi)(\gamma'). \end{aligned} \tag{3.14}$$

Thus

$$\mathcal{L}_{m,\nu}^k(\phi)(\gamma\gamma') = \mathcal{L}_{m,\nu}^k(\phi)(\gamma) + \rho_{2(k+\nu-m-1)}(\gamma) \mathcal{L}_{m,\nu}^k(\phi)(\gamma'),$$

and therefore $\mathcal{L}_{m,\nu}^k(\phi)$ satisfies the cocycle condition (3.3). □

From Theorem 3.1 we obtain the complex linear map

$$\bigoplus_{k=0}^m \mathcal{L}_{m,\nu}^k : \mathcal{QM}_{2\nu}^m(\Gamma) \rightarrow \bigoplus_{k=0}^m H^1(\Gamma, \mathbb{C}^{\otimes 2(k+\nu-m-1)}(\mathbb{C}^2)) \tag{3.15}$$

for each $\nu > m$.

4. Hecke operators

We extend the formulas for \mathfrak{J} and \mathfrak{R} in (2.1) from $SL(2, \mathbb{R})$ to the group $GL^+(2, \mathbb{R})$ of 2×2 real matrices of positive determinant. We also extend the operations $|_\lambda$ and $\|_\lambda$ of $SL(2, \mathbb{R})$ in (2.8) and (2.9) to those of $GL^+(2, \mathbb{R})$ by setting

$$\begin{aligned} (f |_\lambda \alpha)(z) &= (\det \alpha)^{\lambda/2} \mathfrak{J}(\alpha, z)^{-\lambda} f(\alpha z) \\ (F \|_\lambda \alpha)(z, X) &= \det(\alpha)^{\lambda/2} \mathfrak{J}(\alpha, z)^{-\lambda} \\ &\quad \times F(\alpha z, \det(\alpha)^{-1} \mathfrak{J}(\alpha, z)^2 (X - \mathfrak{R}(\alpha, z))) \end{aligned} \tag{4.1}$$

for all $z \in \mathcal{H}$, $\alpha \in GL^+(2, \mathbb{R})$, $f \in R$ and $F(z, X) \in R_m[X]$.

Let Γ be a discrete subgroup of $SL(2, \mathbb{R})$ as in Section 2, and let $\widetilde{\Gamma}$ be its commensurator, that is, the set of elements $g \in GL^+(2, \mathbb{R})$ such that $g\Gamma g^{-1} \cap \Gamma$ has finite index in both Γ and $g\Gamma g^{-1}$. Given $\alpha \in \widetilde{\Gamma}$, the double coset $\Gamma\alpha\Gamma$ has a decomposition of the form

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^s \Gamma\alpha_i \tag{4.2}$$

for some $\alpha_i \in GL^+(2, \mathbb{R})$ with $i = 1, \dots, s$. For the same α and an integer k , the corresponding Hecke operator

$$T_k(\alpha) : M_k(\Gamma) \rightarrow M_k(\Gamma)$$

on modular forms is given by

$$T_k(\alpha)f = \sum_{i=1}^s (f |_k \alpha_i), \tag{4.3}$$

for $f \in M_k(\Gamma)$ (see, for example, [9]). Such operators can be used to introduce the Hecke operator

$$T_\lambda^M(\alpha) : MP_\lambda^m(\Gamma) \rightarrow MP_\lambda^m(\Gamma)$$

on modular polynomials by

$$(T_\lambda^M(\alpha)F)(z, X) = \sum_{r=0}^m (T_{\lambda+2r}(\alpha)\Pi_r^m F)(z)X^r \tag{4.4}$$

for $F(z, X) \in MP_\lambda^m(\Gamma)$, where the projection maps Π_r^m are as in (2.14). On the other hand, for quasimodular polynomials the Hecke operator

$$T_\lambda^P(\alpha) : QP_\lambda^m(\Gamma) \rightarrow QP_\lambda^m(\Gamma)$$

is given by

$$(T_\lambda^P(\alpha)F)(z, X) = \sum_{i=1}^s (F \parallel_\lambda \alpha_i)(z, X) \tag{4.5}$$

for $F(z, X) \in QP_\lambda^m(\Gamma)$, where \parallel_λ is as in (4.1) (see [1]). Using this and the isomorphism in (2.13), we can also introduce the corresponding Hecke operator

$$T_\lambda^Q(\alpha) : QM_\lambda^m(\Gamma) \rightarrow QM_\lambda^m(\Gamma)$$

on quasimodular forms by setting

$$T_\lambda^Q(\alpha)f = \sum_{i=1}^s (\mathcal{Q}_\lambda^m)^{-1}((\mathcal{Q}_\lambda^m f) \parallel_\lambda \alpha_i) = ((\mathcal{Q}_\lambda^m)^{-1} \circ T_\lambda^P(\alpha) \circ \mathcal{Q}_\lambda^m)f$$

for $f \in QM_\lambda^m(\Gamma)$, so that

$$T_\lambda^P(\alpha) \circ \mathcal{Q}_\lambda^m = \mathcal{Q}_\lambda^m \circ T_\lambda^Q(\alpha). \tag{4.6}$$

THEOREM 4.1. *The linear isomorphisms Ξ_λ^m and Λ_λ^m in (2.10) satisfy*

$$T_\lambda^P(\alpha) \circ \Xi_\lambda^m = \Xi_\lambda^m \circ T_{\lambda-2m}^M(\alpha), \quad T_{\lambda-2m}^M(\alpha) \circ \Lambda_\lambda^m = \Lambda_\lambda^m \circ T_\lambda^P(\alpha) \tag{4.7}$$

for each $\alpha \in \bar{\Gamma}$.

PROOF. Given a modular polynomial of the form

$$F(z, X) = \sum_{r=0}^m f_r(z)X^r \in MP_\lambda^m(\Gamma),$$

from (2.5) we see that

$$(\Xi_\lambda^m F)(z, X) = \sum_{r=0}^m f_r^\Xi(z)X^r,$$

where

$$f_r^{\Xi} = \sum_{j=0}^{m-r} \frac{1}{r!j!(\lambda - 2r - j - 1)!} f_{m-r-j}^{(j)}$$

For $\alpha \in \widetilde{\Gamma}$, using (4.1) and (4.5),

$$\begin{aligned} & ((T_\lambda^P(\alpha) \circ \Xi_\lambda^m)F)(z, X) \\ &= \sum_{i=1}^s (\det \alpha_i)^{\lambda/2} \mathfrak{Z}(\alpha_i, z)^{-\lambda} \\ &\quad \times (\Xi_\lambda^m F)(\alpha_1 z, (\det \alpha_i)^{-1} \mathfrak{Z}(\alpha_i, z)^2 (X - \mathfrak{R}(\alpha_i, z))) \\ &= \sum_{i=1}^s \sum_{r=0}^m (\det \alpha_i)^{\lambda/2-r} \mathfrak{Z}(\alpha_i, z)^{2r-\lambda} (X - \mathfrak{R}(\alpha_i, z))^r f_r^{\Xi}(\alpha_i z) \\ &= \sum_{i=1}^s \sum_{r=0}^m \sum_{k=0}^r (\det \alpha_i)^{\lambda/2-r} \mathfrak{Z}(\alpha_i, z)^{2r-\lambda} \binom{r}{k} (-\mathfrak{R}(\alpha_i, z))^{r-k} X^k f_r^{\Xi}(\alpha_i z) \\ &= \sum_{i=1}^s \sum_{k=0}^m \sum_{r=k}^m (-1)^{r-k} \binom{r}{k} (\det \alpha_i)^{\lambda/2-r} \mathfrak{Z}(\alpha_i, z)^{2r-\lambda} \mathfrak{R}(\alpha_i, z)^{r-k} f_r^{\Xi}(\alpha_i z) X^k. \end{aligned}$$

Hence

$$\begin{aligned} ((T_\lambda^P(\alpha) \circ \Xi_\lambda^m)F)(z, X) &= \sum_{i=1}^s \sum_{k=0}^m \sum_{r=k}^m \sum_{j=0}^{m-r} \binom{r}{k} \frac{(\det \alpha_i)^{\lambda/2-r} (-1)^{r-k}}{r!j!(\lambda - 2r - j - 1)!} \\ &\quad \times \mathfrak{Z}(\alpha_i, z)^{2r-\lambda} \mathfrak{R}(\alpha_i, z)^{r-k} f_{m-r-j}^{(j)}(\alpha_i z) X^k \\ &= \sum_{i=1}^s \sum_{k=0}^m \sum_{r=0}^{m-k} \sum_{j=0}^{m-k-r} \binom{r+k}{k} \frac{(\det \alpha_i)^{\lambda/2-r-k} (-1)^r}{(r+k)!j!(\lambda - 2r - 2k - j - 1)!} \\ &\quad \times \mathfrak{Z}(\alpha_i, z)^{2r+2k-\lambda} \mathfrak{R}(\alpha_i, z)^r f_{m-r-k-j}^{(j)}(\alpha_i z) X^k, \end{aligned}$$

which can be written in the form

$$\begin{aligned} & ((T_\lambda^P \circ \Xi_\lambda^m)F)(z, X) \\ &= \sum_{i=1}^s \sum_{k=0}^m \sum_{\ell=0}^{m-k} \sum_{j=0}^{\ell} \binom{\ell - j + k}{k} \frac{(-1)^{\ell-j} (\det \alpha_i)^{\lambda/2-k-\ell+j}}{(\ell - j + k)!j!(\lambda + 2k - 2\ell + j - 1)!} \\ &\quad \times \mathfrak{Z}(\alpha_i, z)^{2k+2\ell-2j-\lambda} \mathfrak{R}(\alpha_i, z)^{\ell-j} f_{m-k-\ell}^{(j)}(\alpha_i z) X^k \end{aligned} \tag{4.8}$$

by using $\ell = j + r$. On the other hand, from (4.4),

$$\begin{aligned} T_{\lambda-2m}^M(\alpha)F(z, X) &= \sum_{k=0}^m (T_{\lambda+2k-2m}(\alpha)f_k)(z)X^k \\ &= \sum_{i=1}^s \sum_{k=0}^m (f_k |_{\lambda+2k-2m} \alpha_i)(z)X^k. \end{aligned}$$

Thus, if we set

$$\phi_k = \sum_{i=1}^s (f_k |_{\lambda+2k-2m} \alpha_i)$$

and use (2.5), then

$$\begin{aligned} (\Xi_\lambda^m \circ T_{\lambda-2m}^M(\alpha))F(z, X) &= \sum_{k=0}^m \sum_{\ell=0}^{m-k} \frac{1}{k! \ell! (\lambda - 2k - \ell - 1)!} \phi_{m-k-\ell}^{(\ell)}(z) X^k \\ &= \sum_{i=1}^s \sum_{k=0}^m \sum_{\ell=0}^{m-k} \frac{1}{k! \ell! (\lambda - 2k - \ell - 1)!} \frac{d^\ell}{dz^\ell} (f_{m-k-\ell} |_{\lambda-2k-2\ell} \alpha_i) X^k. \end{aligned}$$

Note, however, that

$$\begin{aligned} \frac{d^\ell}{dz^\ell} (f_{m-k-\ell} |_{\lambda-2k-2\ell} \alpha_i) &= \sum_{j=0}^{\ell} (-1)^{\ell-j} \frac{\ell!}{j!} \binom{\lambda - 2k - \ell - 1}{\ell - j} (\det \alpha_i)^{\lambda/2-\ell+j-k} \\ &\quad \times \mathfrak{J}(\alpha_i, z)^{2k+2\ell-2j-\lambda} \mathfrak{R}(\alpha_i, z)^{\ell-j} f_{m-k-\ell}^{(j)}(\alpha_i z). \end{aligned}$$

Thus

$$\begin{aligned} (\Xi_\lambda^m \circ T_{\lambda-2m}^M(\alpha))F(z, X) &= \sum_{i=1}^s \sum_{k=0}^m \sum_{\ell=0}^{m-k} \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j} (\det \alpha_i)^{\lambda/2-\ell+j-k} \binom{\lambda - 2k - \ell - 1}{\ell - j}}{k! j! (\lambda - 2k - \ell - 1)!} \\ &\quad \times \mathfrak{J}(\alpha_i, z)^{2k+2\ell-2j-\lambda} \mathfrak{R}(\alpha_i, z)^{\ell-j} f_{m-k-\ell}^{(j)}(\alpha_i z) X^k, \end{aligned}$$

which can easily be shown to coincide with (4.8); hence we obtain the first relation in (4.7). The second relation follows from this and (2.7). \square

From (4.4), (4.6) and (4.7) we see that the diagram

$$\begin{array}{ccccccc} QM_\lambda^m(\Gamma) & \xrightarrow{Q_\lambda^m} & QP_\lambda^m(\Gamma) & \xrightarrow{\Lambda_\lambda^m} & MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\Pi_r^m} & M_{\lambda-2m+2r}^m(\Gamma) \\ T_\lambda^Q(\alpha) \downarrow & & T_\lambda^Q(\alpha) \downarrow & & T_{\lambda-2m}^M(\alpha) \downarrow & & T_{\lambda-2m+2r}(\alpha) \downarrow \\ QM_\lambda^m(\Gamma) & \xrightarrow{Q_\lambda^m} & QP_\lambda^m(\Gamma) & \xrightarrow{\Lambda_\lambda^m} & MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\Pi_r^m} & M_{\lambda-2m+2r}^m(\Gamma) \end{array} \tag{4.9}$$

is commutative for each $\alpha \in \widetilde{\Gamma}$ and $r \in \{0, 1, \dots, m\}$.

We now describe Hecke operators on cohomology. Given an element $\alpha \in \widetilde{\Gamma} \subset GL^+(2, \mathbb{R})$ such that the corresponding double coset has a decomposition as in (4.2), the corresponding Hecke operator on the cohomology group $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ in (3.4) can be determined as follows. If $\gamma \in \Gamma$, since $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma$, the decomposition in (4.2) can be written as

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^s \Gamma\alpha_i\gamma;$$

hence, for $1 \leq i \leq s$,

$$\alpha_i \gamma = \xi_i(\gamma) \alpha_{i(\gamma)} \tag{4.10}$$

for some element $\xi_i(\gamma) \in \Gamma$. We note that the set $\{\alpha_{1(\gamma)}, \dots, \alpha_{s(\gamma)}\}$ is a permutation of $\{\alpha_1, \dots, \alpha_s\}$. Then the Hecke operator $T_n^H(\alpha)$ on the cohomology space $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ is given by

$$(T_n^H(\alpha)(\phi))(\gamma) = \sum_{i=1}^s (\det \alpha_i)^{n/2} \rho_n(\alpha_i) \phi(\xi_i(\gamma)) \tag{4.11}$$

for each 1-cocycle ϕ and $\gamma \in \Gamma$ (see [11]).

THEOREM 4.2. *The linear map $\mathcal{L}_{m,v}^k$ with $k, v \in \mathbb{Z}$ and $0 \leq k \leq m < v$ in (3.8) satisfies*

$$\mathcal{L}_{m,v}^k \circ T_{2v}^Q = T_{2(k+v-m)}^H(\alpha) \circ \mathcal{L}_{m,v}^k$$

for each $\alpha \in \tilde{\Gamma}$.

PROOF. Given $\alpha \in \tilde{\Gamma}$ and $k, v \in \mathbb{Z}$ with $0 \leq k \leq m < v$, from the commutative diagram (4.9) we obtain

$$T_{2(k+v-m)}(\alpha) \circ \Pi_k^m \circ \Lambda_{2v}^m \circ \mathcal{Q}_{2v}^m = \Pi_k^m \circ \Lambda_{2v}^m \circ \mathcal{Q}_{2v}^m \circ T_{2v}^Q(\alpha). \tag{4.12}$$

Let $\phi \in QM_{2v}^m(\Gamma)$, and let $f_k \in M_{2(k+v-m)}(\Gamma)$ be the associated modular form given by (3.11). Using (3.10), (3.12) and (4.12) and assuming that the double coset $\Gamma \alpha \Gamma$ has a decomposition as in (4.2),

$$\begin{aligned} &\mathcal{L}_{m,v}^k(T_{2v}^Q(\alpha)\phi)(\gamma) \\ &= \int_{z_0}^{\gamma z_0} (T_{2(k+v-m)}(\alpha)f_k)(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ &= \sum_{i=1}^s (\det \alpha_i)^{k+v-m} \mathfrak{S}(\alpha_i, z)^{-2(k+v-m)} \int_{z_0}^{\gamma z_0} f_k(\alpha_i z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ &= \sum_{i=1}^s (\det \alpha_i)^{k+v-m} \rho_{2(k+v-m-1)}(\alpha_i)^{-1} \int_{z_0}^{\gamma z_0} f_k(\alpha_i z) \mathbf{v}_{2(k+v-m-1)}(\alpha_i z) d(\alpha_i z) \\ &= \sum_{i=1}^s (\det \alpha_i)^{k+v-m} \rho_{2(k+v-m-1)}(\alpha_i)^{-1} \int_{\alpha_i z_0}^{\alpha_i \gamma z_0} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \end{aligned}$$

for all $\gamma \in \Gamma$. Since $\alpha_i \gamma = \xi_i(\gamma) \alpha_{i(\gamma)}$ by (4.10), the above integral can be written in the form

$$\begin{aligned} \int_{\alpha_i z_0}^{\alpha_i \gamma z_0} &= \int_{z_0}^{\xi_i(\gamma) \alpha_{i(\gamma)} z_0} - \int_{z_0}^{\alpha_i z_0} \\ &= \left(\int_{z_0}^{\xi_i(\gamma) \alpha_{i(\gamma)} z_0} - \rho_{2(k+v-m-1)}(\xi_i(\gamma)) \int_{z_0}^{\alpha_{i(\gamma)} z_0} \right) \\ &\quad + \left(\rho_{2(k+v-m-1)}(\xi_i(\gamma)) \int_{z_0}^{\alpha_{i(\gamma)} z_0} - \int_{z_0}^{\alpha_i z_0} \right). \end{aligned}$$

However, using (3.12), (3.13) and (3.14),

$$\begin{aligned} & \left(\int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)z_0}} -\rho_{2(k+v-m-1)}(\xi_i(\gamma)) \int_{z_0}^{\alpha_{i(\gamma)z_0}} \right) f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ &= \int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)z_0}} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz - \int_{z_0}^{\alpha_{i(\gamma)z_0}} f_k(\xi_i(\gamma)z) \mathbf{v}_{2(k+v-m-1)}(\xi_i(\gamma)z) d(\xi_i(\gamma)z) \\ &= \int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)z_0}} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz - \int_{\xi_i(\gamma)z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)z_0}} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ &= \int_{z_0}^{\xi_i(\gamma)z_0} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz = \mathcal{L}_{m,v}^k(\phi)(\xi_i(\gamma)). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left(\rho_{2(k+v-m-1)}(\xi_i(\gamma)) \int_{z_0}^{\alpha_{i(\gamma)z_0}} - \int_{z_0}^{\alpha_i z_0} \right) f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ &= \rho_{2(k+v-m-1)}(\alpha_i) \rho_{2(k+v-m-1)}(\gamma) \rho_{2(k+v-m)}(\alpha_{i(\gamma)})^{-1} \\ & \quad \times \int_{z_0}^{\alpha_{i(\gamma)z_0}} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ & \quad - \int_{z_0}^{\alpha_i z_0} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz. \end{aligned}$$

Using the above relations, (4.11) and the fact that $\det \alpha_{i(\gamma)} = \det \alpha_i$,

$$\begin{aligned} & \mathcal{L}_{m,v}^k(T_{2v}^Q(\alpha)\phi)(\gamma) \\ &= \sum_{i=1}^s (\det \alpha_i)^{k+v-m} \rho_{2(k+v-m-1)}(\alpha_i)^{-1} \mathcal{L}_{m,v}^k(\phi)(\xi_i(\gamma)) \\ & \quad + \rho_{2(k+v-m-1)}(\gamma) \sum_{i=1}^s (\det \alpha_{i(\gamma)})^{k+v-m} \rho_{2(k+v-m-1)}(\alpha_{i(\gamma)})^{-1} \\ & \quad \times \int_{z_0}^{\alpha_{i(\gamma)z_0}} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ & \quad - \sum_{i=1}^s (\det \alpha_i)^{k+v-m} \rho_{2(k+v-m-1)}(\alpha_i)^{-1} \int_{z_0}^{\alpha_i z_0} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz \\ &= (T_{2(k+v-m-1)}^H(\alpha)\mathcal{L}_{m,v}^k(\phi))(\gamma) + (\rho_{2(k+v-m-1)}(\gamma) - 1)u, \end{aligned}$$

where 1 is the identity map on $\mathbb{C}^{2(k+v-m)+1}$, and

$$u = \sum_{i=1}^s (\det \alpha_i)^{k+v-m} \rho_{2(k+v-m-1)}(\alpha_i)^{-1} \int_{z_0}^{\alpha_i z_0} f_k(z) \mathbf{v}_{2(k+v-m-1)}(z) dz.$$

Hence $\mathcal{L}_{m,v}^k(T_{2v}^Q(\alpha)\phi)$ and $(T_{2(k+v-m-1)}^H(\alpha)\mathcal{L}_{m,v}^k(\phi))$ belong to the same cohomology class, and therefore the proof of the theorem is complete. \square

If the Hecke operator $\bigoplus_{k=0}^m T_{2(k+v-m-1)}^H(\alpha)$ on $\bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+v-m-1)}(\mathbb{C}^2))$ is defined componentwise, then we see from Theorem 4.2 that it is compatible with the Hecke operator $T_{2v}^Q(\alpha)$ on $QM_{2v}^m(\Gamma)$ under the map $\bigoplus_{k=0}^m \mathcal{L}_{m,v}^k$ in (3.15). Thus we obtain the commutative diagram

$$\begin{array}{ccc}
 QM_{2v}^m(\Gamma) & \xrightarrow{\bigoplus_{k=0}^m \mathcal{L}_{m,v}^k} & \bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+v-m-1)}(\mathbb{C}^2)) \\
 T_{2v}^Q(\alpha) \downarrow & & \downarrow \bigoplus_{k=0}^m T_{2(k+v-m-1)}^H(\alpha) \\
 QM_{2v}^m(\Gamma) & \xrightarrow{\bigoplus_{k=0}^m \mathcal{L}_{m,v}^k} & \bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+v-m-1)}(\mathbb{C}^2))
 \end{array}$$

for each $\alpha \in \tilde{\Gamma}$ and $v > m$.

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